The Variety of Subdivision Schemes

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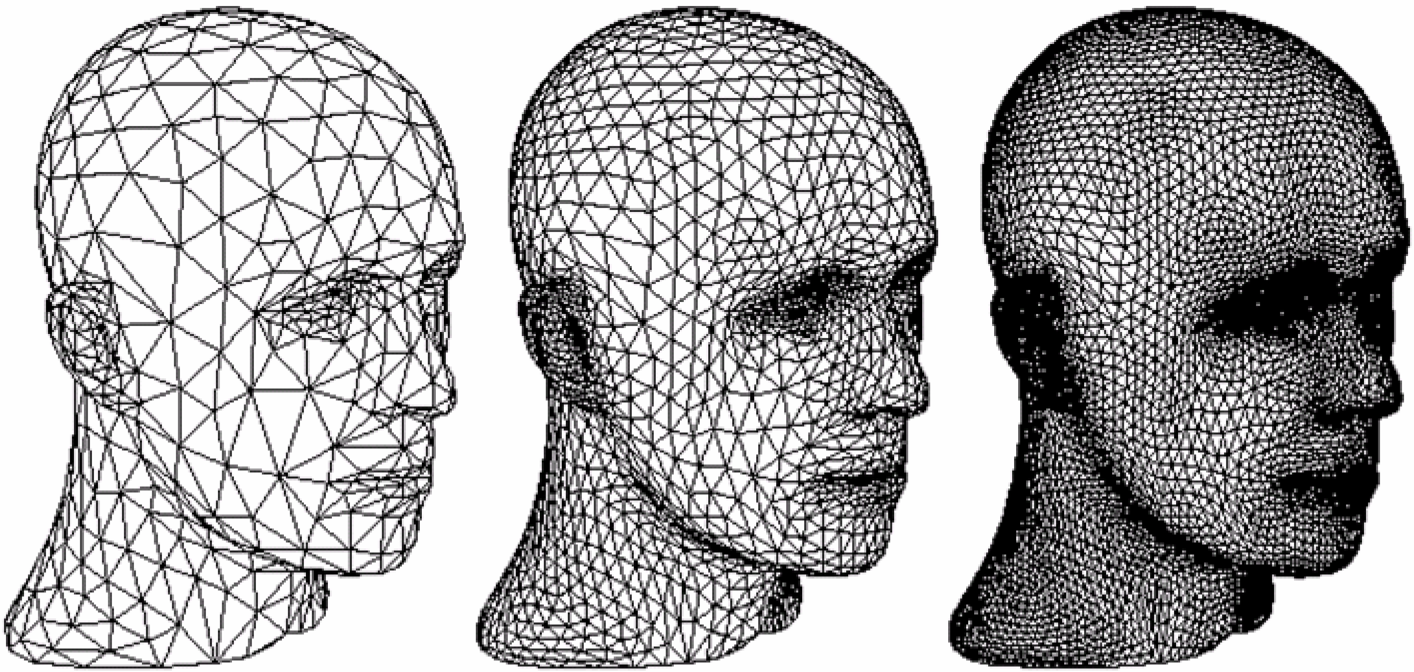
What is Subdivision?

- Define a smooth curves/surface as the limit of a sequence of successive refinements

$$p^{j+1} = Sp^j$$

$$(Sp)_a = \sum_b \alpha_{a-2b} p_b$$

Subdivision Surfaces



Why Subdivision?

- Arbitrary topology
- Multiresolution
- Simple code
- Efficient code
- Construct Wavelet

Chaikin's Algorithm :

$$p_{2i}^{k+1} = \frac{3}{4}p_i^k + \frac{1}{4}p_{i+1}^k$$

$$p_{2i+1}^{k+1} = \frac{1}{4}p_i^k + \frac{3}{4}p_{i+1}^k$$

converges to the quadratic B−spline.

Cubic spline Algorithm :

$$p_{2i}^{k+1} = \frac{1}{2}p_i^k + \frac{1}{2}p_{i+1}^k$$

$$p_{2i+1}^{k+1} = \frac{1}{8}p_i^k + \frac{3}{4}p_{i+1}^k + \frac{1}{8}p_{i+2}^k$$

converges to the cubic B-spline.

4-point interpolatory scheme (N. Dyn, D.Levin and J.Gregory):

$$p_{2i}^{k+1} = p_i^k$$

$$p_{2i+1}^{k+1} = \left(\frac{1}{2} + w\right)(p_i^k + p_{i+1}^k) - w(p_{i-1}^k + p_{i+2}^k)$$

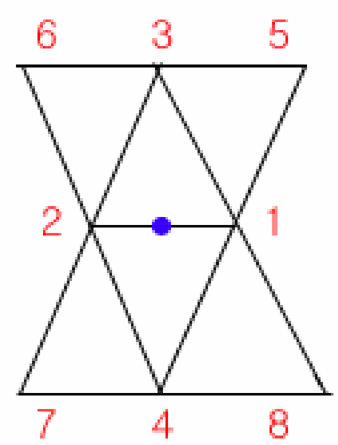
-continuous for $|w| < \frac{1}{4}$

-C^1 for $0 < w < \frac{1}{8}$

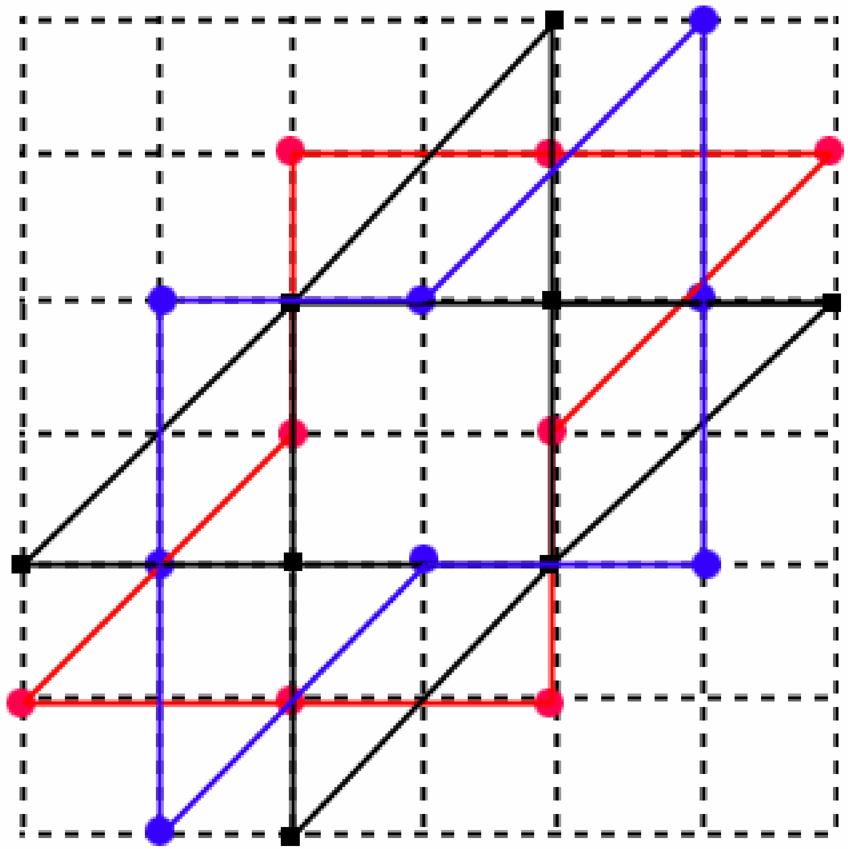
Butterfly scheme :

$$q_e^k = \frac{1}{2}(p_{e,1}^k + p_{e,2}^k) + 2w(p_{e,3}^k + p_{e,4}^k) - w\sum_{j=5}^{8} p_{e,j}^k$$

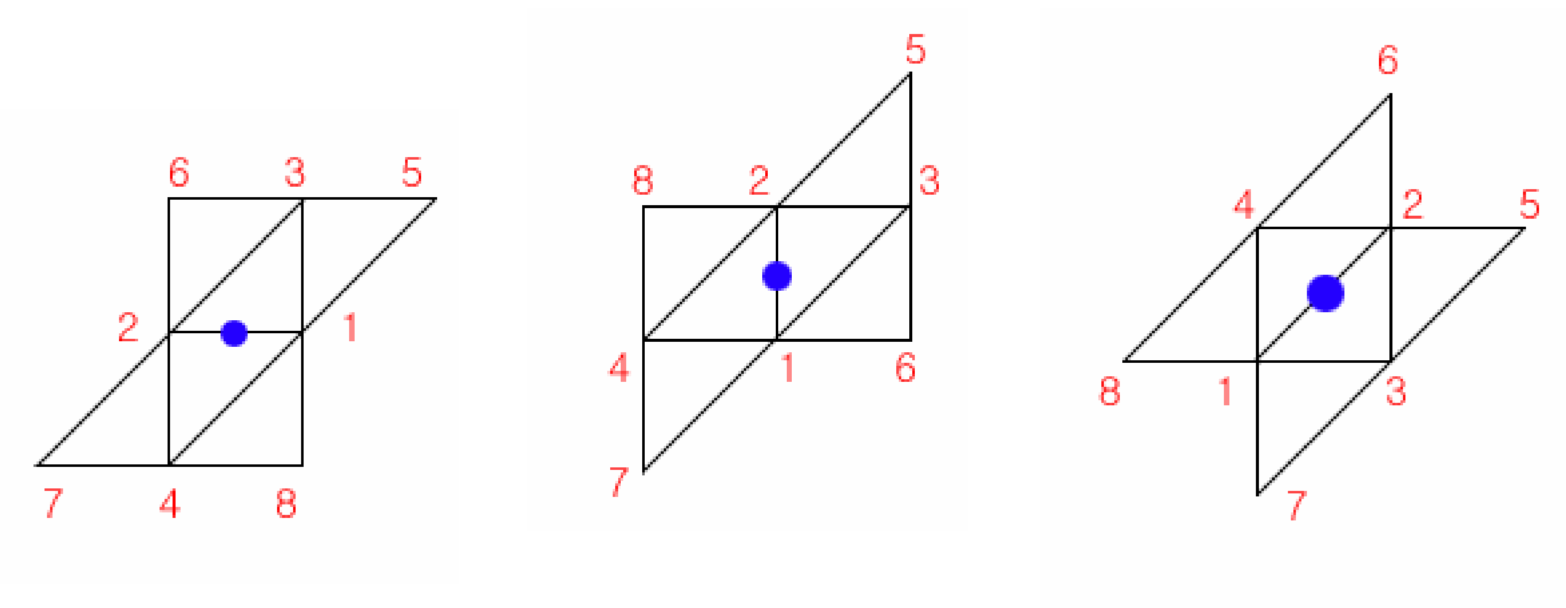
$$p_i^{k+1} = p_i^k \cup q_e^k$$



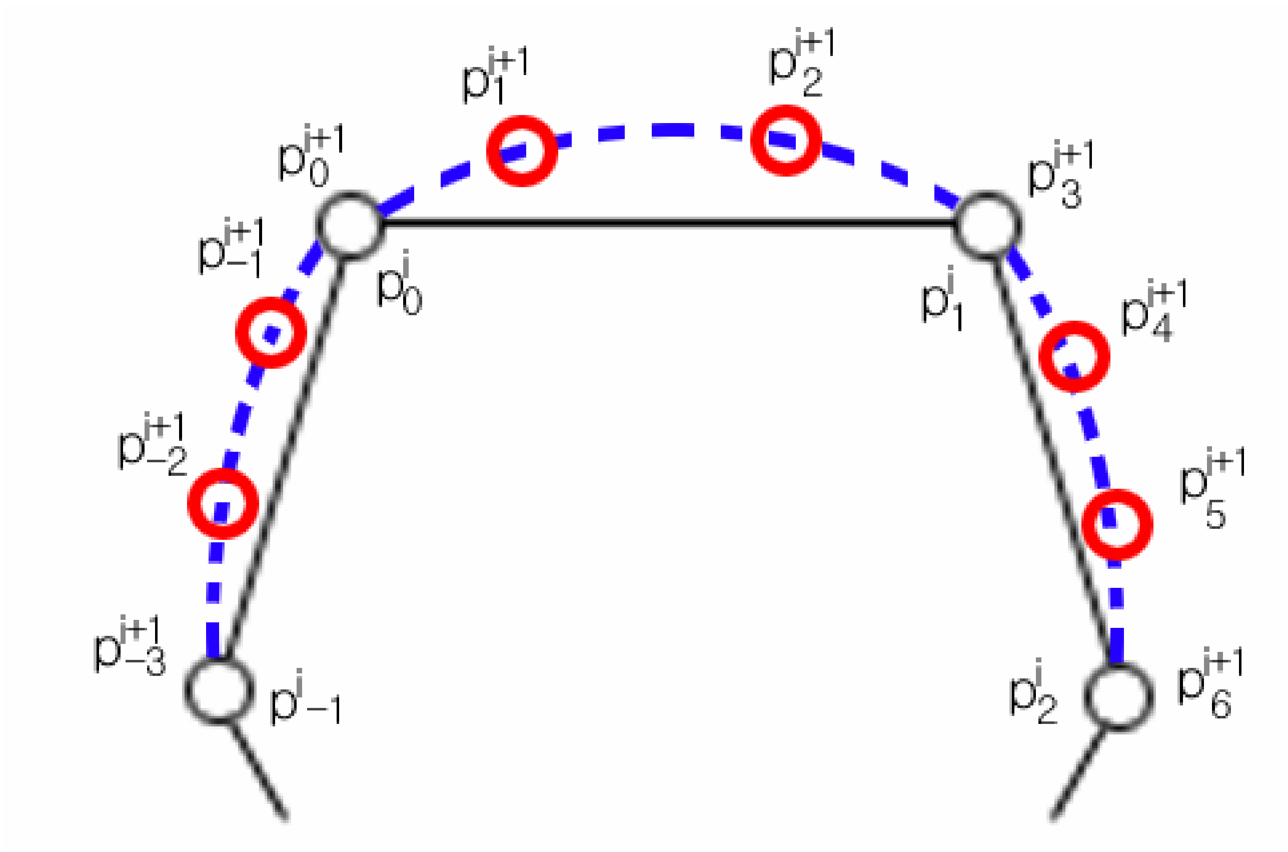
The mask of the butterfly scheme:



The mask of the butterfly scheme:



Ternary Subdivision Scheme:



$$p_{3j}^{j+1} = p_j^i$$

$$p_{3j+1}^{j+1} = ap_{j-1}^i + bp_j^i + cp_{j+1}^i + dp_{j+2}^i$$

$$p_{3j+2}^{j+1} = dp_{j-1}^i + cp_j^i + bp_{j+1}^i + ap_{j+2}^i$$

where the weights are given by

$$a = -\frac{1}{18} - \frac{1}{6}w, b = \frac{13}{18} + \frac{1}{2}w, c = \frac{7}{18} - \frac{1}{2}w, d = -\frac{1}{18} + \frac{1}{6}w.$$

— C^2 for $\frac{1}{15} < w < \frac{1}{9}$

Subdivision Zoo

Classification :

- stationary or non-stationary
- binary or ternary
- type of mesh(triangle or quadrilateral)
- approximating or interpolating
- linear or non-linear

Face split (primal type)

	Triangular meshes	Quad.meshes
approximating	Loop (C^2)	Catmull−Clark (C^2)
interpolating	Butterfly (C^1)	Kobbelt (C^1)

Vertex split (dual type)

	Quad. meshes
approximating	Doo−Sabin (C^1) , Midedge (C^1)

Binary Subdivision of B-splines

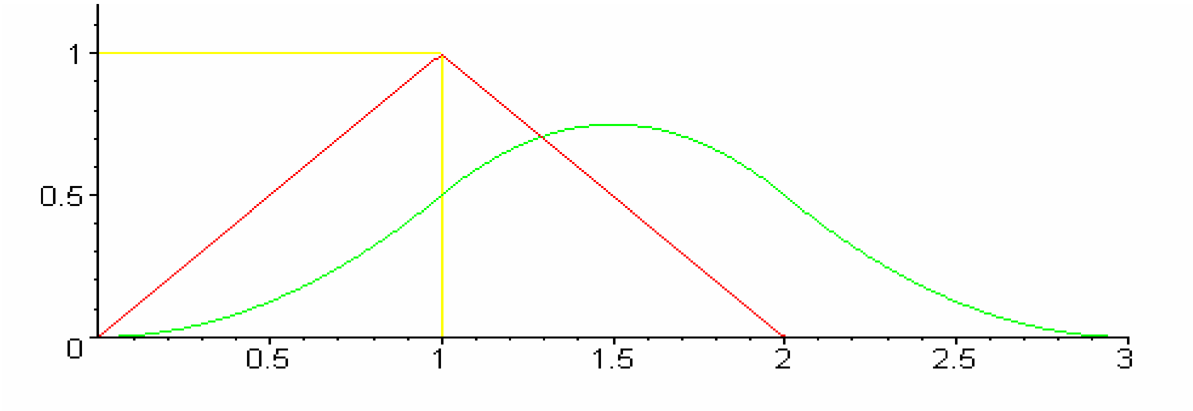
- Univariate B-splines :

$$(*) \quad S^r(u) = \sum_{i \in Z} d_i^r N^r(u-i),$$

where N^r : normalized B-spline of degree r.

Properties :

- Partition of unity : $\sum_i N^r(u-i) = 1$
- Positivity : $N^r(u) \geq 0$
- Local support : $N^r(u-i) = 0 \text{ if } u \notin [i, i+r+1]$
- Continuity : $N^r(u) \in C^{(r-1)}$
- Recursion : $N^r(u-i) = \frac{u-i}{r} N^{r-1}(u-i) + \frac{i+r+1-u}{r} N^{r-1}(u-i-1)$



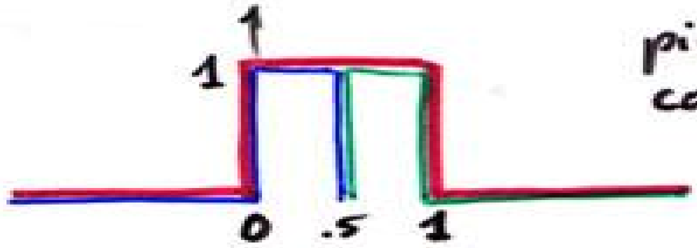
The idea behind a Subdivision

- Rewrite the curve (*) as a curve over a refined knot sequence $Z/2$.
- (*) becomes

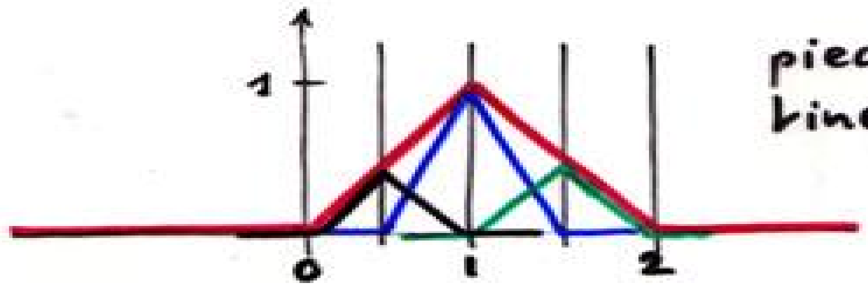
$$S^r(u) = \sum_{j \in Z/2} \hat{d}_j^r N^r(2(u-j))$$

- Determine $\hat{d}_j^r$ ⟵ A single B-spline can be decomposed into similar B-spline of half the support.

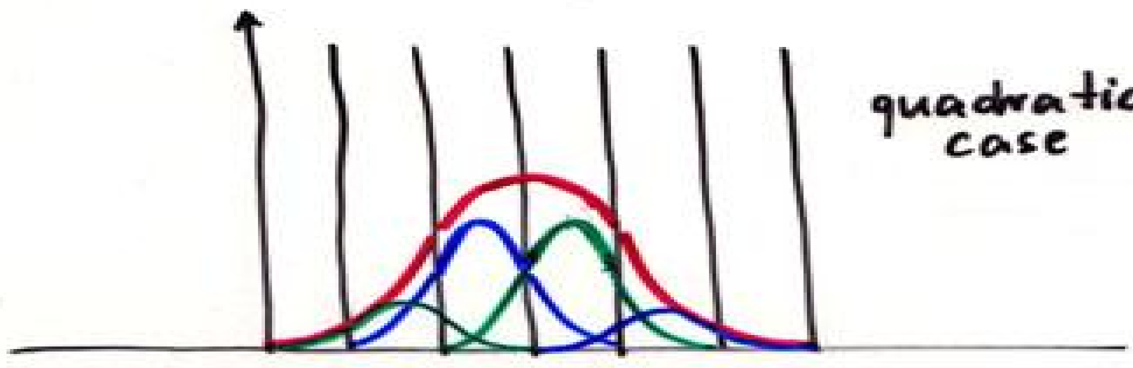
Refinement (Subdivision) of B-Spline Basis Functions



piecewise constant



piecewise linear



quadratic case

- This results in

$$N^r(u) = \sum_{j \in Z/2} c_j^r N^r(2(u-j)),$$

where $c_j^r = 2^{-r}\binom{r+1}{2j}$

$$\longrightarrow \quad \hat{d}_j^r = \sum_{i \in Z} c_{j-i}^r d_i^r, \ j \in Z/2$$

- For example $r = 2$

$$\hat{d}_0^2 = \sum_{i \in Z} c_{-i}^2 d_i^2 = \cdots + (3/4)d_{-1}^2 + (1/4)d_0^2 + \cdots$$

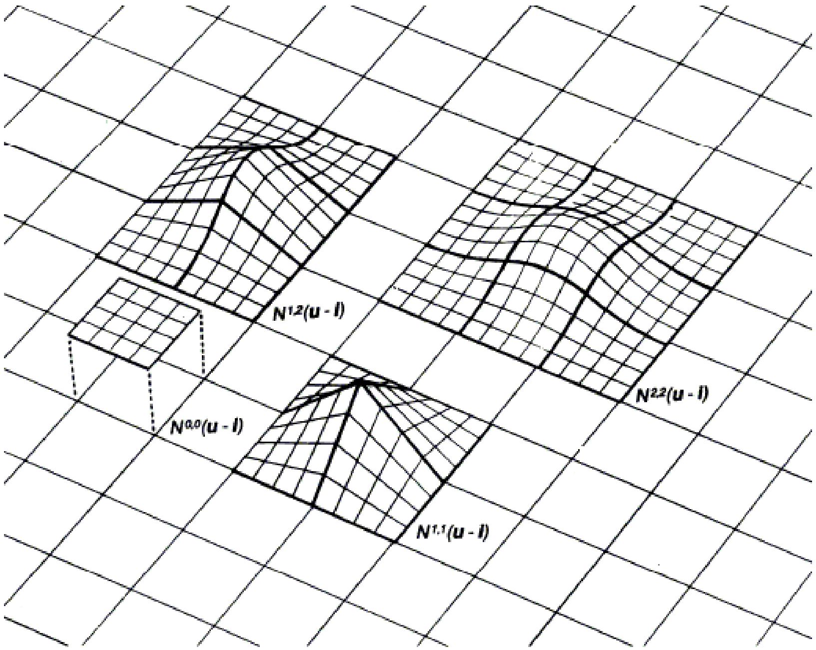
$$\hat{d}_{1/2}^2 = \cdots + (1/4)d_{-1}^2 + (3/4)d_0^2 + \cdots$$

$\longrightarrow$ Chaikin’s algorithm

Tensor Product B–spline Surfaces

$$S^{r,s}(\mathbf{u}) = \sum_{\mathbf{i} \in Z^2} d_i^{r,s} N^{r,s}(\mathbf{u} - \mathbf{i})$$

A tensor product B–spline is the product of two independently univariate B–splines, i.e



$$N^{r,s}(\mathbf{u} - \mathbf{i}) = N^r(u - i) N^s(v - j)$$

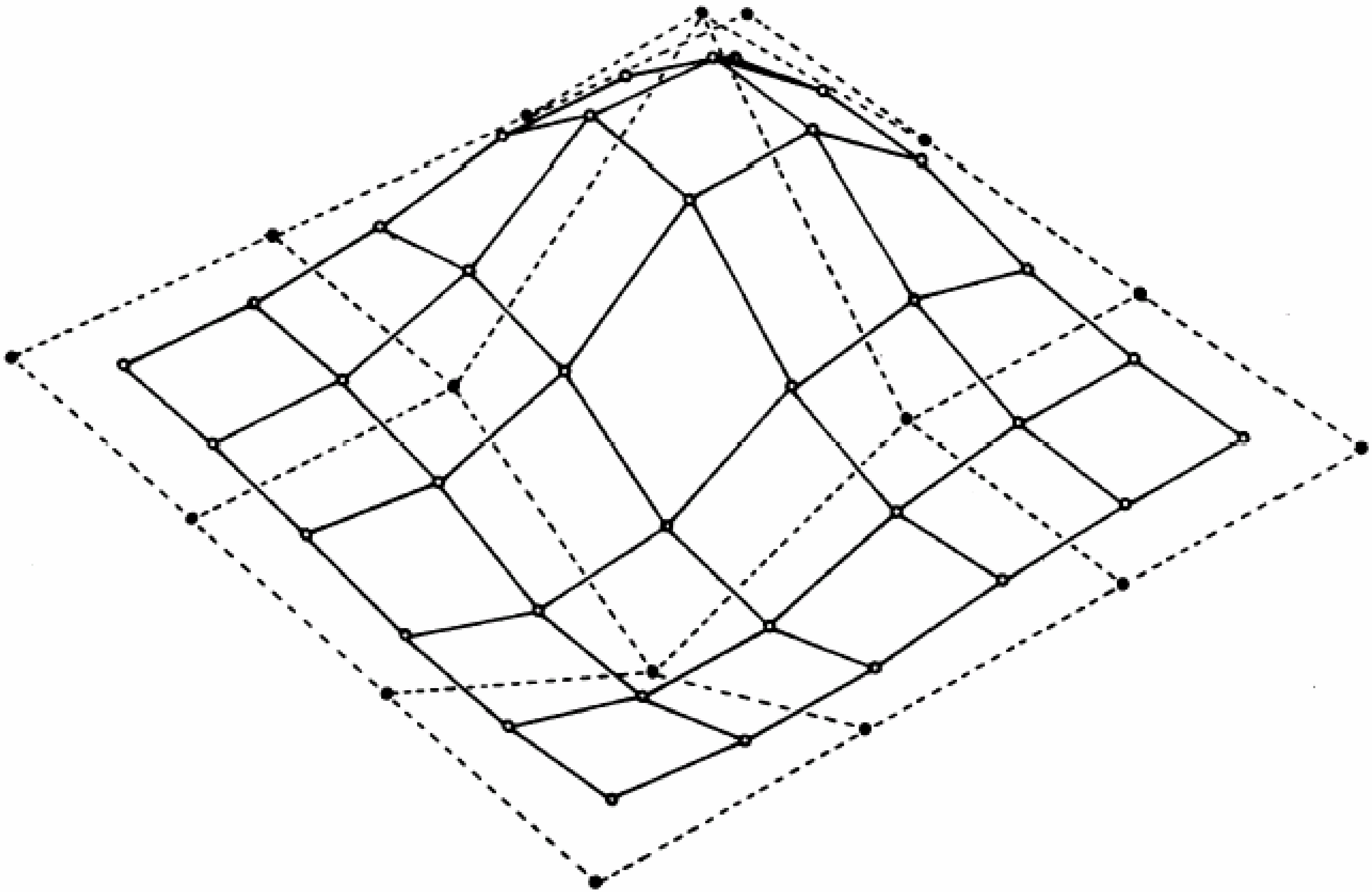
$$\longrightarrow \quad \hat{d}_j^{r,s} = \sum_{i \in Z^2} c_{j-i}^{r,s} d_i^{r,s}, \; j \in Z^2/2$$

$$\begin{aligned} c_{\mathbf{j}}^{r,s} &= c_i^r c_j^s \\ &= 2^{-(r+s)} \binom{r+1}{2i} \binom{s+1}{2j} \end{aligned}$$

- Example 1 $\quad r = s = 2$

$$\begin{aligned} \hat{d}_{0,0}^{2,2} &= \cdots + (\frac{9}{16}) d_{-1,-1}^{2,2} + (\frac{3}{16}) + d_{0,-1}^{2,2} + \cdots \\ & \cdots + (\frac{3}{16}) d_{-1,0}^{2,2} + (\frac{1}{16}) + d_{0,0}^{2,2} + \cdots \end{aligned}$$

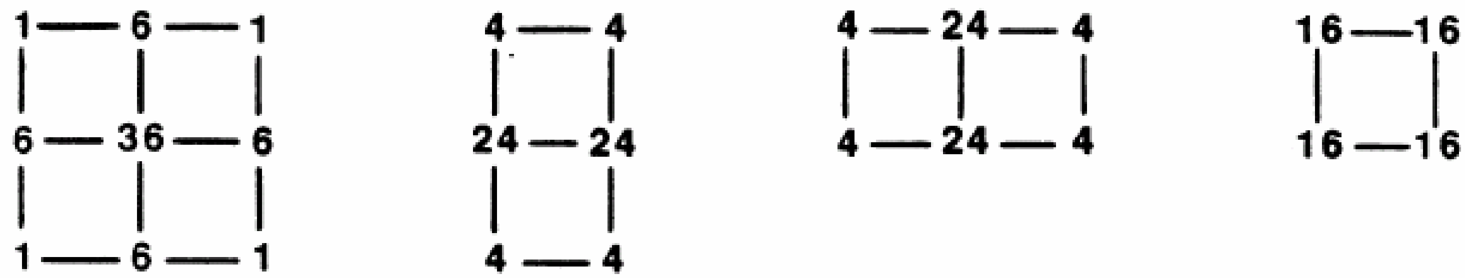
- The mask set :

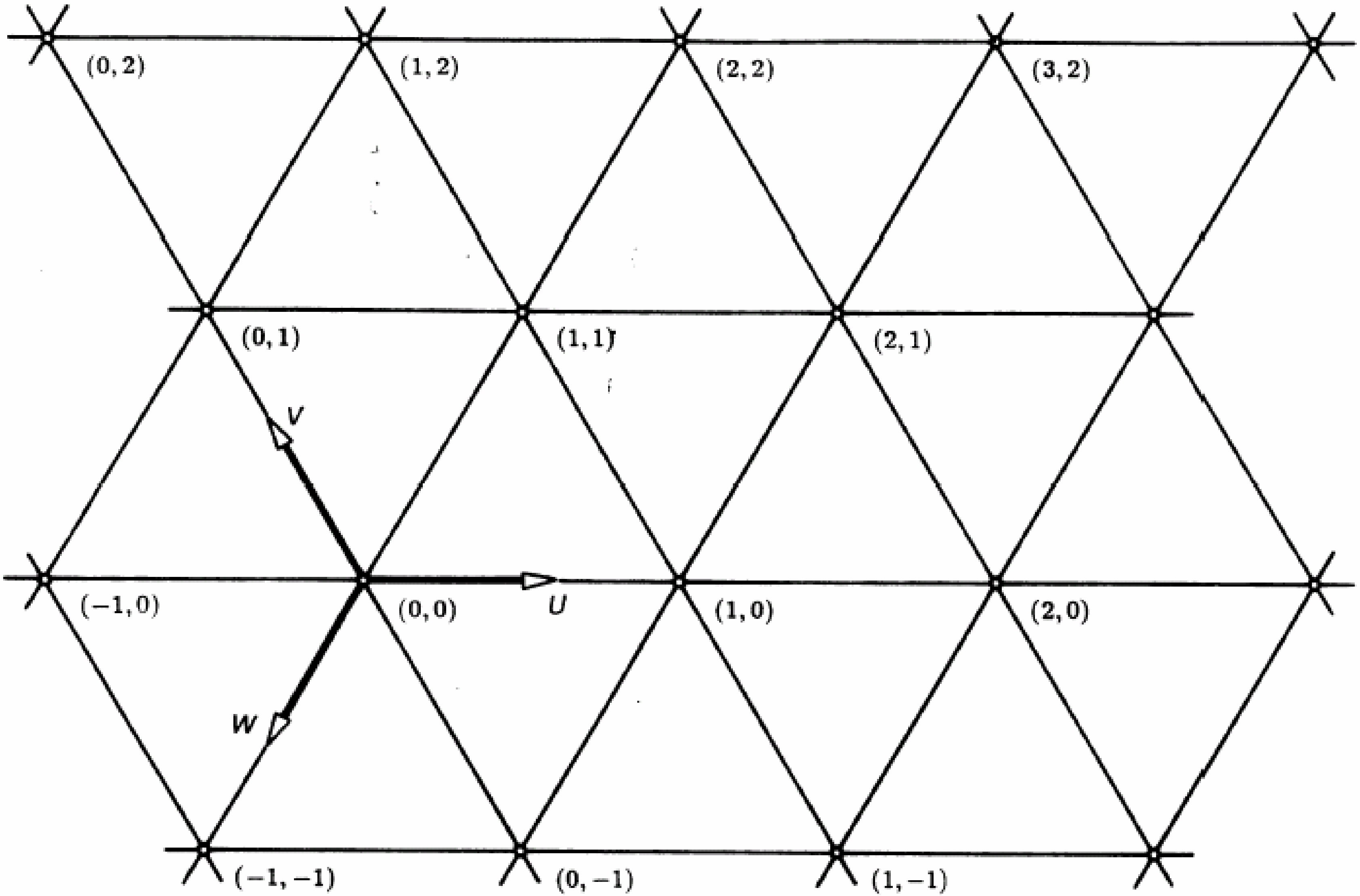


Construction of the refined de Boor net for $S^{2,2}(\mathbf{u})$.

- Example 2 $r = s = 3$

The mask for bicubic tensor product B-splines





Triangular grid.

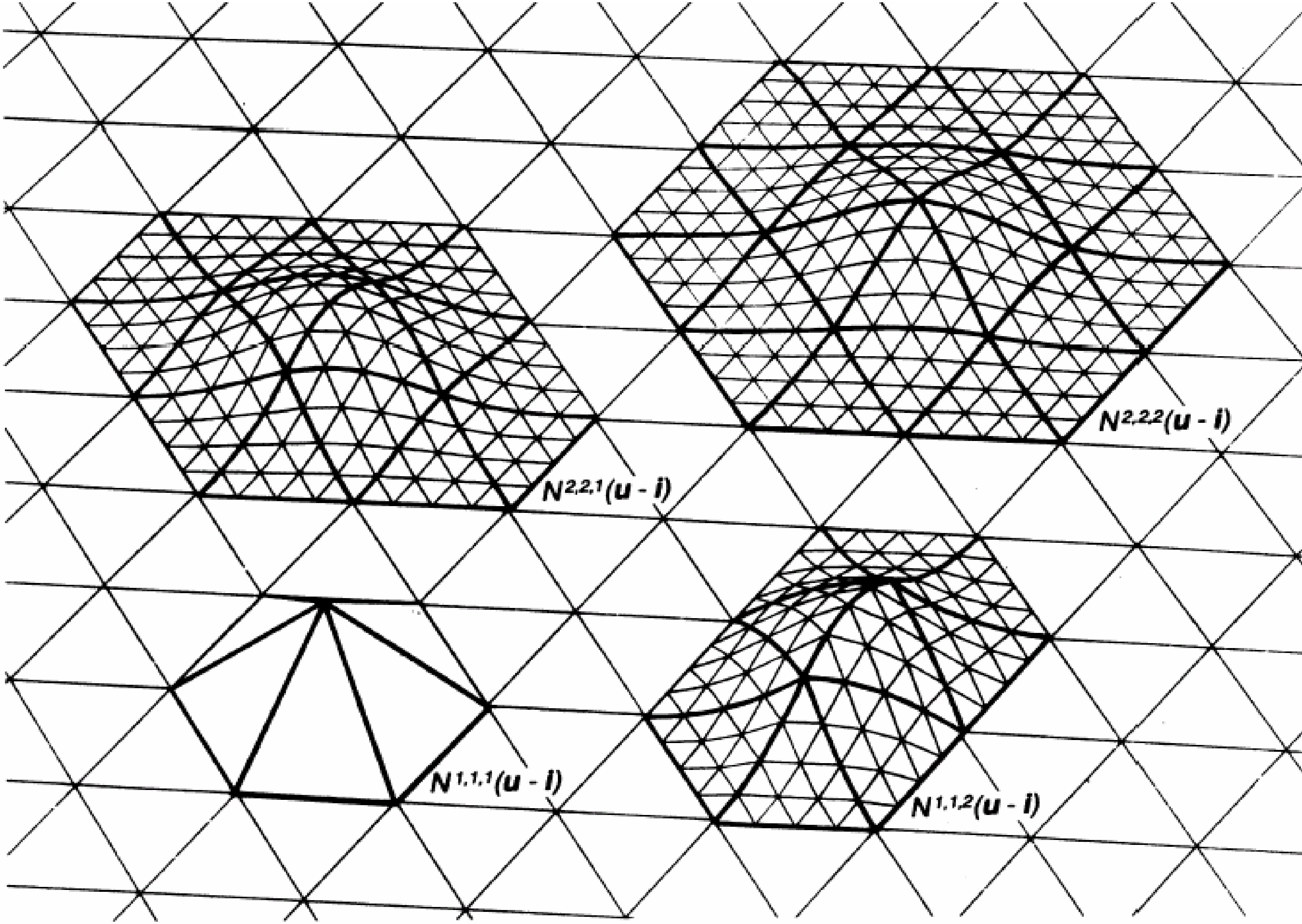
Triangular Splines

- A triangular spline surface :

$$(*) \quad S^{r,s,t}(\mathbf{u}) = \sum_{\mathbf{i}} d_{\mathbf{i}}^{r,s,t} N^{r,s,t}(\mathbf{u}-\mathbf{i}), \quad \mathbf{u} \in R^2, \mathbf{i} \in Z^2$$

where

$N^{r,s,t}(\mathbf{u})$: a normalized triangular spline of degree (r+s+t−2)



Examples of triangular splines.

Subdivision of Triangular Splines

- The surface (*) can be rewritten over the refine grid.

$$S^{r,s,t}(\mathbf{u}) = \sum_{\mathbf{j}} \hat{d}_{\mathbf{j}}^{r,s,t} N^{r,s,t} 2(\mathbf{u} - \mathbf{j})$$

- A single triangular spline is decomposed into splines of identical degree over the refined grid.

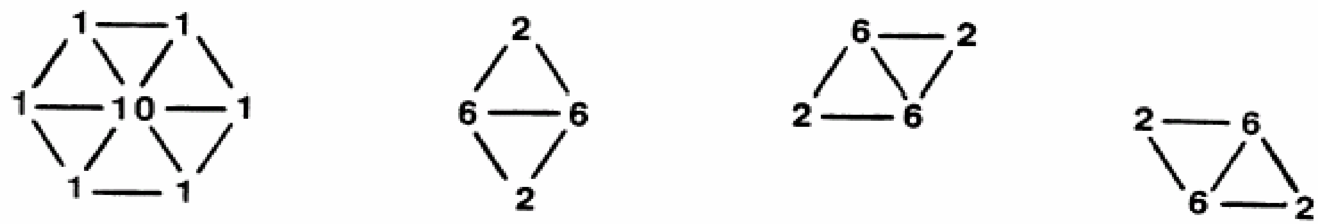
$$N^{r,s,t}(\mathbf{u}) = \sum_{\mathbf{j} \in Z/2} c_{\mathbf{j}}^{r,s,t} N^{r,s,t}(2(\mathbf{u} - \mathbf{j})),$$

$$\longrightarrow \quad \hat{d}_{\mathbf{j}}^{r,s,t} = \sum_{\mathbf{i}\in Z} c_{\mathbf{j}-\mathbf{i}}^{r,s,t} d_{\mathbf{i}}^{r,s,t}$$

Where

$$c_{\mathbf{j}}^{r,s,t} = 2^{-(r+s+t)} \sum_{k=0}^{t} \binom{r}{2i-k} \binom{s}{2j-k} \binom{t}{k}$$

The subdivision masks for the triangular spline $N^{2,2,2}(u)$



The Doo/Sabin algorithm

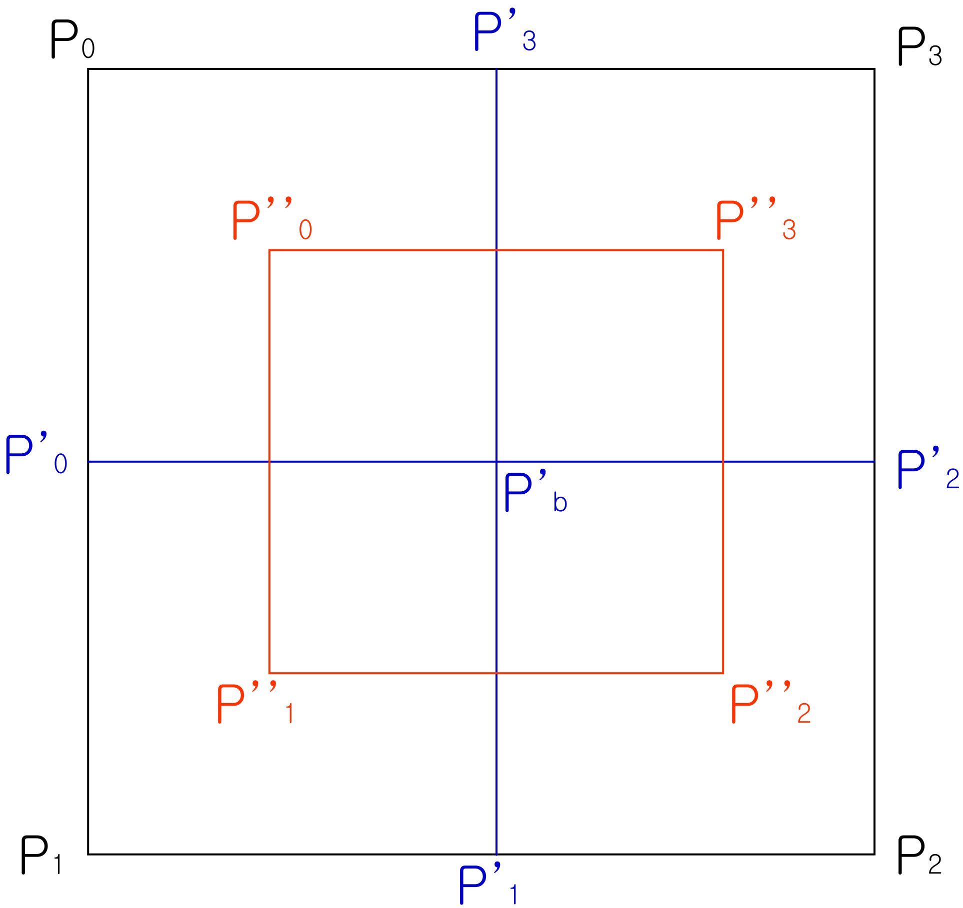
(Problem)

Subdivision for tensor product quadratic B-spline surface has rigid restrictions on the topology.

⟷ Each vertex of the net must order 4.

→ This restriction makes the design of many surfaces difficult

Doo/Sabin presented an algorithm that eliminated this restriction by generalizing the bi-quadratic B-spline subdivision rules to include arbitrary topologies.



Doo-Sabin scheme

$$(P_0'', P_1'', P_2'', P_3'')^T = S_4^{DS}(P_0, P_1, P_2, P_3)^T$$

$$S_4^{DS} = \frac{1}{16}\begin{bmatrix} 9 & 3 & 1 & 3 \\ 3 & 9 & 3 & 1 \\ 1 & 3 & 9 & 3 \\ 3 & 1 & 3 & 9 \end{bmatrix}$$

- The subdivision masks for bi-quadratic B-spline :

- Geometric view of bi-quadratic B-spline subdivision :

  the new points are centroids of the sub-face formed by the face centroid, a corner vertex and the two mid-edge points next to the corner.

Generalization

arbitrary topology

- For an n-sided face Doo/Sabin used subdivision matrix $S_n^{DS} = (\alpha_{ij})_{n \times n}$,

$$\alpha_{ij} = \frac{5+n}{4n} \quad i = j$$

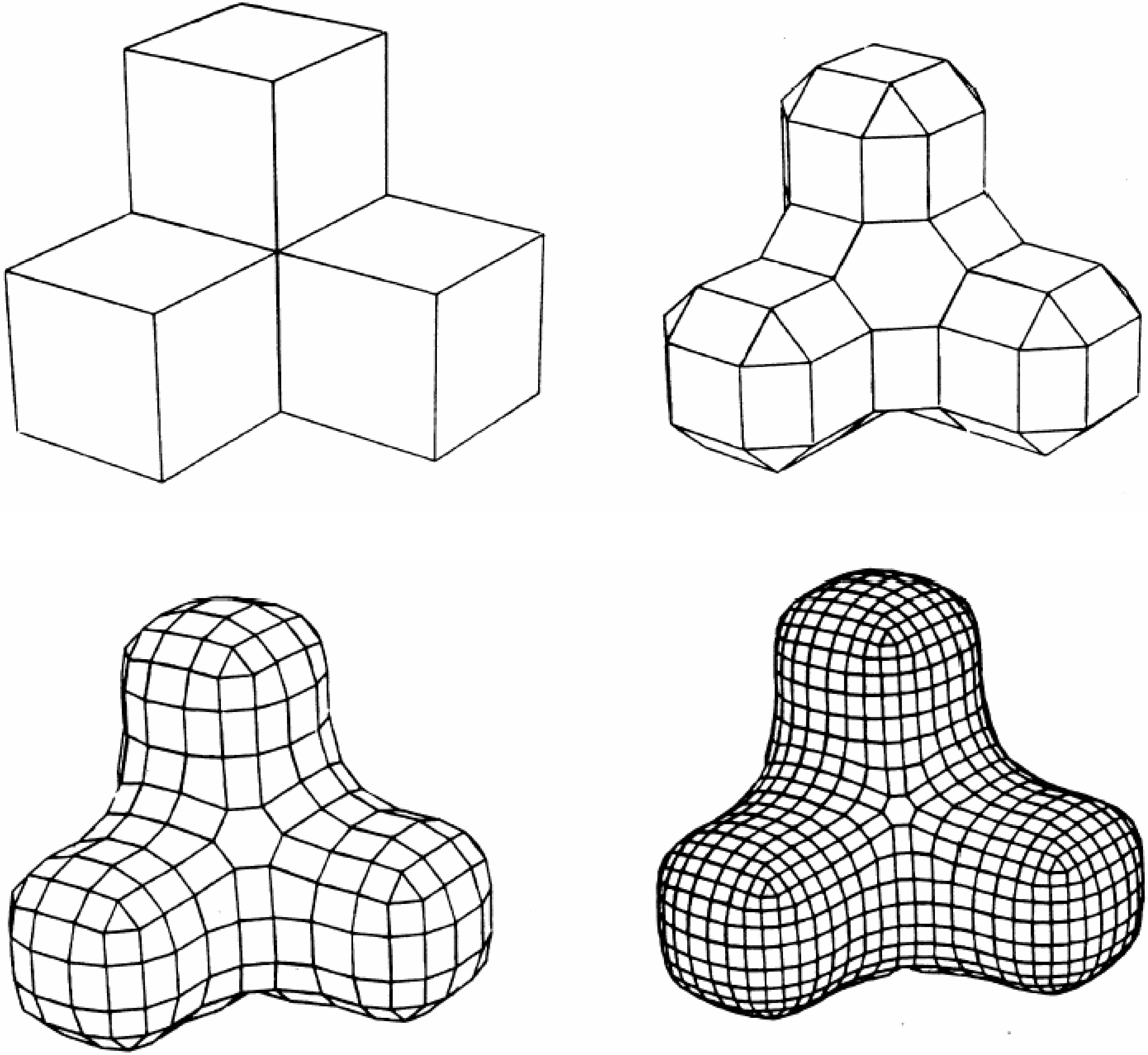
$$\alpha_{ij} = \frac{3 + 2\cos\left(\frac{2\pi(i-j)}{n}\right)}{4n} \quad i \neq j$$

- As subdivision proceeds, the refined control point mesh becomes locally rectangular everywhere except at <u>a fixed number of points</u>.

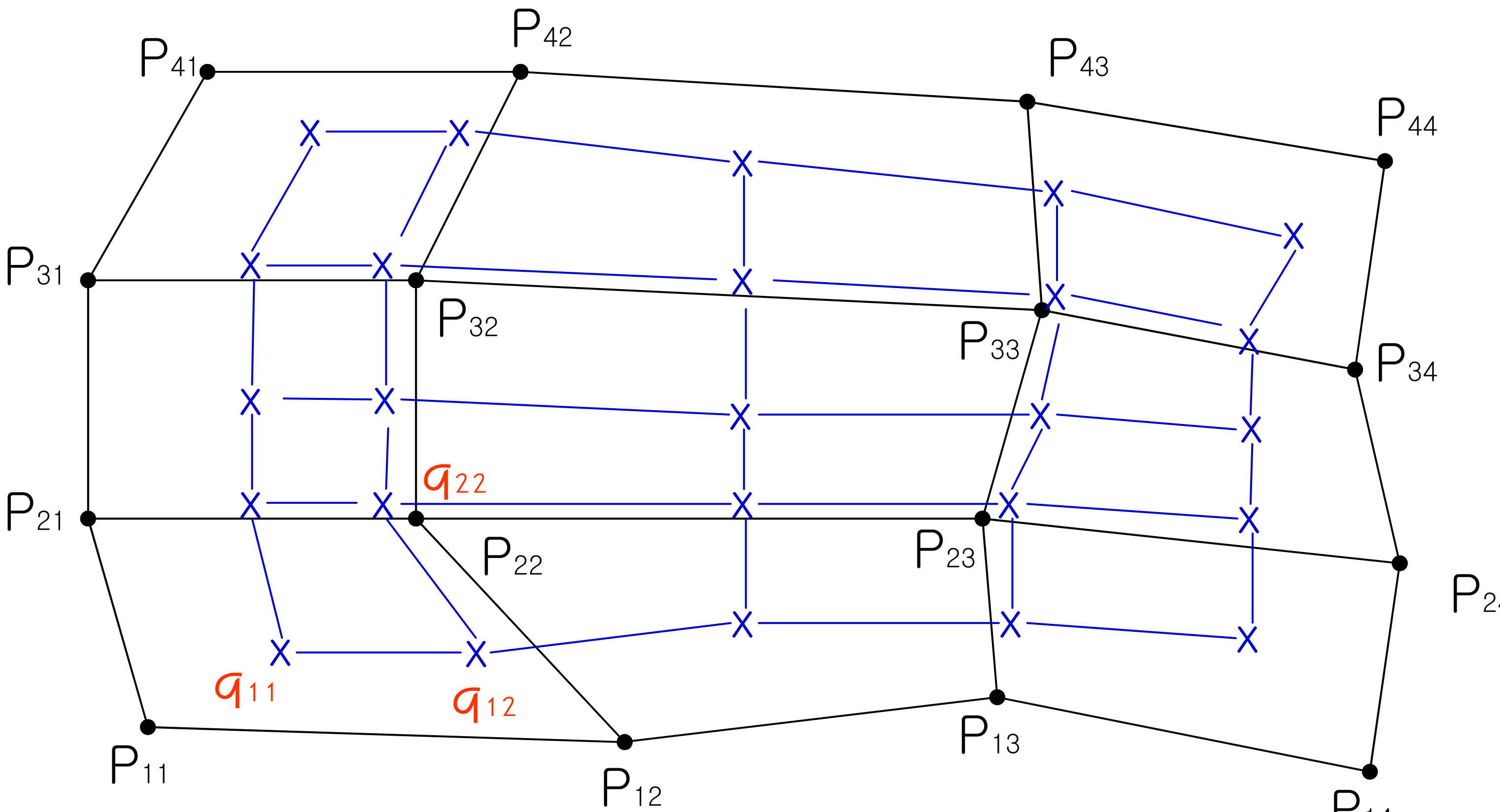
↓

extraordinay points

- Since bi-quadratic B-splines are C^1, the surfaces generated by the Doo/Sabin algorithm are locally C^1



The Catmull/Clark algorithm :



q_{11} : New face points

q_{12} : New edge points

q_{22} : New vertex points

- New face points：

$$q_{11} = \frac{P_{11} + P_{12} + P_{21} + P_{22}}{4}$$

- New edge points：

$$q_{12} = \frac{\frac{C+D}{2} + \frac{P_{12}+P_{22}}{2}}{2}, \quad C = q_{11}, D = q_{13}$$

- New vertex points：

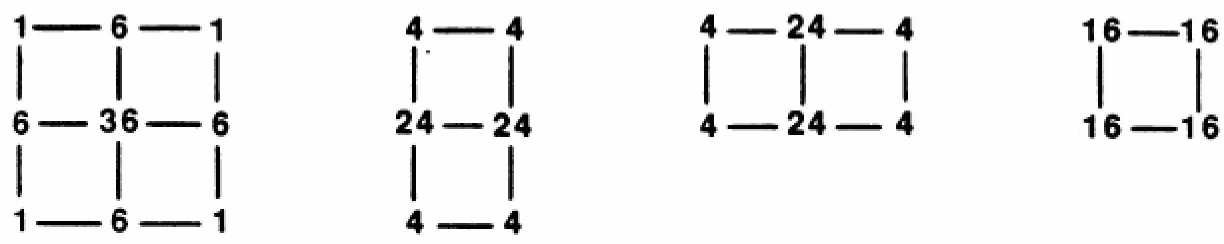
$$q_{22} = \frac{Q}{4} + \frac{R}{2} + \frac{P_{22}}{4}$$

$$Q = \frac{q_{11} + q_{13} + q_{31} + q_{33}}{4}$$

$$R = \frac{1}{4}\left[\left(\frac{p_{22} + p_{12}}{2}\right) + \left(\frac{p_{22} + p_{21}}{2}\right) + \left(\frac{p_{22} + p_{32}}{2}\right) + \left(\frac{p_{22} + p_{23}}{2}\right)\right]$$

- The subdivision masks for bi-cubic B-spline :

  Approach : generalization of bi-cubic B-spline



Generalization

arbitrary topology

- New face points : the average of all of the old points defining the face.

- New edge points : the average of the mid points of the old edge with average of the two new face.

- New vertex points : $\frac{1}{4}Q + \frac{1}{2}R + \frac{1}{4}S,$

  Q : the average of the new face points of all faces sharing an old vertex point.

  R : the average of the midpoints of all old edges incident on the old vertex point.

  S : the old vertex point.

Note : tangent plane continuity was not maintained at extraordinary points.

↓

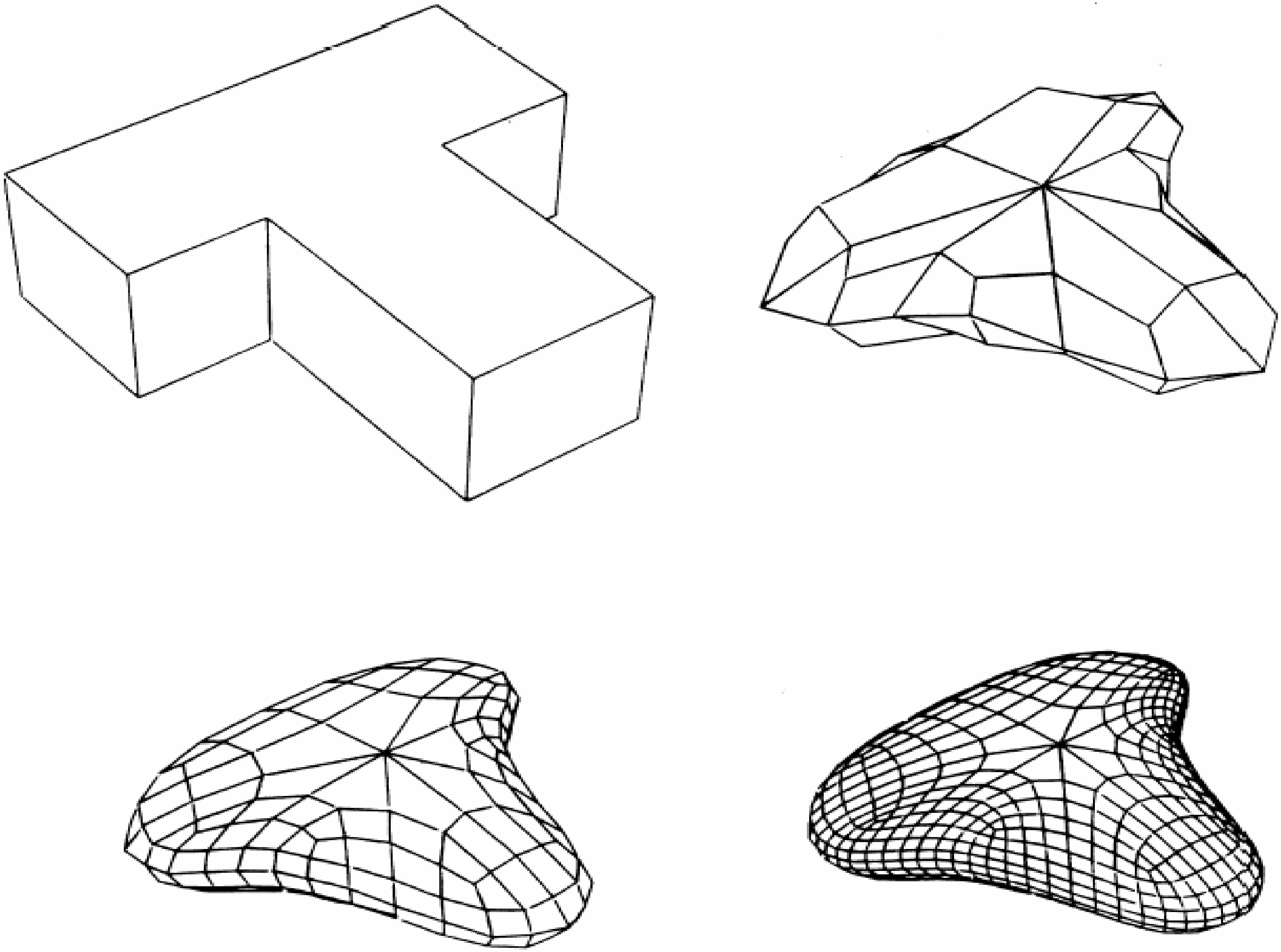
Modified Catmull/Clark rule

$$\hat{S} = \frac{1}{N}Q + \frac{2}{N}R + \frac{N-3}{N}S$$

N : order of the vertex

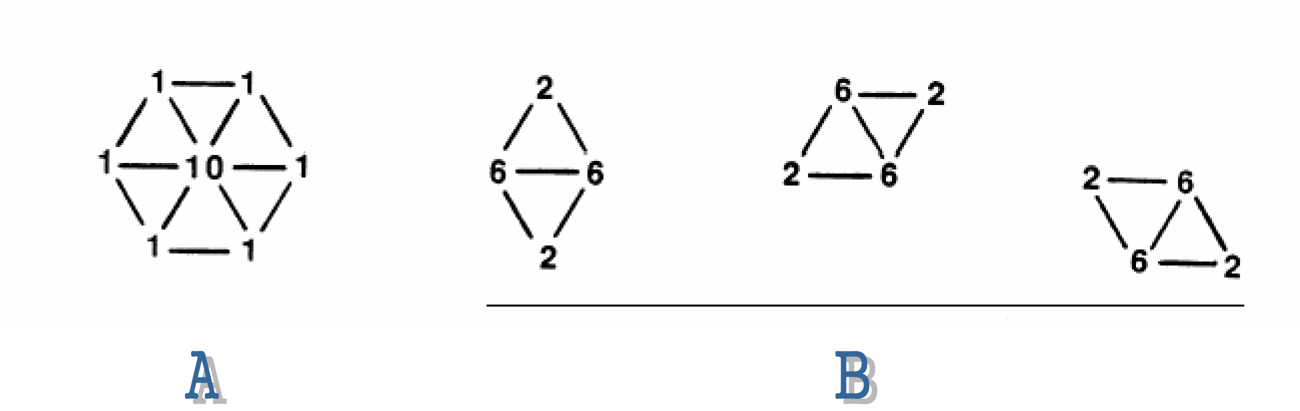
↓

tangent plane continuity at extraordinary points.



The Loop Scheme

- A generalized triangular subdivision surface.
- The subdivision masks for $N^{2,2,2}(u)$



- mask A generates new control points for each vertex
- mask B generates new control points for edge of the original regular triangular mesh.

- Mask B : generalization is to leave this subdivision rule intact (why?)
- Mask A : The new vertex point can be computed as a convex combination of the old vertex and all old vertices that share an edge with it.

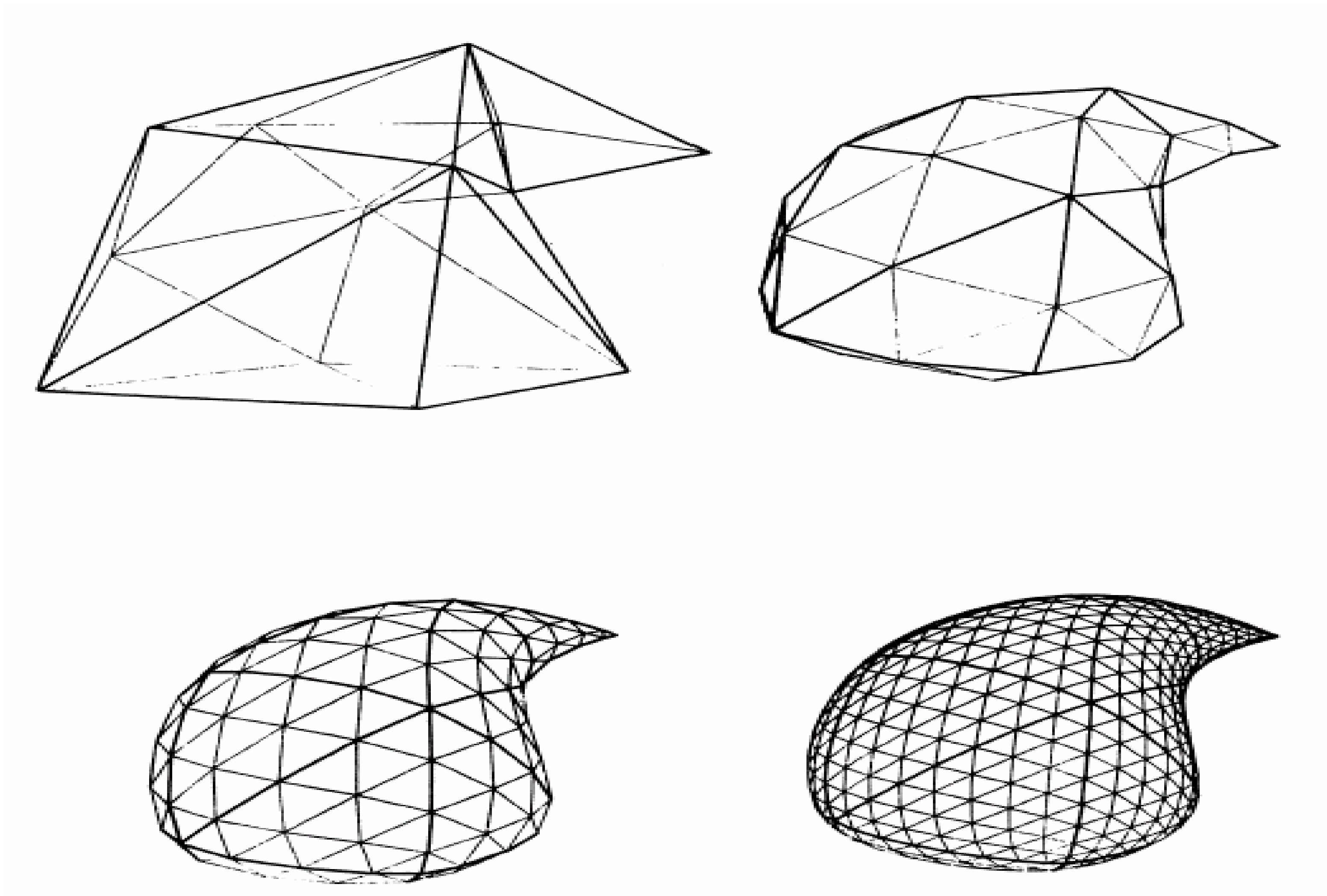
$$\hat{V} = \frac{5}{8}V + \frac{3}{8}Q,$$

V : the old vertex point.

Q : the average of the old points that share an edge with V.

⟶ This same idea may be applied to an arbitrary triangular mesh.

- *Note* : tangent plane continuity is lost at the extraordinary points.

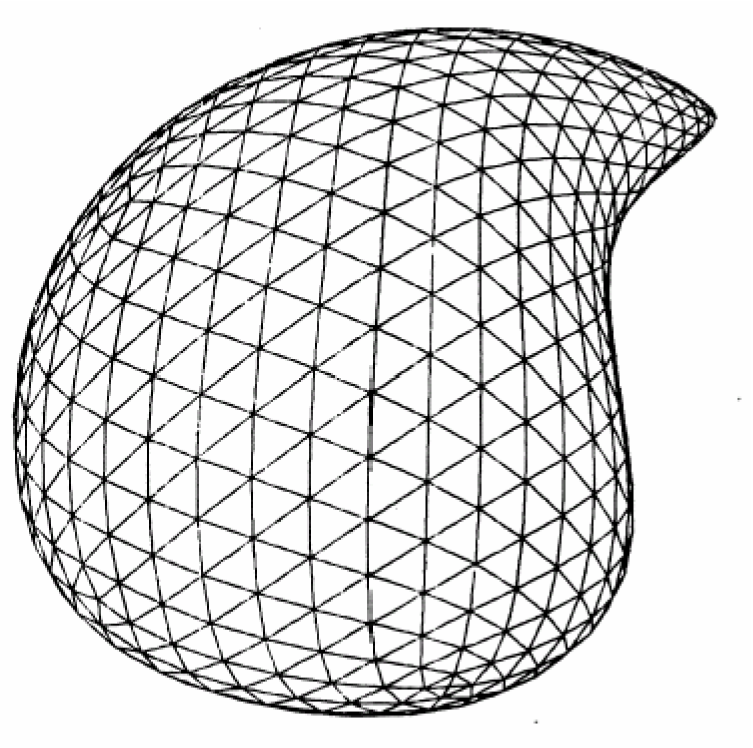


- Loop scheme :

$$\hat{V} = \alpha_N V + (1 - \alpha_N) Q,$$

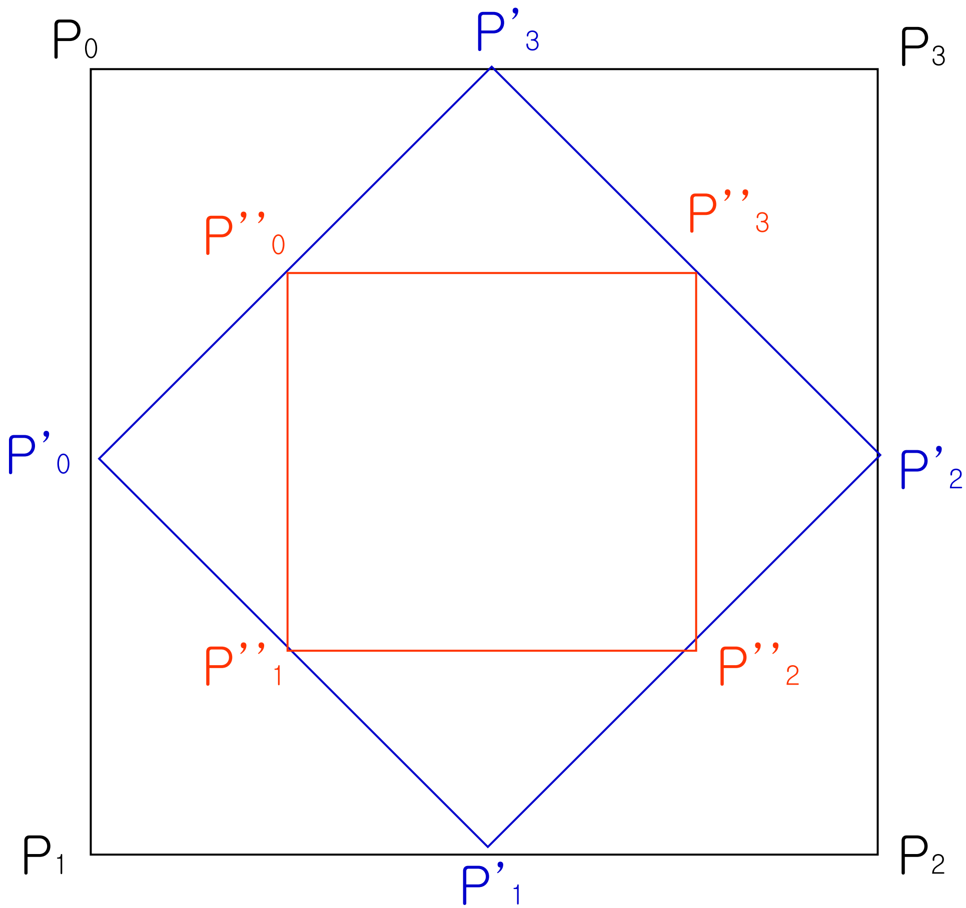
where

$$\alpha_N = \left(\frac{3}{8} + \frac{1}{4}\cos\frac{2\pi}{N}\right)^2 + \frac{3}{8}$$



- curvature continuity at regular point.
- tangent plane continuity at extaordinary point.

Mid-Edge Scheme



$$(P_0``, P_1``, P_2``, P_3``)^T = S_4^{ME}(P_0, P_1, P_2, P_3)^T$$

$$S_4^{ME} = \frac{1}{4}\begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

- For an n-sided face, subdivision matrix :

$$S_n^{ME} = (\beta_{ij})_{n \times n},$$

$$\beta_{ij} = \frac{1 + \cos\frac{2\pi(i-1)}{n}}{n}$$

Circulant matrix

$$S_n^G = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{bmatrix}$$

→ eigenvalues of S_n^G can be calculated by evaluating the polynomial.

$$P_n(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1},$$

$$z = w^j, \quad w = e^{\frac{i2\pi}{n}}, \quad j = 0, 1, \cdots, n-1$$

Further Subdivision Schemes

- Non-uniform scheme.
- Shape preserving scheme.
- Hermite-type scheme.
- Variational scheme.
- Quasi-linear scheme.
- Poly-scale scheme.
- Non-stationary scheme.
- Reverse subdivision scheme.

Convexity Preserving ISS

- A constructive approach is used to derive convexity preserving subdivision scheme.

$$
\begin{aligned}
f_{2i}^{k+1} &= f_i^k \\
f_{2i+1}^{k+1} &= \frac{1}{2}\left(f_i^k + f_{i+1}^k\right) - F_1\left(f_{i-1}^k, f_i^k, f_{i+1}^k, f_{i+2}^k\right)
\end{aligned}
$$

1. interpolatory.
2. local : four points scheme.

define the first and second differences.

$$
\begin{aligned}
df_i &= f_{i+1} - f_i \\
d_i &= d^2 f_i = f_{i+1} - 2f_i + f_{i-1}
\end{aligned}
$$

$$\begin{aligned} f_{2i}^{k+1} &= f_i^k \\ f_{2i+1}^{k+1} &= \frac{1}{2}\left(f_i^k + f_{i+1}^k\right) - F_2\left(\frac{1}{2}(f_i^k + f_{i+1}^k), df_i^k, d_i^k, d_{i+1}^k\right) \end{aligned}$$

3. invariant under addition of affine functions.

(★)
$$\begin{aligned} f_{2i}^{k+1} &= f_i^k \\ f_{2i+1}^{k+1} &= \frac{1}{2}\left(f_i^k + f_{i+1}^k\right) - F(d_i^k, d_{i+1}^k) \end{aligned}$$

F : subdivision function

4. continuous
5. homogeneous, i.e. , $F(\lambda x, \lambda y) = \lambda F(x, y)$
6. symmetric

- Theorem 1 (Convexity)

  A subdivision scheme of type (*) satisfying conditions 1 to 6 is convex preserving for all convex data ⟷ F satisfies

$$0 \le F(x,y) \le \frac{1}{4}\min\{x,y\}, \forall x,y \ge 0$$

Note : if F=0 ⟶ linear SS. ⟶ only C^0

(Question)

under what conditions, SS(*) with conditions 1 to 6 and convexity condition generate continuously differentiable limit functions.

Answer ⟶

- Theorem 2 (Smoothness)

  Under the same conditions hold as in Theorem 1.

  The scheme given by

$$
\begin{aligned}
f_{2i}^{k+1} &= f_i^k \\
f_{2i+1}^{k+1} &= \frac{1}{2}\left(f_i^k + f_{i+1}^k\right) - \frac{1}{4}\frac{1}{\frac{1}{d_i^k} + \frac{1}{d_{i+1}^k}}
\end{aligned}
$$

→ continuously differentiable function which is convex and interpolate the data

- Theorem 3 (Approximation order)

  The convexity preserving subdivision scheme has approximation order 4.