Dubuc-Deslauriers Subdivision for Finite Sequences and Interpolation Wavelets on an Interval

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Abstract

In this paper we consider a method of adapting Dubuc-Deslauriers subdivision, which is defined for bi-infinite sequences, to accommodate sequences of finite length. After deriving certain useful properties of the Dubuc-Deslauriers refinable function on $\mathbb{R}$, we define a multi-scale finite sequence of functions on a bounded interval, which are then proved to be refinable. Using this fact, the resulting adapted interpolatory subdivision scheme for finite sequences is then shown to be convergent. Corresponding interpolation wavelets on an interval are defined, and explicit formulations of the resulting decomposition and reconstruction algorithms are calculated. Finally, we give two numerical examples on signature smoothing and two-dimensional feature extraction of the subdivision and wavelet algorithms.

Keywords:

subdivision, interpolation, refinable functions, wavelets, bounded interval

AMS Classification:

65D10, 65T60, 42C40, 41A05

1 Introduction

Consider the following simple iterative procedure: Let M be the linear space of bi-infinite vector-valued sequences $c = \{c_j\} = \{c_j : j \in \mathbb{Z}\}$, and begin with an ordered sequence

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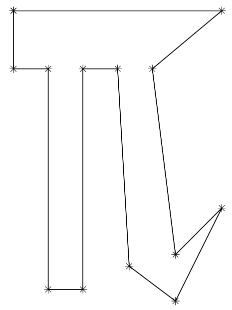
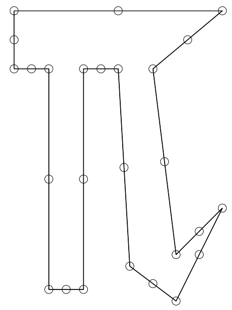
$c^{(0)} \in M$, called the *initial control points*. From these points, generate a new sequence $c^{(1)}$ of control points in M where the even-indexed new control points interpolate the old ones. In contrast to standard interpolation procedures, subdivision schemes generate the new points by taking a linear combination of the old control points. For example if one generates the new points using

$$c_{2j}^{(1)} = c_j^{(0)} \text{ and } c_{2j+1}^{(1)} = \tfrac{1}{2}(c_j^{(0)} + c_{j+1}^{(0)}), \quad j \in \mathbb{Z}, \tag{1.1}$$

then the odd-indexed new points are generated halfway between the old ones. This step can of course be repeated indefinitely, roughly 'doubling' the number of points at each step. In this case the points fill in or converge to the straight lines connecting the initial control points (see Figure 1). Thus we obtain a continuous piecewise linear curve. In general, the existence and smoothness of this limit curve depend on the choice of the coefficients of the linear combination. The initial task is to find suitable choices for these coefficients which: (i) ensure convergence to a limit curve, (ii) yield appropriate smoothness of the limit curve.



(a) The original control points ($\star$).



(b) The new control points ($\circ$) after one iteration.

Figure 1: Illustration of the iterative procedure (1.1).

In general, given a finitely supported real-valued sequence a, we define the corresponding subdivision operator $S : M \to M$ by

$$(Sc)_j = \sum_k a_{j-2k} c_k, \quad j \in \mathbb{Z}, \quad c \in M. \tag{1.2}$$

Unless stated otherwise we sum over all the integers. The resulting subdivision scheme is then defined, for a given initial sequence $c \in M$, by

$$c^{(0)} = c, \quad c^{(r+1)} = Sc^{(r)}, \quad r = 0, 1, \ldots, \tag{1.3}$$

or, equivalently,

$$c^{(0)} = c, \quad c^{(r)} = S^r c, \quad r = 1, \ldots. \tag{1.4}$$

The sequence $a = \{a_j\}$ is called the *mask* of the subdivision scheme.

There is no unique or best way of obtaining a mask. One possibility is to demand that if the original control points fall on a polynomial of degree $2N + 1$, then the newly generated control points must also lie on the same polynomial. This is, in fact, the idea behind Dubuc-Deslauriers subdivision [15, 12], which is a symmetric interpolatory scheme.

In Section 2.1, we develop this idea to eventually calculate an explicit formulation of the resulting mask sequence $\{a_j\}$.

To investigate the convergence of Dubuc-Deslauriers subdivision, we first refer to a result by Micchelli [24], according to which there exists a compactly supported fundamental interpolant $\phi \in C(\mathbb{R})$ which is refinable with respect to the sequence $\{a_j\}$. We then deduce, in Theorem 2.2, further properties of ϕ, including polynomial reproduction and information on its zeros, which prove to be necessary for our arguments in the subsequent two sections. It can then be shown as in Theorem 2.3 below that Dubuc-Deslauriers subdivision converges in the sense that, at each step of the iteration, the subdivision sequence lies entirely on the limit curve.

The Dubuc-Deslauriers subdivision operator has the form (1.2) and is therefore formulated for bi-infinite sequences. In Section 3 we consider the case of finite sequences and propose a method to adapt Dubuc-Deslauriers subdivision to this situation. We base our construction on a multi-scale finite sequence of fundamental interpolants $\{\phi_j^r\}$ defined on a bounded interval. Away from the boundaries the integer shifts of the original functions ϕ suffice, while the adjustments in the proximity of the boundaries preserve the polynomial reproduction property of ϕ. The resulting adapted interpolatory subdivision scheme, which corresponds to Dubuc-Deslauriers subdivision away from the boundaries, then also preserves the abovementioned polynomial filling property. In fact, most results derived in this paper depend on these polynomial filling or polynomial reproduction properties.

Our main result, as proved in Theorem 3.2, is that, on each fixed level r, the sequence $\{\phi_j^r\}$ is refinable in the sense that the function ϕ_j^r can be expressed as a finite linear combination of the (finite) sequence $\{\phi_j^{r+1}(2\,\bullet)\}$. Using this result, it is then a simple matter to prove, in Theorem 3.3, that our adapted Dubuc-Deslauriers subdivision for finite sequences converges. We end Section 3 by calculating an explicit formulation for this adapted scheme.

The refinability of the sequence $\{\phi_j^r\}$ allows a multiresolutional construction of interpolation wavelets on an interval, and we pursue this idea in Section 4. The main result of this section is the direct sum (non-orthogonal) space decomposition of Theorem 4.1, by virtue of which we obtain compactly supported symmetric interpolation wavelets. By giving up orthogonality we describe here the construction of symmetric interpolation wavelets resulting in finite decomposition and reconstruction formulas with rational coefficients given by the values of certain Lagrange polynomials at half integers. Also, a connection between interpolation wavelet decomposition on an interval and the adapted subdivision of Section 3 is explained.

Alternative approaches to the construction of wavelets on an interval include work by Daubechies [11, Section 10.7] and Cohen, Daubechies and Vial [9], in which periodization and related methods are used for the construction of orthonormal wavelet bases on an interval. In the spline setting, explicit constructions of symmetric biorthogonal spline wavelets on an interval, as well as the corresponding decomposition and reconstruction algorithms, appear in Chui and Quak [8], Quak and Weyrich [25], Chui and de Villiers [6] and Chui [5, Section 7.3.2].

The connection between the compactly supported orthonormal wavelets of Daubechies [10, 11] and Dubuc-Deslauriers subdivision has been noted by several authors (see e.g. [24, Section 3]), and exploited for the construction of biorthogonal interpolatory wavelets by Beylkin and Saito [4] and Bertoluzza and Naldi [2, 3]. A further study of the relationship between interpolation processes and wavelets construction appear in the paper [21] by Lee, Sharma and Tan.

The idea, as used in our Section 4, to construct interpolation wavelets by means of a non-orthogonal linear space decomposition and an interpolation operator has been studied for wavelets on $\mathbb{R}$ by Chui and Li [7], and used in the context of the Sweldens lifting scheme in [27].

Our interpolation wavelet decomposition and reconstruction algorithms for finite data sets, as given by (4.32), (4.33) and (3.34) – (3.38) below, are identical to those derived by Aràndiga, Donat and Harten [1, equations (55) and (56)]. Making use of ideas developed by Harten [17, 18], these authors derive their equations from a general framework—our approach has the advantage that it allows an explicit construction of the underlying refinable sequence $\{\phi_j^r\}$ as demonstrated by our equation (3.12) below. This is perhaps closer to the unpublished work of Donoho [13], in the sense that both are based on the principle of polynomial extrapolation. An essential difference however, lies in the way in which the associated nested sequence of linear spaces $\{V_r\}$ is defined. While Donoho's construction is consistently based on a polynomial extrapolation operator, we define the linear space V_r as the span of the sequence $\{\phi_j^r\}$. The nesting property of the $\{V_r\}$ then follows from the refinability of $\{\phi_j^r\}$, as proved in Theorem 3.2 below.

Note that the wavelets constructed in this manner do not have vanishing moments—the mean of our wavelet is not zero. In this regard our wavelets are similar to those mentioned by Mallat [22, p 301]. We show however that the corresponding interpolation wavelet space can be characterized in terms of a projection operator which is exact on polynomials, thereby causing the interpolation wavelet coefficients of a function to be relatively small in those regions where the function exhibits local polynomial-like behavior. For the construction of (non-orthogonal) interpolation wavelets on the interval with vanishing moments, we refer to Donoho [14].

In [26], Schröder and Sweldens develop algorithms implementing interpolation wavelets on an interval without providing detailed proofs.

The main algorithms are illustrated in the final section using examples from signature verification and two-dimensional image processing.

2 Dubuc-Deslauriers subdivision for bi-infinite sequences

In this section we introduce Dubuc-Deslauriers subdivision as an optimally local curve filling iterative procedure which reproduces polynomials of a given odd degree. We next prove the existence of an associated refinable function, which is then shown to provide a limit curve for the Dubuc-Deslauriers subdivision scheme.

2.1 Construction of the mask

For a given non-negative integer N, consider the problem of finding a minimally supported mask a such that the $(2N+1)$-th degree polynomial filling property

$$\sum_k a_{j-2k} p(k) = p\left(\frac{j}{2}\right), \quad j \in \mathbb{Z}, \quad p \in \pi_{2N+1}, \tag{2.1}$$

holds. Here π_{2N+1} denotes the linear space of polynomials of degree $\leq 2N+1$.

For this purpose we introduce the Lagrange fundamental polynomials $L_k \in \pi_{2N+1}$, $k = -N, \ldots, N+1$, as defined by

$$L_k(x) = \prod_{k \neq j = -N}^{N+1} \frac{x-j}{k-j}, \quad x \in \mathbb{R}, \quad k = -N, \ldots, N+1, \tag{2.2}$$

for which

$$L_k(j) = \delta_{k,j}, \quad k, j = -N, \ldots, N+1, \tag{2.3}$$

and

$$\sum_{k=-N}^{N+1} p(k) L_k(x) = p(x), \quad x \in \mathbb{R}, \quad p \in \pi_{2N+1}. \tag{2.4}$$

Setting $j = 0$ and $j = 1$ in (2.1), and using (2.3) and (2.4), the equation (2.1) implies

$$\left.\begin{aligned} a_{2j} + \sum_{k \notin \{-N,\ldots,N+1\}} a_{2k} L_{-j}(k) &= \delta_{j,0}, \\ a_{2j+1} + \sum_{k \notin \{-N,\ldots,N+1\}} a_{1-2k} L_{-j}(k) &= L_{-j}\left(\tfrac{1}{2}\right), \end{aligned}\right\} j = -N-1, \ldots, N. \tag{2.5}$$

A necessary condition for a minimally supported mask $a = \{a_j\}$ to satisfy (2.1) is thus

$$a_{2j} = \delta_{j,0}, \qquad j \in \mathbb{Z}, \tag{2.6a}$$
$$a_{2j+1} = L_{-j}\left(\tfrac{1}{2}\right), \quad j = -N-1, \ldots, N, \tag{2.6b}$$
$$a_{2j+1} = 0, \qquad j \geq N+1 \text{ or } j \leq -N-2. \tag{2.6c}$$

The choice (2.6) is also sufficient to fulfil (2.1). In fact, if $j = 2m$, $m \in \mathbb{Z}$, then, for $p \in \pi_{2N+1}$, equation (2.6a) implies

$$\sum_k a_{j-2k} p(k) = \sum_k a_{2m-2k} p(k) = \sum_k a_{2k} p(m-k) = p(m) = p\left(\frac{j}{2}\right),$$

whereas, if $j = 2m+1,\ m \in \mathbb{Z}$, then (2.6b),(2.6c) and (2.4) give

$$\sum_k a_{j-2k}p(k) = \sum_k a_{2k+1}p(m-k) = \sum_{k=-N-1}^{N} L_{-k}\left(\tfrac{1}{2}\right)p(m-k) = p\left(m+\tfrac{1}{2}\right) = p\left(\frac{j}{2}\right).$$

Hence the subdivision scheme corresponding to the mask (2.6), which was introduced by Deslauriers and Deslauriers in [12], is indeed a minimally supported mask sequence $a = \{a_j\}$ for which (2.1) holds.

Observe that (1.2) implies, with reference to (1.3), the interpolatory property

$$c_{2j}^{(r+1)} = c_j^{(r)}, \quad j \in \mathbb{Z}, \quad r = 0, 1, ..., \tag{2.7}$$

by virtue of which Dubuc-Deslauriers subdivision is called an *interpolatory* scheme.

We now derive an explicit expression for the mask. Since, for $k \in \{-N, \ldots, N+1\}$, we have

$$\prod_{k \neq j = -N}^{N+1} \left(\tfrac{1}{2} - j\right) = \frac{1}{2^{2N+1}} \frac{1}{1-2k} \prod_{j=-N-1}^{N} (2j+1) = \frac{(-1)^N}{2^{4N+1}} \frac{1}{2k-1} \left[\frac{(2N+1)!}{N!}\right]^2$$

and

$$\prod_{k \neq j = -N}^{N+1} (k-j) = (-1)^{N+1+k}(N+k)!(N+1-k)!,$$

we deduce from (2.6) and (2.2) that the Dubuc-Deslauriers mask sequence $a = \{a_j\}$ has the explicit formulation

$$\left.\begin{array}{lcll} a_{2j+1} & = & \dfrac{N+1}{2^{4N+1}}\dbinom{2N+1}{N}\dfrac{(-1)^j}{2j+1}\dbinom{2N+1}{N+j+1}, & j = -N-1, \ldots, N, \\ a_{2j} & = & \delta_{j,0}, & j = -N, \ldots, N, \\ a_j & = & 0, & |j| \geq 2N+2. \end{array}\right\} \tag{2.8}$$

For example, if $N = 1$, (2.8) gives

$$a_{2j+1} = \begin{cases} -\frac{1}{16}, & j = -2, \\ \frac{9}{16}, & j = -1, \\ \frac{9}{16}, & j = 0, \\ -\frac{1}{16}, & j = 1, \\ 0, & \text{otherwise}. \end{cases}$$

From (2.8) observe that the mask coefficients are symmetric, i.e.

$$a_j = a_{-j}, \quad j \in \mathbb{Z}. \tag{2.9}$$

Therefore we call Dubuc-Deslauriers subdivision a *symmetric* scheme.

For $N = 0$, note that equations (2.8), (1.2) and (1.3) yield, for $r = 0$, the iteration procedure in (1.1). This subdivision scheme converges to a continuous piecewise linear function which interpolates the original control points (see Figure 1). In contrast, subdivision with the mask obtained by setting $N = 1$ in (2.8) converges to a smoother function, while still interpolating the original control points (see Figure 2). Incidentally, this example is not completely arbitrary. In fact Knuth based his construction of $\TeX$ fonts on ideas remarkably similar to subdivision schemes [19, Chapter 2], more than 10 years before the Dubuc-Deslauriers scheme was introduced in [12].

(a) Original control points ($\star$) and one step of the subdivision algorithm ($\circ$).

(b) Original control points with limit curve.

Figure 2: Illustration of Dubuc-Deslauriers subdivision for $N = 1$.

We now show that, for any given initial sequence $c \in M$, the Dubuc-Deslauriers subdivision sequence $\{c^{(r)} : r = 0, 1, \ldots\} \subset M$ converges to a limit curve which will be specified in terms of an associated refinable function.

2.2 The corresponding refinable function

Before investigating the convergence of Dubuc-Deslauriers subdivision, we establish the existence and properties of the corresponding refinable function , following an approach employed by Micchelli in [24] (see also [16]).

THEOREM 2.1. *For any given non-negative integer N, let the sequence $a = \{a_j\}$ be chosen as in (2.6). Then there exists a compactly supported function $\phi \in C(\mathbb{R})$ such that*

$$\text{(i)} \qquad \phi(x) = \sum_j a_j \phi(2x - j), \quad x \in \mathbb{R}; \tag{2.10}$$

$$\text{(ii)} \qquad \phi(j) = \delta_{j,0}, \quad j \in \mathbb{Z}; \tag{2.11}$$

$$\text{(iii)} \qquad \phi(x) = 0, \quad x \notin (-2N-1, 2N+1); \tag{2.12}$$

$$\text{(iv)} \qquad \sum_j p(j)\phi(x-j) = p(x), \quad x \in \mathbb{R}, \quad p \in \pi_{2N+1}; \tag{2.13}$$

$$\text{(v)} \qquad \phi(x) = \phi(-x), \quad x \in \mathbb{R}; \tag{2.14}$$

$$\text{(vi)} \qquad \phi\left(\frac{j}{2}\right) = a_j, \quad j \in \mathbb{Z}; \tag{2.15}$$

$$\text{(vii)} \qquad \phi(2N+1-2^{-j}(N-\tfrac{1}{2}-k)) = 0, \quad k = 0, 1, \ldots, \quad j = 0, 1, \ldots. \tag{2.16}$$

Proof. The existence of a compactly supported function of $\phi \in C(\mathbb{R})$ satisfying the properties (i), (ii), and (iii), was proven in [24, Lemma 3.1, Theorem 4.1 and Corollary 4.1].

To prove (iv), suppose $\ell \in \{0, 1, \ldots, 2N+1\}$, $k \in \mathbb{Z}$ and $r \in \{0, 1, \ldots\}$. We shall prove that

$$\sum_j j^\ell \phi\left(\frac{k}{2^r} - j\right) = \left(\frac{k}{2^r}\right)^\ell, \tag{2.17}$$

which then implies (iv), since the set $\{\frac{k}{2^r} : k \in \mathbb{Z}, \quad r = 0, 1, \ldots\}$ is dense in $\mathbb{R}$, and since ϕ is a compactly supported continuous function on $\mathbb{R}$.

Noting that (2.17) is an immediate consequence of (2.11) if $r = 0$, we assume next that $r \geq 1$. Then, using consecutively (2.10), (2.1) and (2.11), we get

$$\begin{aligned}
\sum_j j^\ell \phi\left(\frac{k}{2^r} - j\right) &= \sum_j j^\ell \sum_m a_m \phi\left(\frac{k}{2^{r-1}} - 2j - m\right) \\
&= \sum_m \left[\sum_j a_{m-2j} j^\ell\right] \phi\left(\frac{k}{2^{r-1}} - m\right) \\
&= \frac{1}{2^\ell} \sum_m m^\ell \phi\left(\frac{k}{2^{r-1}} - m\right) \\
&\ \ \vdots \\
&= \frac{1}{2^{\ell r}} \sum_m m^\ell \phi(k-m) = \left(\frac{k}{2^r}\right)^\ell.
\end{aligned}$$

Similarly, the property (v) will be proved if we can show that, for $k \in \mathbb{Z}$ and $r \in \{0, 1, \ldots\}$,

$$\phi\left(\frac{k}{2^r}\right) = \phi\left(-\frac{k}{2^r}\right). \tag{2.18}$$

For $r = 0$, (2.18) follows from (2.11), whereas for $r = 1$, it follows from (2.10), (2.9) and (2.11) that $\phi(-\frac{k}{2}) = \sum_j a_j \phi(-k-j) = \sum_j a_j \phi(-k+j) = \sum_j a_j \phi(k-j) = \phi(\frac{k}{2})$. If $r \geq 2$, we also use (1.2) to deduce that

$$\begin{aligned}
\phi\left(-\frac{k}{2^r}\right) &= \sum_j a_j \phi\left(-\frac{k}{2^{r-1}} - j\right) \\
&= \sum_j a_j \phi\left(-\frac{k}{2^{r-1}} + j\right) \\
&= \sum_j a_j \sum_\ell a_{-\ell} \phi\left(-\frac{k}{2^{r-2}} + 2j - \ell\right) \\
&= \sum_\ell \left[\sum_j a_{\ell-2j} a_j\right] \phi\left(-\frac{k}{2^{r-2}} + \ell\right) \\
&= \sum_\ell (Sa)_\ell \phi\left(-\frac{k}{2^{r-2}} + \ell\right) \\
&\vdots \\
&= \sum_\ell \left(S^{r-1}a\right)_\ell \phi(-k+\ell) = (S^{r-1}a)_k.
\end{aligned} \tag{2.19}$$

A similar strategy shows that

$$\phi\left(\frac{k}{2^r}\right) = (S^{r-1}a)_k,$$

which, together with (2.19), proves (2.18) for $r \geq 2$.

Property (vi) is an immediate consequence of (2.10) and (2.11).

Finally, we prove (vii) by induction. For $j = 0$, property (2.16) follows from (2.15) and (2.6c). To advance the inductive hypothesis from j to $j + 1$, we use (2.10) and (2.6c) to deduce that, for $k \in \{0, 1, ...\}$,

$$\begin{aligned}
\phi(2N + 1 - 2^{-(j+1)}(N - \tfrac{1}{2} - k)) &= \sum_\ell a_\ell \phi(4N + 2 - 2^{-j}(N - \tfrac{1}{2} - k) - \ell) \\
&= \sum_{\ell=0}^{4N+2} a_{2N+1-\ell} \phi(2N + 1 - 2^{-j}(N - \tfrac{1}{2} - [k + 2^j \ell])).
\end{aligned}$$

Using (2.12) the proof is complete. ■

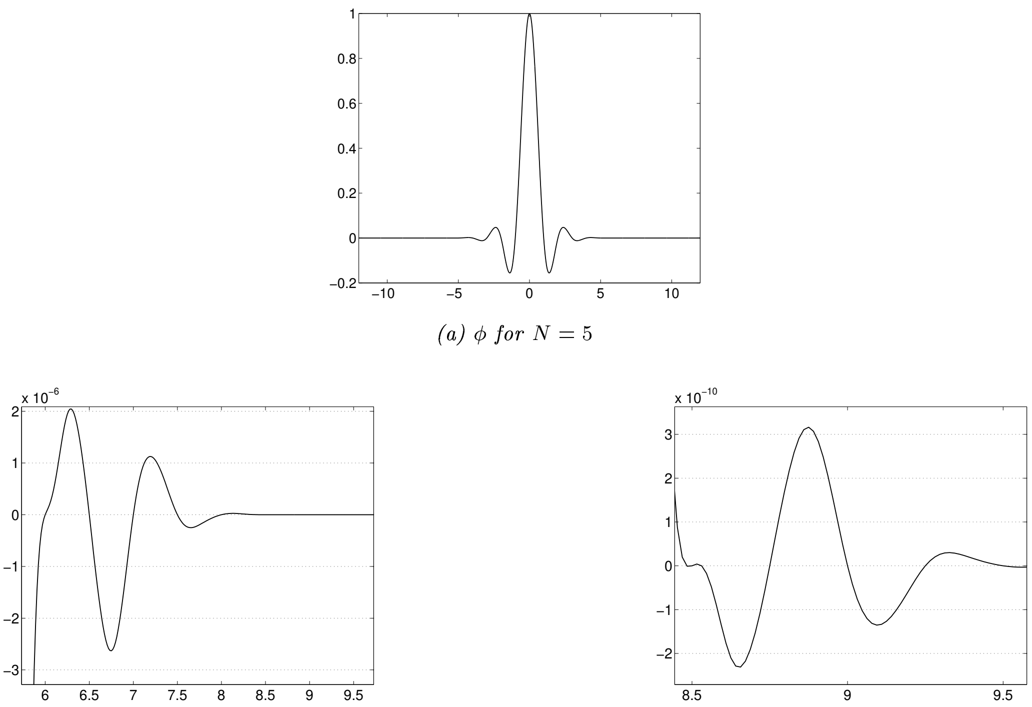
Remarks:

1. Equations (2.10) and (2.15) imply
$$\phi(x) = \sum_j \phi\left(\frac{j}{2}\right)\phi(2x-j), \quad x \in \mathbb{R}. \tag{2.20}$$

2. Equations (2.14) and (2.16) imply
$$\phi(-2N-1+2^{-j}(N-\tfrac{1}{2}-k)) = 0, \quad k = 0,1,\ldots, \quad j = 0,1,\ldots. \tag{2.21}$$

3. For $k \geq N$, (2.16) and (2.21) hold because the argument falls outside the support of ϕ, while for $k = 0, 1, ..., N-1$ the argument falls within the support. This in turn, implies that ϕ has an infinite number of zeros within its support and that these zeros are clustered more densely towards the edges of the support, as illustrated in Figure 3.

4. A detailed study of the regularity of ϕ as a function of N is given in [12].



(a) ϕ for $N = 5$

(b) Magnification of ϕ

(c) Further magnification of ϕ

Figure 3: Illustration of the refinable function with clustered zeros

2.3 Convergence of the subdivision scheme

We now show that the Dubuc-Deslauriers subdivision scheme converges in the sense that, for each $r \in \{0, 1, \ldots, \}$, the sequence $c^{(r)}$ lies entirely on the limit curve.

Theorem 2.2. *For each initial sequence $c = \{c_j\} \in M$, the Dubuc-Deslauriers subdivision scheme (1.3), (1.2), (2.6) converges to the function*

$$f(x) = \sum_j c_j \phi(x-j), \quad x \in \mathbb{R}, \tag{2.22}$$

with ϕ defined as in Theorem 2.2, in the sense that

$$c_k^{(r)} = f\left(\frac{k}{2^r}\right), \quad k \in \mathbb{Z}, \quad r = 0, 1, \ldots. \tag{2.23}$$

Proof. If $r = 0$, then (2.23) follows from (2.11). For $r \geq 1$, we use (2.10), (1.2), (1.3) and (2.11) to deduce, for $k \in \mathbb{Z}$,

$$\begin{aligned} f\left(\frac{k}{2^r}\right) &= \sum_j c_j \phi\left(\frac{k}{2^r} - j\right) \\ &= \sum_j c_j \sum_\ell a_{\ell-2j} \phi\left(\frac{k}{2^{r-1}} - \ell\right) \\ &= \sum_\ell (Sc)_k \phi\left(\frac{k}{2^{r-1}} - \ell\right) \\ &\ \ \vdots \\ &= \sum_\ell (S^r c)_\ell \phi(k-\ell) = (S^r c)_k = c_k^{(r)}, \end{aligned}$$

the last equality by virtue of (1.4). ■

3 A modified subdivision scheme for finite sequences

The algorithms for bi-infinite sequences, as described in the previous sections, are applied mainly in the case of periodic sequences. For finitely supported sequences these algorithms must be modified to accommodate the boundaries. We consider here a method of adapting the Dubuc-Deslauriers subdivision scheme of Section 2 to the situation where the initial sequence c is finite.

3.1 Construction of a modified scheme

We first show that refinable functions defined on an interval allow one to construct a subdivision scheme for finite sequences. We derive the specific properties of these refinable functions from those of the refinable (on $\mathbb{R}$) function ϕ of Section 2, with appropriate modifications near the boundaries.

With N as in Section 2, assume n is a positive integer with $n \geq 4N+2$, and let r be a non-negative integer. On the basis of Theorem 2.2, and the subsequent equation (2.20), we seek to construct a sequence $\{\phi_j^r\} = \{\phi_j^r : j = 0, 1, \ldots, 2^r n, \quad r = 0, 1, \ldots\}$ such that, for each fixed r,

$$\text{(a)} \qquad \phi_j^r \in C[0, 2^r n], \quad j = 0, 1, \ldots, 2^r n; \tag{3.1}$$

$$\text{(b)} \qquad \phi_j^r(x) = 0,\ x \notin \begin{cases} [0, 2N+1+j), & j = 0, 1, \ldots, 2N+1, \\ (-2N-1+j, 2N+1+j), & j = 2N+2, \ldots, 2^r n - 2N - 2, \\ (-2N-1+j, 2^r n], & j = 2^r n - 2N - 1, \ldots, 2^r n; \end{cases} \tag{3.2}$$

$$\text{(c)} \qquad \phi_j^r(k) = \delta_{j,k}, \quad j, k = 0, 1, \ldots, 2^r n; \tag{3.3}$$

$$\text{(d)} \qquad \sum_{j=0}^{2^r n} p(j)\phi_j^r(x) = p(x), \quad x \in [0, 2^r n], \quad p \in \pi_{2N+1}; \tag{3.4}$$

$$\text{(e)} \qquad \phi_j^r(x) = \sum_{k=0}^{2^{r+1} n} \phi_j^r\left(\frac{k}{2}\right) \phi_k^{r+1}(2x), \quad x \in [0, 2^r n], \quad j = 0, 1, \ldots, 2^r n. \tag{3.5}$$

Denoting the linear space of vector-valued sequences

$$c = \{c_j\} = \{c_j : j = 0, 1, \ldots, 2^r n\}$$

by M_r, the subdivision operator sequence $\{S_r : r = 0, 1, \ldots, \ldots\}$ for $S_r : M_r \to M_{r+1}$, is then defined by

$$(S_r c)_j = \sum_{k=0}^{2^r n} \phi_k^r\left(\frac{j}{2}\right) c_k, \quad j = 0, 1, \ldots, 2^{r+1} n. \tag{3.6}$$

The corresponding subdivision scheme is defined by

$$c^{(0)} = c \in M_0, \quad c^{(r+1)} = S_r c^{(r)}, \quad r = 0, 1, \ldots, \tag{3.7}$$

or, equivalently,

$$c^{(0)} = c, \quad c^{(r+1)} = S_r(\ldots(S_1(S_0 c))\ldots), \quad r = 0, 1, \ldots. \tag{3.8}$$

Observe that (3.3), (3.6) and (3.7) imply

$$c_{2j}^{(r+1)} = c_j^{(r)}, \quad j = 0, 1, \ldots, 2^r n, \quad r = 0, 1, \ldots, \tag{3.9}$$

i.e. the subdivision scheme (3.6) – (3.7) is interpolatory, whereas (3.4) implies

$$\sum_{k=0}^{2^r n} \phi_k^r\left(\frac{j}{2}\right) p(k) = p\left(\frac{j}{2}\right), \quad j = 0, 1, \ldots, 2^{r+1} n, \quad p \in \pi_{2N+1}, \tag{3.10}$$

according to which the subdivision scheme (3.6) – (3.7) has the $(2N+1)$-th degree polynomial filling property. In (3.6) – (3.10) we have obtained the analogues of, respectively, (1.2), (1.3), (1.4), (2.7) and (2.1).

To find a sequence $\{\phi_j^r\}$ satisfying (3.1) – (3.5), we observe from (2.12) and (2.13) that

$$\sum_{j=0}^{2^r n} p(j)\phi(x-j) = \sum_j p(j)\phi(x-j) = p(x), \quad x \in [2N, 2^r n - 2N], \quad p \in \pi_{2N+1}. \quad (3.11)$$

Thus, the sequence $\{\phi(\cdot - j)\}$ provides suitable refinable functions away from the boundaries. The boundary modifications can again be based on property (2.1). Using arguments similar to the ones which led to the construction of the mask (2.6), we define, for each fixed $r \in \{0, 1, \ldots.\}$, the sequence $\{\phi_j^r\}$ on the interval $[0, 2^r n]$ by

$$\phi_j^r(x) = \begin{cases} \phi(x-j) + \displaystyle\sum_{k=-2N}^{-1} L_{j-N}(k-N)\phi(x-k), & j = 0, 1, \ldots, 2N+1, \quad (3.12a) \\ \phi(x-j), & j = 2N+2, \ldots, 2^r n - 2N - 2, \quad (3.12b) \\ \phi_{2^r n - j}^r(2^r n - x), & j = 2^r n - 2N - 1, \ldots, 2^r n, \quad (3.12c) \end{cases}$$

where the sequence $\{L_k : k = -N, \ldots, N+1\} \subset \pi_{2N+1}$ is given by (2.2). Examples are plotted in Figure 4.

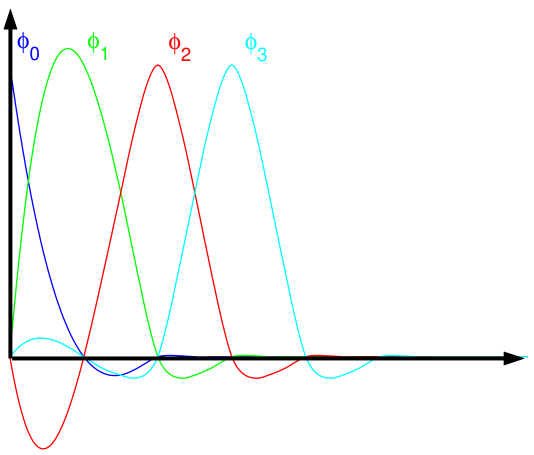


Figure 4: Illustration of definition (3.12) with $N = 1$.

Using

$$\phi(N + \frac{3}{2} + k) = 0, \quad k = 0, 1, 2, \ldots, \quad (3.13)$$

and

$$\phi(-N - \frac{3}{2} - k) = 0, \quad k = 0, 1, 2, \ldots. \quad (3.14)$$

which follow from (2.16) and (2.21), we prove the following useful properties of the sequence $\{\phi_j^{(r)}\}$.

PROPOSITION 3.1. *For given non-negative integers N and r, let n be an integer with $n \geq 4N+2$. Then the sequence $\{\phi_j^r\}$, as defined by (3.12), satisfies*

$$\text{(a)} \qquad \phi_j^r(x) = \phi(x-j), \qquad x \in [2N, 2^r n - 2N], \qquad j = 0,1,\ldots,2^r n, \tag{3.15}$$

$$\text{(b)} \qquad \phi_j^r(x) = L_{j-N}(x-N), \quad x \in [0,1], \qquad j = 0,\ldots,2N+1, \tag{3.16}$$

$$\text{(c)} \qquad \phi_j^r\left(\frac{k}{2}\right) = \begin{cases} L_{j-N}(\frac{k}{2}-N), & k = 0,\ldots,2N+1, \\ \phi(\frac{k}{2}-j), & k = 2N,\ldots, \end{cases} \qquad j = 0,\ldots,2N+1. \tag{3.17}$$

Proof. (a) For $j = 0,\ldots,2^r n - 2N - 2$, (3.15) follows from the definition (3.12) and the finite support (2.12) of ϕ. It therefore suffices to prove (3.15) for $j = 2^r n - 2N - 1, \ldots, 2^r n$. But then, from (3.12c) and (3.12a),

$$\phi_j^r(x) = \phi_{2^r n - j}^r(2^r n - x) = \phi(-x+j) = \phi(x-j),$$

by virtue of (2.14).

(b) To prove (3.16), suppose $x \in [0,1]$ and $j \in \{0,1,\ldots,2N+1\}$. Then (3.12a), (2.3) and (2.12) yield

$$\begin{aligned} \phi_j^r(x) &= \sum_{k=-2N}^{2N+1} L_{j-N}(k-N)\phi(x-k) \\ &= \sum_k L_{j-N}(k-N)\phi(x-k) = L_{j-N}(x-N), \end{aligned}$$

by virtue of (2.13).

(c) For the proof of (3.17), we first let $k \in \{0,1,\ldots,2N+1\}$ and $j \in \{0,1,\ldots,2N+1\}$. Then, using consecutively (3.12a), (2.3), (3.14), (2.12) and (2.13), we get

$$\begin{aligned} \phi_j^r\left(\frac{k}{2}\right) &= \phi\left(\frac{k}{2}-j\right) + \sum_{\ell=-2N}^{-1} L_{j-N}(\ell-N)\phi\left(\frac{k}{2}-\ell\right) && (3.18) \\ &= \sum_{\ell=-2N}^{2N+1} L_{j-N}(\ell-N)\phi\left(\frac{k}{2}-\ell\right) \\ &= \sum_{\ell=-2N}^{3N+1} L_{j-N}(\ell-N)\phi\left(\frac{k}{2}-\ell\right) \\ &= \sum_{\ell} L_{j-N}(\ell-N)\phi\left(\frac{k}{2}-\ell\right) \\ &= L_{j-N}\left(\frac{k}{2}-N\right). && (3.19) \end{aligned}$$

Next, for $k \in \{2N+2,\ldots\}$ and $j \in \{0,1,\ldots,2N+1\}$, we see that (3.18) holds again, and the bottom part of (3.17) therefore follows since $\phi(\frac{k}{2}-\ell) = 0$, $\ell = -2N,\ldots,-1$, by virtue of (2.11) and (3.13).

To complete the proof of (3.17), it remains to show that for $j \in \{0, 1, \ldots, 2N+1\}$, we have

$$\phi\left(\frac{k}{2} - j\right) = L_{j-N}\left(\frac{k}{2} - N\right) \quad \text{for} \quad k = 2N \text{ and } k = 2N+1. \tag{3.20}$$

For $k = 2N$, the property (3.20) is a consequence of (2.11) and (2.3), whereas for $k = 2N+1$ we use (2.15) and (2.6b) to deduce that

$$\phi\left(\frac{k}{2} - j\right) = a_{2N+1-2j} = L_{j-N}\left(\frac{1}{2}\right) = L_{j-N}\left(\frac{k}{2} - N\right). \qquad \blacksquare$$

Remark: Observe from (3.15) and (3.12a), (3.12b), together with the support properties (3.2) and (2.12), that

$$\phi_j^r(x) = \phi(x - j), \quad x \in [2N, 2^r n], \quad j = 0, 1, \ldots, 2^r n - 2N - 2. \tag{3.21}$$

3.2 The refinability of the sequence $\{\phi_j^r\}$

Analogous to the bi-infinite case, our proof in Section 3.3 below of the convergence of the subdivision scheme will depend on the refinability of the sequence $\{\phi_j^r\}$, as proved in this section.

THEOREM 3.2. *The sequence $\{\phi_j^r\}$, as defined in (3.12), satisfies the properties (3.1) – (3.5).*

Proof. The properties (3.1) – (3.3) are immediate consequences of (2.11) and (2.12).

To prove (3.4), we choose $p \in \pi_{2N+1}$, and assume first that $x \in [0, 2N)$. Using (3.2), (3.12a), (3.12b), (2.4), (2.12) and (2.13) consecutively, one has

$$\begin{aligned}
\sum_{j=0}^{2^r n} p(j)\phi_j^r(x) &= \sum_{j=0}^{4N} p(j)\phi_j^r(x) \\
&= \sum_{j=0}^{4N} p(j)\phi(x-j) + \sum_{k=-2N}^{-1}\left[\sum_{j=0}^{2N+1} p(j)L_{j-N}(k-N)\right]\phi(x-k) \\
&= \sum_{j=-2N}^{4N} p(j)\phi(x-j) = \sum_{j} p(j)\phi(x-j) = p(x).
\end{aligned}$$

For $x \in [2N, 2^r n - 2N]$, the property (3.4) follows from (3.15) and (3.11). Similar arguments establish polynomial reproduction for $x \in (2^r n - 2N, 2^r n]$.

To prove the refinability (3.5), assume first that $j \in \{0, 1, \ldots, 2N+1\}$ and $x \in [0, 2N)$. Then, (3.12a) and (2.3) imply

$$\phi_j^r(x) = \sum_{k=-2N}^{2N+1} L_{j-N}(k-N)\phi(x-k).$$

Now, use the refinement equation (2.20), together with the support property (2.12), to obtain

$$\phi_j^r(x) \quad = \quad \sum_{\ell=-2N}^{6N} \left[\sum_{k=-2N}^{2N+1} L_{j-N}(k-N)\phi\left(\frac{\ell}{2}-k\right)\right] \phi(2x-\ell). \tag{3.22}$$

For the first part of the sum in (3.22) we get, from (2.11), (3.13), (3.14), the polynomial reproduction (2.13), and definition (3.12a),

$$\begin{aligned}
&\sum_{\ell=-2N}^{2N+1} \left[\sum_{k=-2N}^{2N+1} L_{j-N}(k-N)\phi\left(\frac{\ell}{2}-k\right)\right] \phi(2x-\ell) \\
= & \sum_{\ell=-2N}^{2N+1} \left[\sum_{k} L_{j-k}(k-N)\phi\left(\frac{\ell}{2}-k\right)\right] \phi(2x-\ell) \\
= & \sum_{\ell=-2N}^{2N+1} L_{j-N}\left(\frac{\ell}{2}-N\right) \phi(2x-\ell) \\
= & \sum_{\ell=-2N}^{-1} L_{j-N}\left(\frac{\ell}{2}-N\right) \phi(2x-\ell) + \sum_{\ell=0}^{2N+1} L_{j-N}\left(\frac{\ell}{2}-N\right) \phi_\ell^{r+1}(2x) \\
& \qquad - \sum_{k=-2N}^{-1} \left[\sum_{\ell=0}^{2N+1} L_{j-N}\left(\frac{\ell}{2}-N\right) L_{\ell-N}(k-N)\right] \phi(2x-k) \\
= & \sum_{\ell=0}^{2N+1} L_{j-N}\left(\frac{\ell}{2}-N\right) \phi_\ell^{r+1}(2x) \; = \; \sum_{\ell=0}^{2N+1} \phi_j^r\left(\frac{\ell}{2}\right) \phi_\ell^{r+1}(2x),
\end{aligned} \tag{3.23}$$

having also used the polynomial reproduction (2.4), and (3.17).

For the remaining part of the sum in (3.22), we use the interpolatory properties (2.3) and (2.11), as well as (3.13) and (3.17), to obtain

$$\begin{aligned}
&\sum_{\ell=2N+2}^{6N} \left[\sum_{k=-2N}^{2N+1} L_{j-N}(k-N)\phi\left(\frac{\ell}{2}-k\right)\right] \phi(2x-\ell) \\
= & \sum_{\ell=2N+2}^{6N} \phi\left(\frac{\ell}{2}-j\right) \phi(2x-\ell) \\
= & \sum_{\ell=2N+2}^{6N} \phi_j^r\left(\frac{\ell}{2}\right) \phi_\ell^{r+1}(2x) = \sum_{\ell=2N+2}^{2^{r+1}n} \phi_j^r\left(\frac{\ell}{2}\right) \phi_\ell^{r+1}(2x),
\end{aligned} \tag{3.24}$$

where we also used the definition (3.12b), and the inequality $n \geq 4N+2$. Combining (3.22), (3.23) and (3.24) then yields the result (3.5) for $j \in \{0, 1, \ldots, 2N+1\}$ and $x \in [0, 2N)$.

Next, for $j \in \{0, 1, \ldots, 2N+1\}$ and $x \in [2N, 2N+1+j)$, we use Proposition 3.1, Theorem 2.2 and (3.13) to get

$$\begin{aligned}
\phi_j^r(x) = \phi(x-j) &= \sum_k \phi\left(\frac{k}{2} - j\right) \phi(2x-k) \\
&= \sum_{k=2N}^{4N+2j+2} \phi\left(\frac{k}{2} - j\right) \phi(2x-k) \\
&= \sum_{k=2N}^{4N+2j+2} \phi_j^r\left(\frac{k}{2}\right) \phi_k^{r+1}(2x) \\
&= \sum_{k=0}^{2^{r+1}n} \phi_j^r\left(\frac{k}{2}\right) \phi_k^{r+1}(2x),
\end{aligned}$$

since $n \geq 4N+2$, thereby establishing (3.5) for this subcase.

For $j \in \{0, 1, \ldots, 2N+1\}$ and $x \in [2N+1+j, 2^r n]$, we deduce from (3.2), (3.17), (2.11) and (3.13) that

$$\begin{aligned}
\sum_{k=0}^{2^{r+1}n} \phi_j^r\left(\frac{k}{2}\right) \phi_k^{r+1}(2x) &= \sum_{k=2N+2+2j}^{4N+1+2j} \phi_j^r\left(\frac{k}{2}\right) \phi_k^{r+1}(2x) \\
&= \sum_{k=2N+2+2j}^{4N+1+2j} \phi\left(\frac{k}{2} - j\right) \phi_k^{r+1}(2x) \;=\; 0 = \phi_j^r(x),
\end{aligned}$$

by virtue of the top part of (3.2). Hence, we established (3.5) for all $j \in \{0, 1, \ldots, 2N+1\}$.

Now consider the case $j \in \{2N+2, \ldots, 2^r n - 2N - 2\}$ and $x \in (-2N-1+j, 2N+1+j)$. We use definition (3.12b), (2.20), (2.12), as well as (3.13) and the finite support property (3.2), to find that

$$\begin{aligned}
\phi_j^r(x) = \phi(x-j) &= \sum_{k=-2N-1+2j}^{2N+1+2j} \phi\left(\frac{k}{2} - j\right) \phi(2x-k) \\
&= \sum_{k=-2N-1+2j}^{2N+1+2j} \phi\left(\frac{k}{2} - j\right) \phi_k^{r+1}(2x) \\
&= \sum_{k=0}^{2^{r+1}n} \phi\left(\frac{k}{2} - j\right) \phi_k^{r+1}(2x) \\
&= \sum_{k=0}^{2^{r+1}n} \phi_j^r\left(\frac{k}{2}\right) \phi_k^{r+1}(2x),
\end{aligned}$$

based on the inequalities $-2N-1+2j \geq 2N+3$, and $2N+1+2j \leq 2^{r+1}n - 2N - 3$.

For $j \in \{2N+2, \ldots, 2^r n - 2N - 2\}$ and $x \in [0, -2N-1+j]$, we use (3.2), (3.12b), (2.11) and (3.14) to get

$$\sum_{k=0}^{2^{r+1}n} \phi_j^r\left(\frac{k}{2}\right)\phi_k^{r+1}(2x) \;=\; \sum_{k=0}^{-2N-2+2j} \phi\left(\frac{k}{2}-j\right)\phi_k^{r+1}(2x) \;=\; 0 \;= \phi_j^r(x).$$

Similarly, for $j \in \{2N+2, \ldots, 2^r n - 2N - 2\}$ and $x \in [2N+1+j,\ 2^r n]$, with (3.13), we have

$$\sum_{k=0}^{2^{r+1}n} \phi_j^r\left(\frac{k}{2}\right)\phi_k^{r+1}(2x) \;=\; \sum_{k=2N+2+2j}^{2^{r+1}n} \phi\left(\frac{k}{2}-j\right)\phi_k^{r+1}(2x) \;=\; 0 \;= \phi_j^r(x).$$

Hence (3.5) also holds for $j \in \{2N+2, \ldots, 2^r n - 2N - 2\}$.

Finally, let $j \in \{2^r n - 2N - 1, \ldots, 2^r n\}$ and $x \in [0, 2^r n]$. Then (3.12c), together with the fact that (3.5) holds for $j \in \{0, 1, \ldots, 2N+1\}$, gives

$$\begin{aligned}
\phi_j^r(x) &= \phi_{2^r n-j}^r(2^r n - x) \\
&= \sum_{k=0}^{2^{r+1}n} \phi_{2^r n-j}^r\left(\frac{k}{2}\right)\phi_k^{r+1}\left(2^{r+1}n - 2x\right) \\
&= \sum_{k=0}^{2^{r+1}n} \phi_{2^r n-j}^r\left(2^r n - \frac{k}{2}\right)\phi_{2^{r+1}n-k}^{r+1}(2^{r+1}n - 2x) \\
&= \sum_{k=0}^{2^{r+1}n} \phi_j^r\left(\frac{k}{2}\right)\phi_{2^{r+1}n-k}^{r+1}(2^{r+1}n - 2x).
\end{aligned}$$

We claim that

$$\phi_{2^{r+1}n-k}^{r+1}(2^{r+1}n - 2x) = \phi_k^{r+1}(2x), \quad x \in [0, 2^r n], \quad k = 0, 1, \ldots, 2^{r+1}n, \tag{3.25}$$

which, if true, completes the proof of the theorem. Definition (3.12c) implies that (3.25) is true for $k \in \{0, 1, \ldots, 2N+1\} \cup \{2^r n - 2N - 1, \ldots, 2^r n\}$, whereas, if $k \in \{2N+2, \ldots, 2^r n - 2N - 2\}$, we find that (3.12b) and (2.14) yield, for $x \in [0, 2^r n]$,

$$\phi_{2^{r+1}n-k}^{r+1}(2^{r+1}n - 2x) = \phi(k - 2x) = \phi(2x - k) = \phi_k^{r+1}(2x). \qquad \blacksquare$$

3.3 Convergence of the modified subdivision scheme

We now prove the analogue of Theorem 2.2 for *finite* subdivision sequences.

THEOREM 3.3. *For each initial sequence $c = \{c_j\} \in M_0$, the subdivision scheme (3.6) – (3.7) converges to the function*

$$g(x) = \sum_{j=0}^{n} c_j \phi_j(x), \quad x \in [0, n], \tag{3.26}$$

with $\phi_j = \phi_j^0$ defined as in (3.12), in the sense that

$$c_k^{(r)} = g\left(\frac{k}{2^r}\right), \quad k = 0, 1, \ldots, 2^r n, \quad r = 0, 1, \ldots. \tag{3.27}$$

Proof. Repeatedly using (3.5), (3.6), (3.7), and eventually (3.3), for $k \in \{0, 1, \ldots, 2^r n\}$ and $r \in \{0, 1, \ldots\}$, we obtain

$$\begin{aligned}
g\left(\frac{k}{2^r}\right) &= \sum_{j=0}^{n} c_j \phi_j^0 \left(\frac{k}{2^r}\right) \\
&= \sum_{j=0}^{n} c_j \sum_{\ell=0}^{2n} \phi_j^0 \left(\frac{\ell}{2}\right) \phi_\ell^1 \left(\frac{k}{2^{r-1}}\right) \\
&= \sum_{\ell=0}^{2n} c_\ell^{(1)} \phi_\ell^1 \left(\frac{k}{2^{r-1}}\right) \\
&\;\;\vdots \\
&= \sum_{\ell=0}^{2^r n} c_\ell^{(r)} \phi_\ell^r(k) = c_k^{(r)}.
\end{aligned}$$

■

3.4 An explicit formulation

We derive an explicit formulation for the subdivision scheme (3.6) – (3.7).

Let $c \in M_r$. Then, from the subdivision operator definition (3.6), we obtain

$$(S_r c)_{2j+1} = \sum_{k=0}^{2^r n} \phi_k^r \left(j + \tfrac{1}{2}\right) c_k, \quad j = 0, 1, \ldots, 2^r n - 1, \tag{3.28}$$

and thus, using also (3.12b), (3.14) and the top of (3.17), we get

$$(S_r c)_{2j+1} = \sum_{k=0}^{2N+1} L_{k-N}\left(j + \tfrac{1}{2} - N\right) c_k, \quad j = 0, 1, \ldots, N. \tag{3.29}$$

Next, we claim that

$$(S_r c)_{2j+1} = \sum_{k=-N+j}^{N+1+j} L_{k-j}\left(\tfrac{1}{2}\right) c_k, \quad j = N+1, \ldots, 2^r n - 2N - 2. \tag{3.30}$$

Indeed, if $j \in \{N+1, \ldots, 2N-1\}$, then (3.28), (3.2), (3.17), (3.12b), (2.15) and (2.6c) yield

$$(S_r c)_{2j+1} = \sum_{k=0}^{4N} \phi\left(j + \tfrac{1}{2} - k\right) c_k = \sum_{k=-N+j}^{N+1+j} a_{2j+1-2k} c_k,$$

and (3.30) then follows from (2.6b) for that range of j. Similarly, if $j \in \{2N, \ldots, 2^r n - 2N - 2\}$, we additionally use (3.15) to get

$$(S_r c)_{2j+1} = \sum_{k=0}^{2^r n} \phi\left(j + \tfrac{1}{2} - k\right) c_k = \sum_{k=-N+j}^{N+1+j} a_{2j+1-2k} c_k,$$

which, together with (2.6b), then proves (3.30).

For $j \in \{2^r n - 2N - 1, \ldots, 2^r n - 1\}$, we first note the symmetry

$$\phi_k^r(x) = \phi_{2^r n-k}^r(2^r n - x), \quad x \in [0, 2^r n], \quad k = 0, 1, \ldots, 2^r n, \tag{3.31}$$

which follows from (3.12c), and the fact that, for $k \in \{2N+2, \ldots, 2^r n - 2N - 2\}$, (3.12b) and (2.14) give

$$\phi_{2^r n-k}^r(2^r n - x) = \phi(k - x) = \phi(x - k) = \phi_k^r(x).$$

Thus, from (3.31),

$$\phi_k^r\left(j + \tfrac{1}{2}\right) = \phi_{2^r n-k}^r\left((2^r n - 1 - j) + \tfrac{1}{2}\right), \qquad \begin{array}{l} k = 0, 1, \ldots, 2^r n, \\ j = 2^r n - 2N - 1, \ldots, 2^r n - 1. \end{array} \tag{3.32}$$

Combining (3.29), (3.30), (3.32), and using (3.9), we find that the subdivision scheme (3.6) – (3.7) has, for a given initial sequence $c^{(0)} = c \in M_0$, the explicit formulation

$$\left.\begin{array}{ll} c_{2j}^{(r+1)} = c_j^{(r)}, & j = 0, 1, \ldots, 2^r n, \\ c_{2j+1}^{(r+1)} = \displaystyle\sum_{k=0}^{2^r n} a_{j,k}^{(r)} c_k^{(r)}, & j = 0, 1, \ldots 2^r n - 1, \end{array}\right\} \quad r = 0, 1, \ldots, \tag{3.33}$$

where

$$a_{j,k}^{(r)} = \begin{cases} L_{k-N}\left(j + \tfrac{1}{2} - N\right), & k = 0, 1, \ldots, 2N+1, \\ 0, & k = 2N+2, \ldots, 2^r n, \end{cases} \quad \begin{array}{c} j = 0, 1, \ldots, N-1, \\ (\text{if } N \geq 1), \end{array} \tag{3.34}$$

$$a_{j,k}^{(r)} = \begin{cases} L_{k-j}(\tfrac{1}{2}), & k = -N + j, \ldots, N + 1 + j, \\ 0, & k \in [0, -N - 1 + j] \cup [N + 2 + j, 2^r n], \end{cases} \quad j = N, \ldots, 2^r n - N - 1, \tag{3.35}$$

$$a_{j,k}^{(r)} = a_{2^r n-1-j, 2^r n-k}^{(r)}, \quad k = 0, 1, \ldots, 2^r n, \qquad j = 2^r n - N, \ldots, 2^r n - 1. \tag{3.36}$$

Explicit formulations of (3.34) and (3.35) are now obtained by using a calculation similar to the one which yielded the top of (2.8). We find that, for $k = 0, 1, \ldots, 2N+1$, and $j = 0, 1, \ldots, N-1$,

$$L_{k-N}(j + \tfrac{1}{2} - N) = \frac{(-1)^{j+k}}{2^{4N+1}} \frac{1}{2j + 1 - 2k} \frac{(2j+1)!(4N+1-2j)!}{(2N-j)!(2N+1-k)!j!k!}, \tag{3.37}$$

whereas for $k = -N + j, \ldots, N + 1 + j$, and $j = N, \ldots, 2^r n - N - 1$,

$$L_{k-j}(\tfrac{1}{2}) = \frac{N+1}{2^{4N+1}} \binom{2N+1}{N} \frac{(-1)^{j+k}}{2j+1-2k} \binom{2N+1}{N+1+j-k}. \tag{3.38}$$

For example, if $N = 1$, (3.34) and (3.38) give

$$a_{0,k}^{(r)} = \begin{cases} \frac{5}{16}, & k = 0, \\ \frac{15}{16}, & k = 1, \\ -\frac{5}{16}, & k = 2, \\ \frac{1}{16}, & k = 3, \\ 0, & k = 4, \ldots, 2^r n. \end{cases}$$

4 Interpolation Wavelets on an Interval

We show how the refinable sequence $\{\phi_j^r : j = 0, 1, \ldots, 2^r n,\ r = 0, 1, \ldots\}$ of Section 3 can be used to explicitly construct interpolation wavelets on an interval. Our definition in (4.10) below coincides, in an inner region bounded away from the endpoints, with the definition of interpolation wavelets on $\mathbb{R}$ as given in e.g. [7, equation (1.11)], [22, p 300] and [27, p 193].

4.1 Decomposition based on interpolation

Let the integers N and $n \geq 4N + 2$ be as in Section 3, and let R be a given positive integer. We define the linear space sequence $\{V_r : r = 0, 1, \ldots, R\}$ by

$$V_r = \operatorname{span}\{\phi_j^r(2^{r-R}\bullet) : j = 0, 1, \ldots, 2^r n\}, \quad r = 0, 1, \ldots, R, \tag{4.1}$$

and the linear operator sequence $\{P_r : r = 0, 1, \ldots, R\}$ for $P_r : C[0, 2^R n] \to V_r$, by

$$(P_r f)(x) = \sum_{j=0}^{2^r n} f(2^{R-r} j) \phi_j^r(2^{r-R} x), \quad x \in [0, 2^R n], \quad r = 0, 1, \ldots, R. \tag{4.2}$$

It follows from (3.3) that P_r is an interpolation operator, which means that, for each $f \in C[0, 2^R n]$,

$$(P_r f)(2^{R-r} j) = f(2^{R-r} j), \quad j = 0, \ldots, 2^r n, \quad r = 0, 1, \ldots, R. \tag{4.3}$$

Also, a proof based on (3.3) shows that P_r is a projection on V_r. Thus,

$$P_r f = f, \quad f \in V_r. \tag{4.4}$$

Furthermore, (3.4) gives

$$\sum_{j=0}^{2^r n} p(2^{R-r}j)\phi_j^r(2^{r-R}x) = p(x), \quad x \in [0, 2^R n], \quad p \in \pi_{2N+1}, \quad r = 0, 1, \ldots, R, \tag{4.5}$$

by virtue of which

$$\pi_{2N+1} \subset V_r, \quad r = 0, 1, \ldots, R, \tag{4.6}$$

where here π_{2N+1} is restricted to the interval $[0, 2^R n]$.

Since (3.5) yields the refinement equation

$$\phi_j^r(2^{r-R}x) = \sum_{k=0}^{2^{r+1}n} \phi_j^r\left(\frac{k}{2}\right)\phi_k^{r+1}(2^{r+1-R}x), \quad x \in [0, 2^R n], \quad j = 0, 1, \ldots, 2^r n, \quad r = 0, \ldots, R-1, \tag{4.7}$$

we have the nesting property

$$V_r \subset V_{r+1}, \quad r = 0, 1, \ldots, R-1. \tag{4.8}$$

Based on (3.3) one can prove that, for each fixed r, the set $\{\phi_j^r(2^{r-R}\bullet) : j = 0, 1, \ldots, 2^r n\}$ is linearly independent on $[0, 2^R n]$, and thus, from (4.1), we have

$$\dim V_r = 2^r n + 1, \quad r = 0, 1, \ldots, R. \tag{4.9}$$

Now define the sequence $\{\psi_j^r\} = \{\psi_j^r : j = 0, 1, \ldots, 2^r n - 1,\ r = 0, 1, \ldots, R-1\}$ by

$$\psi_j^r(x) = \phi_{2j+1}^{r+1}(x), \quad x \in [0, 2^R n], \quad j = 0, 1, \ldots, 2^r n - 1, \quad r = 0, 1, \ldots, R-1, \tag{4.10}$$

with corresponding linear spaces

$$W_r = \operatorname{span}\{\psi_j^r(2^{r+1-R}\bullet) : j = 0, 1, \ldots, 2^r n - 1\}, \quad r = 0, 1, \ldots, R-1. \tag{4.11}$$

From (4.10) and (4.1) follows that

$$W_r \subset V_{r+1}, \quad r = 0, 1, \ldots, R-1. \tag{4.12}$$

If U, V and W are linear spaces, we use the direct sum notation $U = V \oplus W$ to denote that fact that, for each $f \in V$, there exist $g \in V$ and $h \in W$ such that $f = g + h$, and with g and h uniquely determined by f.

The following direct sum decomposition result holds.

THEOREM 4.1.

$$V_{r+1} = V_r \oplus W_r, \quad r = 0, 1, \ldots, R-1, \tag{4.13}$$

with V_r and W_r defined by (4.1) and (4.11).

To prove Theorem 4.1, we first introduce the linear spaces U_r and X_r, where

$$U_r = \{f - P_r f : f \in V_{r+1}\}, \quad r = 0, 1, \ldots, R-1, \tag{4.14}$$

with P_r defined by (4.2), and

$$X_r = \{f \in V_{r+1} : f(2^{R-r}j) = 0, \quad j = 0, 1, \ldots, 2^r n\}, \quad r = 0, 1, \ldots, R-1, \tag{4.15}$$

in terms of which the following preliminary result holds.

PROPOSITION 4.2. *The linear spaces V_r, W_r, U_r and X_r, as defined by (4.1), (4.11), (4.14) and (4.15), satisfy*

$$\text{(a)} \qquad W_r = U_r = X_r, \quad r = 0, 1, \ldots, R-1; \tag{4.16}$$

$$\text{(b)} \qquad V_r \cap W_r = \{0\}, \quad r = 0, 1, \ldots, R-1. \tag{4.17}$$

Proof. (a) First, we show that $U_r = X_r$. If $g \in U_r$, then (4.2) and (4.3) yield $g(2^{R-r}j) = 0$, $j = 0, \ldots, 2^r n$, i.e. $g \in X_r$, so that $U_r \subset X_r$. If $g \in X_r$, then (4.2) and (4.15) imply $P_r g = 0$; hence, $g = f - P_r f$ with $f = g$, and thus, since also $f \in X_r \subset V_{r+1}$ from (4.15), we have from (4.14) that $g \in U_r$. Consequently, $X_r \subset U_r$.

Next, we prove that $W_r \subset X_r$. If $g \in W_r$, there exists a coefficient sequence $\{c_0, \ldots, c_{2^r n-1}\}$ such that $g(x) = \sum_{\ell=0}^{2^r n-1} c_\ell \psi_\ell^r(2^{r+1-R}x)$, $x \in [0, 2^R n]$. But then, from (4.10), we have

$$g(2^{R-r}j) = \sum_{\ell=0}^{2^r n-1} c_\ell \phi_{2\ell+1}^{r+1}(2j) = 0, \quad j \in \{0, 1, \ldots, 2^r n\},$$

by virtue of (3.3). Definition (4.15) then implies that $g \in X_r$. So, $W_r \subset X_r$.

We now show that $U_r \subset W_r$, thereby completing the proof of (4.16). Suppose therefore $g \in U_r$, so that, from (4.14) and (4.1), there exists a coefficient sequence $\{c_j : j = 0, 1, \ldots, 2^{r+1}n\}$ such that the function $f \in V_{r+1}$ given by

$$f(x) = \sum_{j=0}^{2^{r+1}n} c_j \phi_j^{r+1}(2^{r+1-R}x), \quad x \in [0, 2^R n], \tag{4.18}$$

satisfies the equation

$$g(x) = f(x) - (P_r f)(x), \quad x \in [0, 2^R n]. \tag{4.19}$$

But, from (4.2), (4.18), (3.3) and (3.5), we obtain

$$\begin{aligned}
(P_r f)(x) &= \sum_{\ell=0}^{2^{r+1}n} \left[\sum_{k=0}^{2^r n} \phi_\ell^{r+1}(2k) \phi_k^r(2^{r-R}x) \right] c_\ell \\
&= \sum_{\ell=0}^{2^r n} c_{2\ell} \phi_\ell^r (2^{r-R}x) \\
&= \sum_{\ell=0}^{2^r n} c_{2\ell} \sum_{k=0}^{2^{r+1}n} \phi_\ell^r \left(\frac{k}{2}\right) \phi_k^{r+1}(2^{r+1-R}x) \\
&= \sum_{\ell=0}^{2^r n} c_{2\ell} \phi_{2\ell}^{r+1}(2^{r+1-R}x) \\
&\quad + \sum_{\ell=0}^{2^r n} c_{2\ell} \sum_{k=0}^{2^r n-1} \phi_\ell^r \left(k+\tfrac{1}{2}\right) \phi_{2k+1}^{r+1}(2^{r+1-R}x). \qquad (4.20)
\end{aligned}$$

Also, from (4.18),

$$f(x) = \sum_{j=0}^{2^r n} c_{2j} \phi_{2j}^{r+1}(2^{r+1-R}x) + \sum_{j=0}^{2^r n-1} c_{2j+1} \phi_{2j+1}^{r+1}(2^{r+1-R}x). \qquad (4.21)$$

Combining (4.19), (4.20) and (4.21) yields

$$g(x) = \sum_{j=0}^{2^r n-1} \left[c_{2j+1} - \sum_{k=0}^{2^r n} c_{2k} \phi_k^r \left(j+\tfrac{1}{2}\right) \right] \phi_{2j+1}^{r+1}(2^{r+1-R}x), \qquad (4.22)$$

which, together with (4.11), (4.10), implies that $g \in W_r$. Thus $U_r \subset W_r$.

(b) Suppose $f \in V_r \cap W_r$. Then (4.4) and (4.2) imply that, for $x \in [0, 2^R n]$,

$$f(x) = (P_r f)(x) = \sum_{j=0}^{2^r n} f(2^{R-r}j) \phi_j^r (2^{r-R}x) = 0,$$

after noting from (4.16) that $f \in X_r$, and then using the definition (4.15). ■

We can now prove Theorem 4.1.

Proof. Let $r \in \{0, 1, \ldots, R-1\}$ be fixed. Take any $f \in V_{r+1}$ and define $g = P_r f$ and $h = f - P_r f$. Then (4.2) implies that $g \in V_r$, whereas (4.14) and (4.16) imply that $h \in W_r$. We have therefore shown that there exist functions $g \in V_r$ and $h \in W_r$ such that $f = g + h$. It remains to be proven that g and h are uniquely determined by f. But, if $g_0 \in V_r$ and $h_0 \in W_r$ were such that $f = g_0 + h_0$, then $u = g_0 - g \in V_r$ and $v = h - h_0 \in W_r$, with $u = v$. Thus $u \in V_r \cap W_r$ and $v \in V_r \cap W_r$, and the desired uniqueness result follows from (4.17). ■

From (4.13) one concludes that $\dim W_r = \dim V_{r+1} - \dim V_r$, so that (4.9) leads to

$$\dim W_r = 2^r n, \quad r = 0, 1, \ldots, R-1. \tag{4.23}$$

Hence, based on the definition (4.11), we conclude that the set $\{\psi_j^r(2^{r+1-R}\bullet) : j = 0, 1, \ldots, 2^r n - 1\}$ is linearly independent on $[0, 2^R n]$, and therefore is a basis for W_r.

We have therefore established, for each fixed $r \in \{0, 1, \ldots, R-1\}$, an *interpolation wavelet basis* $\{\psi_j^r(2^{r+1-R}\bullet) : j = 0, 1, \ldots, 2^r n - 1\}$ for the *interpolation wavelet space* W_r. The elements of the sequence $\{\psi_j^r\}$ are called *interpolation wavelets*. Examples for $N = 1$ are plotted in Figure 5.

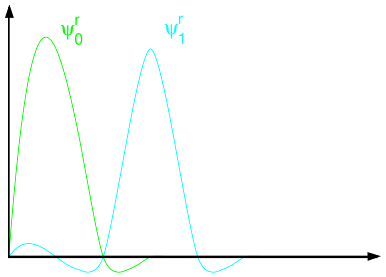


Figure 5: Boundary interpolation wavelets with $N = 1$.

In the next section, we derive the corresponding decomposition and reconstruction algorithms.

4.2 Decomposition and reconstruction algorithms

To obtain the decomposition algorithm, let $r \in \{0, 1, \ldots, R-1\}$ be fixed, and suppose $f_{r+1} \in V_{r+1}$ is given by

$$f_{r+1}(x) = \sum_{j=0}^{2^{r+1}n} c_j^{(r+1)} \phi_j^{r+1}(2^{r+1-R}x), \quad x \in [0, 2^R n]. \tag{4.24}$$

According to Theorem 4.1, and for the bases $\{\phi_j^r\}$ and $\{\psi_j^r\}$ of V_r and W_r, we know that there exist unique coefficient sequences $\{c_j^{(r)} : j = 0, 1, \ldots, 2^r n\}$ and $\{d_j^{(r)} : j = 0, 1, \ldots, 2^r n - 1\}$ such that the functions $f_r \in V_r$ and $g_r \in W_r$ defined by

$$f_r(x) = \sum_{j=0}^{2^r n} c_j^{(r)} \phi_j^r(2^{r-R}x), \quad x \in [0, 2^R n], \tag{4.25}$$

and

$$g_r(x) = \sum_{j=0}^{2^r n - 1} d_j^{(r)} \psi_j^r(2^{r+1-R}x), \quad x \in [0, 2^R n], \tag{4.26}$$

satisfy

$$f_{r+1} = f_r + g_r. \tag{4.27}$$

In particular, observe that we then have the *interpolation wavelet decomposition*

$$f_R = f_0 + \sum_{j=0}^{R-1} g_j.$$

Moreover,

$$f_r = P_r f_{r+1} \tag{4.28}$$

and

$$g_r = f_{r+1} - P_r f_{r+1}. \tag{4.29}$$

The coefficients $\{d_j^{(r)}\}$ in (4.26) are called the *interpolation wavelet coefficients.*

Using (4.24), (4.10) and (4.29), and the argument which led to (4.22), we obtain, for all $x \in [0, 2^R n]$,

$$\sum_{j=0}^{2^r n-1} d_j^{(r)} \psi_j^r (2^{r+1-R} x) = \sum_{j=0}^{2^r n-1} \left[c_{2j+1}^{(r+1)} - \sum_{k=0}^{2^r n} c_{2k}^{(r+1)} \phi_k^r \left(j + \tfrac{1}{2} \right) \right] \psi_j^r (2^{r+1-R} x).$$

Since $\{\psi_j^r (2^{r+1-R} \bullet) : j = 0, 1, \ldots, 2^r n - 1\}$ is a linearly independent set on $[0, 2^R n]$, we have

$$d_j^{(r)} = c_{2j+1}^{(r+1)} - \sum_{k=0}^{2^r n} c_{2k}^{(r+1)} \phi_k^r \left(j + \tfrac{1}{2} \right), \quad j = 0, 1, \ldots, 2^r n - 1. \tag{4.30}$$

Next, using (4.25), (4.28), (4.2) and (3.3) we get, for all $x \in [0, 2^R n]$,

$$\begin{aligned} \sum_{j=0}^{2^r n} c_j^{(r)} \phi_j^r (2^{r-R} x) = f_r(x) &= (P_r f_{r+1})(x) \\ &= \sum_{j=0}^{2^r n} \left[\sum_{\ell=0}^{2^{r+1} n} c_\ell^{(r+1)} \phi_\ell^{r+1}(2j) \right] \phi_j^r (2^{r-R} x) \\ &= \sum_{j=0}^{2^r n} c_{2j}^{(r+1)} \phi_j^r (2^{r-R} x). \end{aligned}$$

Since the set $\{\phi_j^r (2^{r-R} \bullet) : j = 0, 1, \ldots, 2^r n\}$ is linearly independent on $[0, 2^R n]$, we deduce that

$$c_j^{(r)} = c_{2j}^{(r+1)}, \quad j = 0, 1, \ldots, 2^r n. \tag{4.31}$$

Now observe from (3.28), (3.7) and (3.33) that

$$\phi_k^r (j + \tfrac{1}{2}) = a_{j,k}^{(r)}, \quad k = 0, 1, \ldots, 2^r n, \quad j = 0, 1, \ldots, 2^r n - 1,$$

which, together with (4.30) and (4.31), then yield, for a given data sequence $\{c_j^{(R)} : j = 0, 1, \ldots, 2^R n\}$, the following interpolation wavelet algorithms:

Decomposition algorithm:

$$\left.\begin{aligned} c_j^{(r)} &= c_{2j}^{(r+1)}, && j = 0, 1, \ldots, 2^r n, \\ d_j^{(r)} &= c_{2j+1}^{(r+1)} - \sum_{k=0}^{2^r n} a_{j,k}^{(r)} c_{2k}^{(r+1)}, && j = 0, 1, \ldots, 2^r n - 1, \end{aligned}\right\} r = R-1, R-2, \ldots, 0. \tag{4.32}$$

Reconstruction algorithm:

$$\left.\begin{aligned} c_{2j}^{(r+1)} &= c_j^{(r)}, && j = 0, 1, \ldots, 2^r n, \\ c_{2j+1}^{(r+1)} &= d_j^{(r)} + \sum_{k=0}^{2^r n} a_{j,k}^{(r)} c_k^{(r)}, && j = 0, 1, \ldots, 2^r n - 1, \end{aligned}\right\} r = 0, 1, \ldots, R-1. \tag{4.33}$$

Here the coefficient sequence $\{a_{j,k}^{(r)} : k = 0, 1, \ldots, 2^r n,\ j = 0, 1, \ldots, 2^r n - 1,$ $r = 0, 1, \ldots, R-1\}$ is defined by (3.34), (3.35), (3.36), with explicit formulations in (3.37) and (3.38).

Suppose $f \in C[0, 2^R n]$, with the integers R and n suitably chosen. The sequence $\{f_r : r = R-1, R-2, \ldots, 0\}$ is then defined by (4.28), with $f_R = f$, whereas the sequence $\{g_r : r = R-1, R-2, \ldots, 0\}$ is defined by (4.29). The coefficient sequences $\{c_j^{(r)} : j = 0, 1, \ldots, 2^r n,\ r = R, R-1, \ldots, 0\}$ and $\{d_j^{(r)} : j = 0, 1, \ldots, 2^r n - 1,\ r = R-1, R-2, \ldots, 0\}$ are computed recursively by means of (4.32). In particular, observe that since (4.2) and (4.25) yield

$$\sum_{j=0}^{2^R n} c_j^{(R)} \phi_j^R(x) = f_R(x) = (P_R f)(x) = \sum_{j=0}^{2^R n} f(j) \phi_j^R(x),$$

we have $c_j^{(R)} = f(j),\ j = 0, 1, \ldots, 2^R n$, and thus, the interpolation wavelet decomposition algorithm (4.32) can, in this context, be rewritten as

$$\left.\begin{aligned} c_j^{(r)} &= f(2^{R-r} j), && j = 0, 1, \ldots, 2^r n, \\ d_j^{(r)} &= f(2^{R-r-1}(2j+1)) - \sum_{k=0}^{2^r n} a_{j,k}^{(r)} f(2^{R-r} k), && j = 0, 1, \ldots, 2^r n - 1. \end{aligned}\right\}$$
$$r = R-1, R-2, \ldots, 0. \tag{4.34}$$

The reconstruction is then performed by means of (4.33).

At this stage, it is of interest to point out the following relationship between the interpolation wavelet procedure (4.34), (4.33) and the interpolatory subdivision scheme (3.33).

After the decomposition with (4.34) has been performed, suppose that we set the interpolation wavelet coefficients

$$d_j^{(r)} = 0, \quad j = 0, 1, \ldots, 2^r n - 1, \quad r = 0, 1, \ldots, R-1, \tag{4.35}$$

at each successive step of the reconstruction phase (4.33). The final reconstructed (and smoothed) function is then

$$f_R(x) = \sum_{j=0}^{2^R n} c_j^{(R)} \phi_j^R(x), \quad x \in [0, 2^R n], \tag{4.36}$$

with, as is clear from the top parts of (4.33) and (4.34), and (3.3),

$$f_R(2^R k) = c_{2^R k}^{(R)} = f(2^R k), \quad k = 0, 1, \ldots, n.$$

Now observe that the zero values (4.35) substituted into the bottom part of (4.33) yield precisely the bottom part of the subdivision formula (3.33).

Hence the interpolation wavelet decomposition and reconstruction scheme described above is equivalent to the following interpolatory subdivision scheme:

Choose $c = c^{(0)} = f(2^R k), \quad k = 0, 1, \ldots,$ and apply the interpolatory subdivision scheme (3.33). According to Theorem 3.3, this scheme converges to the limit curve

$$g(x) = \sum_{j=0}^{n} f(2^R j) \phi_j^0(x), \quad x \in [0, n]. \tag{4.37}$$

Then, using the refinement equation (3.5), as well as (3.6) – (3.7), we get

$$\begin{aligned} g(x) &= \sum_{j=0}^{n} c_j^{(0)} \phi_j^0(x) \\ &= \sum_{j=0}^{n} c_j^{(0)} \sum_{\ell=0}^{2n} \phi_j^0 \left(\frac{\ell}{2}\right) \phi_j^1(2x) \\ &= \sum_{\ell=0}^{2n} c_\ell^{(1)} \phi_j^1(2x) \\ &\;\;\vdots \\ &= \sum_{\ell=0}^{2^R n} c_\ell^{(R)} \phi_j^R(2^R x) = f_R(2^R x), \end{aligned}$$

from (4.36). Hence,

$$g(x) = f_R(2^R x), \quad x \in [0, n]. \tag{4.38}$$

The subdivision approach therefore has the significant advantage of providing an efficient iterative procedure for the construction of the function f_R in (4.36).

Finally, suppose in the decomposition procedure defined by the algorithm (4.34) we have, for some given polynomial $p \in \pi_{2N+1}$, and for a fixed $r \in \{R-1, R-2, \ldots, 0\}$, that there exists an integer $\mu \in [0, 2^r n]$ such that

$$f(x) = p(x), \quad x \in [\alpha, \beta],$$

where

$$\alpha = \max\{0, 2^{R-r}(\mu - 2N)\}, \qquad \beta = \min\{2^R n, 2^{R-r}(\mu + 2N)\}.$$

Then, using also (4.34), (4.30), (3.2) and (3.4), we find

$$\begin{aligned} d_\mu^{(r)} &= p(2^{R-r-1}(2\mu+1)) - \sum_{k=\max\{0,\mu-2N\}}^{k=\min\{2^r n,\mu+2N\}} p(2^{R-r}k)\phi_k^r\left(\mu+\tfrac{1}{2}\right) \\ &= p(2^{R-r-1}(2\mu+1)) - \sum_{k=0}^{2^r n} p(2^{R-r}k)\phi_k^r\left(\mu+\tfrac{1}{2}\right) = 0. \end{aligned}$$

Hence the interpolation wavelet coefficients $d_j^{(r)}$ corresponding, according to (3.2), to those regions in $[0, 2^R n]$ where the function f has polynomial-like behavior can be expected to be small relative to the interpolation wavelet coefficients $d_j^{(r)}$ corresponding to those regions in $[0, 2^R n]$ where f exhibit non-polynomial-like behavior.

For example, choose the function f in (4.34) as the cubic cardinal B-spline, with an arbitrary choice of the integers R and n. The B-spline consists of four cubic polynomial pieces joined together in such a way that the second derivative of the B-spline is continuous, while the third derivative has jump discontinuities at the nodes. Now choose $N = 1$ to ensure, by the above argument, that the wavelet coefficients $d_j^{(r)}$, $r = R-1, R-2, ..., 0$ are zero in the regions where f is identical to a polynomial and non-zero in the regions where the third derivative has a jump discontinuity.

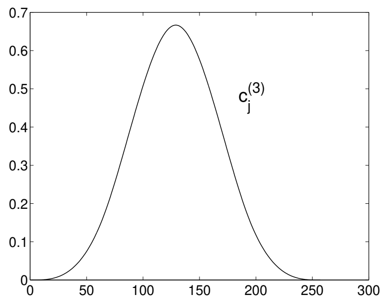
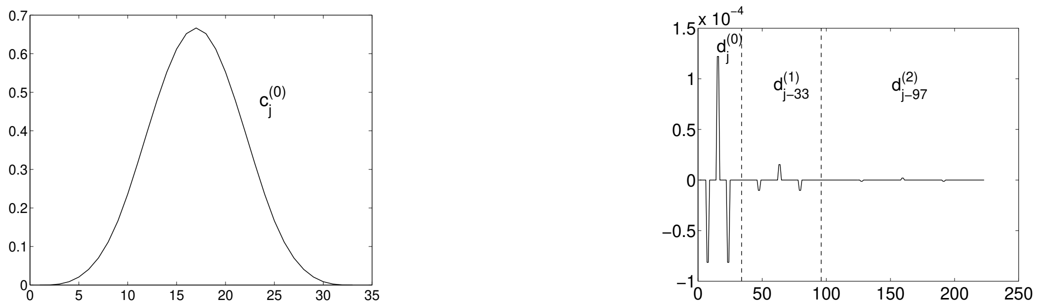


Figure 6: Cubic B-spline, sampled at 257 equally spaced points.

In Figure 6, we have chosen $R = 3$ and $n = 32$ and plotted the sequence of sampled values $c_j^{(3)} = f(j)$ versus their indices. Figure 7 shows the result of the interpolation wavelet decomposition algorithm (4.34), where the resulting coefficients are also plotted versus their indices.

We observe that the average values $c_j^{(0)}$, as shown in Figure 7(a), consist of precisely every eighth value of the original B-spline. What is of more interest is the set of detail components shown in Figure 7(b). Note that, as expected, the interpolation wavelet coefficients $d_j^{(r)}$, $r = 2, 1, 0$, are non-zero only where the B-spline has jump discontinuities in its third derivative.



(a) Average component. *(b) Detail components.*

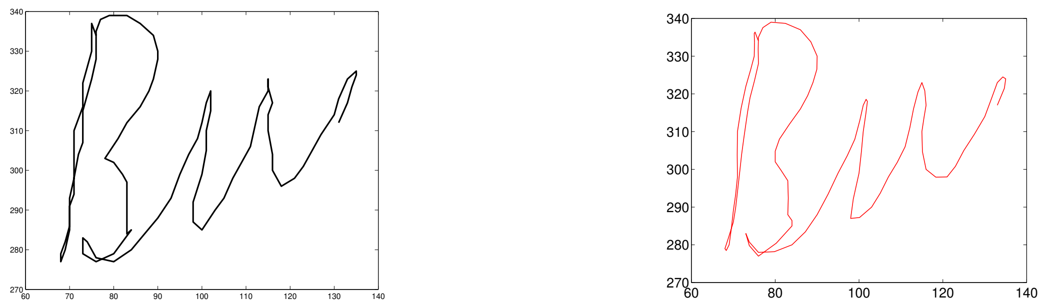
Figure 7: Third level decomposition of the B-spline of Figure 6.

5 Examples

In this section we illustrate the interpolation wavelets by applying them to two practical problems. Of course we do not claim that the schemes described above are more efficient in practice than any of the alternatives. This would require a detailed study which we leave for future consideration. These examples merely serve to illustrate the theory developed above. Note in particular how the interpolation wavelet decomposition on an interval avoids any edge artifacts.

5.1 Signature smoothing

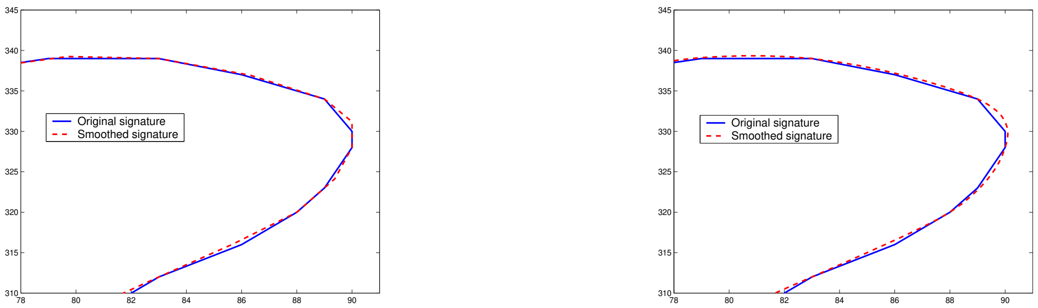
Figure 8(a) shows part of a signature that was captured by a digitized tablet. One clearly sees the quantization effect of the underlying grid of this particular tablet. Almost all applications require that the signature should be smoothed. We therefore applied a single level of the interpolation wavelet decomposition (4.34) with $N = 1$. All the detail coefficients were set equal to zero, and a single reconstruction step was performed using the zero detail coefficients. The result is the smoothed signature in Figure 8(b). Note that we follow standard practice by displaying only the discrete coefficients $c_j^{(R)}$, connected by straight line segments. This is different from the smooth curve given by (4.36).



(a) Original signature sample. *(b) Smoothed signature.*

Figure 8: Illustration of smoothing with interpolation wavelet decomposition.

A closer look at the original and reconstructed signature is given in Figure 9. Note from Figure 9(a) how every other data point stays the same (due to the interpolation property). The remaining points however, are calculated in such a way that the result is a smoother signature. In fact, this procedure is exactly the same as if we discarded every other data point of the original signature and then applied one step of the subdivision scheme with $N = 1$ to the result, as described in Section 4. As mentioned above we plot only the data points connected by straight lines, and not the smooth curve f_R described by (4.36). An efficient way of calculating f_R is by subdivision: according to the argument leading from (4.36) to (4.38), each step of the subdivision scheme doubles the number of points on the curve f_R. The result of two more subdivision steps is shown in Figure 9(b). Note how the reconstructed curve has become noticeably smoother.



(a) A single reconstruction step *(b) After two more subdivision steps.*

Figure 9: Magnification of original and smoothed signature.

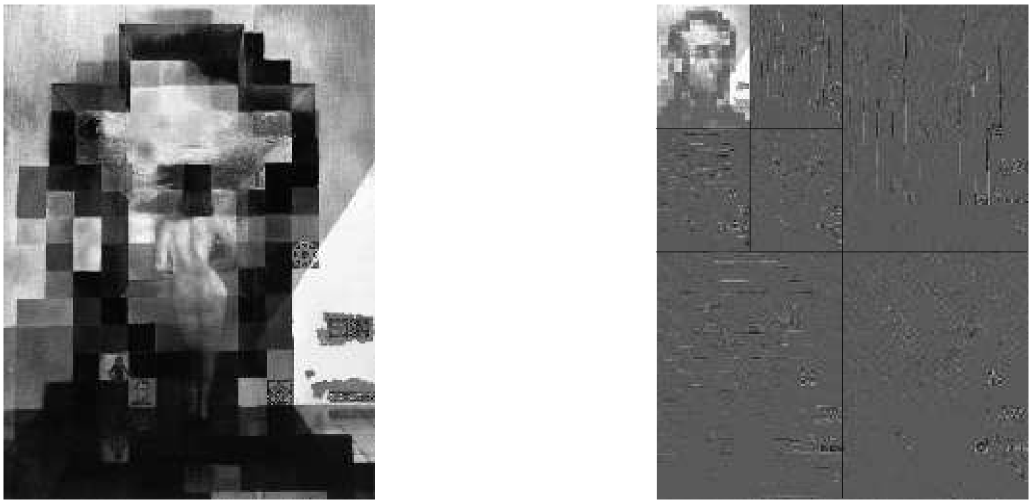
Note that the main problem in this case is that the data points themselves are corrupted by noise. Insisting that the data points are interpolated is therefore not the best way to proceed. In this case non-interpolatory schemes such as Lane-Riesenfeld [20] (see also [23, Chapter 2]). might prove beneficial. Even in this less than ideal situation, the interpolation wavelets provide a surprisingly good smoothing, with no artificial edge effects.

5.2 Two-dimensional interpolation wavelet decomposition

For our second example we use the well-known painting by Salvador Dali, *Gala contemplating the Mediterranean*, Figure 10(a), the original of which can be seen in the Dali museum in Figueres, Spain. Although already painted in 1976, it is a beautiful illustration of Dali's awareness of images on different scales, in this case, a portrait of Abraham Lincoln, the 16th President of the United States of America, and Gala, Dali's wife, looking out to sea.

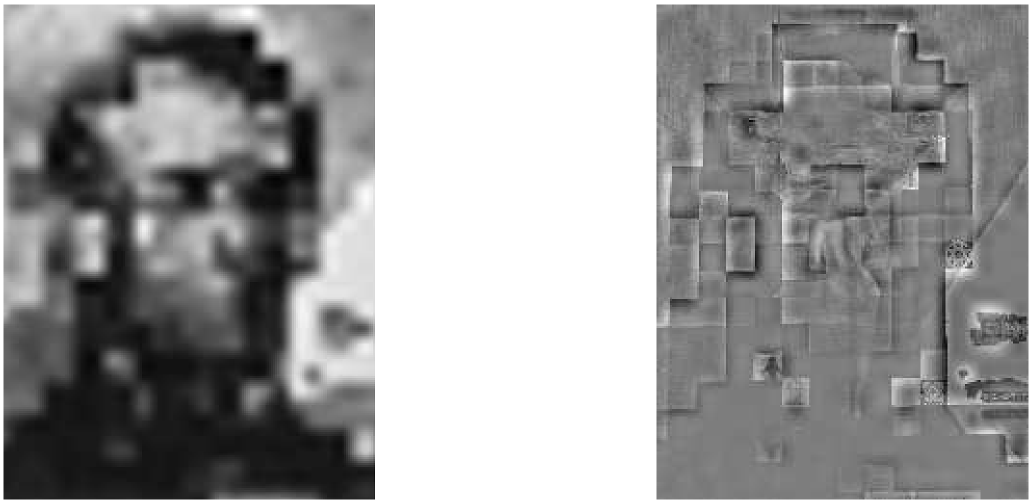
Constructing a two-dimensional tensor product from the interpolation wavelet (4.10) (see e.g. [5, Section 6.4]), allows us to perform a two-dimensional decomposition of Dali's painting, as illustrated in Figure 10(b). Note how clearly the image of Abraham Lincoln is captured by the average component, displayed in the top lefthand corner of Figure 10(b). The remaining part of the figure consists of the various detail components.

Now we set the detail components equal to zero, and then reconstruct according to a



(a) Original image. (b) Decomposition of image.

Figure 10: Illustration of two-dimensional interpolation wavelet decomposition.



(a) Detail removed. (b) Average removed.

Figure 11: Reconstructed images.

two-dimensional reconstruction algorithm also based on the tensor product interpolation wavelet. The result, as shown in Figure 11(a), is the image of Abraham Lincoln interpolated back onto the original grid of Figure 10(a). If we set the average component equal to zero, and then reconstruct, we obtain the image in Figure 11(b), which contains all the areas of sharp transition which are absent from Figure 11(a).

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