SUBDIVISION SCHEMES IN CAGD

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1. Introduction

Computer Aided Geometric Design (CAGD) is a branch of applied mathematics concerned with algorithms for the design of smooth curves and surfaces and for their efficient mathematical representation. The representation is used for the computation of the curves and surfaces, as well as geometrical quantities of importance such as curvatures, intersection curves between two surfaces and offset surfaces.

The general setup is the following: The designer produces a set of points with connectivity relations between them, termed control points. Where the points are arranged according to a univariate index set $p_1, p_2, \ldots, p_k \in \mathbb{R}^3$, the set of points is termed a control polygon and is identified with the piecewise linear curve going through the points. It is then required to represent mathematically a smooth curve in $\mathbb{R}^3$ which has a similar shape to the control polygon. This curve can pass through the points or be in some other sense close to the control polygon.

The control points for the design of a surface can be given in various ways. The simplest connectivity is that of a regular grid, namely each point has two indices: p_{ij}, $i = 1, \ldots, N_1$, $j = 1, \ldots, N_2$. Thus the four points $p_{ij}, p_{i+1,j}, p_{i,j+1}$ and $p_{i+1,j+1}$ constitute a "face". Another possible topology of the connectivity relations is that of a triangular grid, where each face is determined by three control points and each pair of control points can belong to at most two triangles. In this case the control polyhedron consists of the planar faces determined by the triangulation. The most general type of connectivity relation is that of a set of "faces" with a variable number of vertices. The set of control points together with the connectivity relations constitute the control net.

Given the control net it is then required to construct a smooth surface approximating it. Thus to each "face" there corresponds a patch which is determined by the vertices of that "face" and perhaps by several of their direct neighbors.

As a general principal the methods of construction are required to be local, so that changes in a control point affect only a limited number of patches. The functions used in the construction of the curves and surfaces are mainly piecewise polynomials or piecewise relational functions. Polynomials and rational functions of low degree are easily computed and their piecewise nature yields the required flexibility.

One common approach to the design of curves and surfaces which is of great relevance to subdivision is based on the existence of a family of smooth compactly supported functions $B_m(t)$ termed B-splines, with the following properties:

(a) $B_m(t) \in C^{m-1}$ is a piecewise polynomial of degree m,

(b) $B_m(t) > 0$ inside its support, $(0, m+1)$,

(c) $\sum_{i\in\mathbb{Z}} B_m(t-i) = 1\ , \qquad t \in \mathbb{R}\ .$

(d) $\mathrm{span}\{B_m(t-i) \mid i \in \mathbb{Z}\} = \mathrm{S}_m = \{f \mid f \in C^{m-1}\ ,\ f\big|_{(i,i+1)} \in \pi_m\ ,\ i \in \mathbb{Z}\}\ .$

Here π_m denotes the space of all polynomials over $\mathbb{R}$ of degree $\leq m$.

The curve

$$C(t) = \sum_{i=1}^{N} p_i B_m(t-i)\ , \tag{1.1}$$

has the properties

(i) for $t \in (j, j+1)$, $C(t) \in \langle p_{j-m}, \ldots, p_j\rangle^0$

where $\langle p_1 \cdots p_k\rangle = \{x \in \mathbb{R}^3 \mid x = \sum\limits_{i=1}^{k} b_i p_i\ ,\ b_i \geq 0\ ,\ \sum\limits_{i=1}^{k} b_i = 1\}$ is the convex hull of $p_1, \ldots, p_k$ and Ω^0 denotes the interior of Ω.

(ii) The curve $C(t)$ has its components in C^{m-1}.

If $\|C'(t)\|_2 \neq 0$, $t \in I \subset \mathbb{R}$, then $C(t)\big|_I \in C^{m-1}$. The condition $\big\|C'(t_0)\big\|_2 \neq 0$ guarantees that $C_i'(t_0) \neq 0$ for $i = 1$ or 2 or 3. Hence by the Implicit Functions Theorem $t = t(C_i) \in C^{m-1}$ for some $|t - t_0| < \varepsilon$ and $C_j(t) = C_j\big(t(C_i)\big)$, $j \neq i$.

For control points with the topology of a regular grid, the definition of a B-spline surface is very similar

$$S(u,v) = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} p_{ij} B_m(u-i) B_m(v-j)\ . \tag{1.2}$$

This surface has the following properties:

(i) for $(u,v) \in (k,k+1)\times(\ell,\ell+1)$,

$$S(u,v) \in \left\langle p_{ij} \middle| i = k-m,\dots,k\ ,\ j = \ell - m,\dots,\ell \right\rangle^0\ .$$

(ii) The components of $S(u,v)$ are in C^{m-1}.

If the Jacobian matrix $\left(\frac{\partial S}{\partial u}, \frac{\partial S}{\partial v}\right)$ is of rank 2 for $(u,v)\in\Omega$ then $S(u,v)\in C^{m-1}(\Omega)$. Again, using the Implicit Functions Theorem, it is possible to obtain the points of $\{S(u,v) : |u-v_0|, |v-v_0| < \varepsilon\}$ as $S_j\big(u(S_i,S_\ell), v(S_i,S_\ell)\big) = F(S_i,S_\ell)\big) \in C^{m-1}(\Omega)$ where $\{i,j,\ell\} = \{1,2,3\}$, and Ω is a neighborhood of $\big(S_i(u_0,v_0), S_\ell(u_0,v_0)\big)$.

Sufficient conditions on the control points can be given to guarantee that the B-spline curve/surface of the form (1.1)/(1.2) is C^{m-1}.

The simplest continuous B-spline is

$$B_1(t) = 1 - |t-1|\ , \qquad t\in[0,2]\ . \tag{1.3}$$

$B_1(t)$ is a positive, continuous, piecewise linear function on its support $(0,2)$, and has also properties (c),(d) of B-splines.

The curve $C(t) = \sum_{i=1}^N p_i B_1(t-i)$ is the control polygon of $p_1,\dots,p_N$ for $t\in[2,N+1]$: it is linear for $t\in(i,i+1)$ and $C(i+1) = p_i$, $i = 1,\dots,N$.

The surface $S(u,v) = \sum\limits_{i,j=1}^{N} p_{ij} B_1(u-i) B_1(v-j)$ passes through the control points:

$$S(i+1,j+1) = p_{ij}\ , \qquad i,j = 1,\dots,N\ .$$

Higher order B-splines are obtained from lower order ones by repeated integration

$$B_{m+1}(t) = \int_{t-1}^{t} B_m(\tau)d\tau\ . \tag{1.4}$$

It is easy to verify that if B_m has properties (a)-(d) of B-splines then B_{m+1} as given by (1.4) has these properties as well. Since B_1 given by (1.3) is symmetric with respect to $t=1$, then by (1.4), B_m is symmetric relative to $t=(m+1)/2$.

The computation of a given B-spline curve/surface can be done in several ways depending on the representation used.

1. Computation based on the polynomial representation of the curve/surface is each interval/square.
2. Computation based on the B-spline representation using recurrence relations for the evalutation of B-splines.
3. Computation of the representations of the curve/surface relative to the sequence of bases

$$\{B_m(2^k \cdot -j) : j \in \mathbb{Z}\} , \qquad k = 1, 2, \ldots \quad . \tag{1.5}$$

The last method termed "Subdivision" is based on the observation that

$$B_m \in \operatorname{span}\{B_m(2(\cdot - j)) : j \in \tfrac{1}{2}\mathbb{Z}\} = \{f : f\left(\tfrac{\cdot}{2}\right) \in \mathcal{S}_m\} . \tag{1.6}$$

It requires the computation of a sequence of control points. Expressing (1.6) explicitly

$$B_m(t) = \sum_{j\in\mathbb{Z}} a_{j,m} B(2t - j) , \tag{1.7}$$

we get a sequence of representations of $C(t)$,

$$C(t) = \sum_{i\in\mathbb{Z}} p_i^0 B_m(t-i) = \sum_{i\in\mathbb{Z}} p_i^1 B_m(2t-i) = \cdots = \sum_{i\in\mathbb{Z}} p_i^k B_m(2^k t - i) ,$$

where the control points at stage $k+1$ are obtained from those at stage k by the rule

$$p_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-2j,m} p_j^k , \qquad i \in \mathbb{Z} . \tag{1.8}$$

The coefficients in (1.8) constitute the mask of the subdivision scheme. In particular the curve segment corresponding to $t \in 2^{-k}(j, j+1)$ in the k'th-stage representation is given by

$$C(t)\big|_{2^{-k}(j,j+1)} = \sum_{i=j-m}^{j} p_i^k B_m(2^k t - i) , \tag{1.9}$$

and the point p_i^k is related to the parameter value $2^{-k}i$. For $t_0 \in 2^{-\ell}j$ and $k \geq \ell$, the points $\{p^k_{2^{k-\ell}j-m}, \ldots, p^k_{2^{k-\ell}j}\}$ determine the curve segment in $(t_0, t_0 + 2^{-k})$, and as k becomes larger these points tend to $C(t_0)$. Hence for k large enough the control polygon approximates closely the curve $C(t)$. The surface case is similar.

The equality (1.7) for $m = 1$ is easily derived by choosing $\{a_{j,1}\ ,\ j \in \mathbb{Z}\}$ so that both sides agree on $\frac{1}{2}\mathbb{Z}$. Thus

$$B_1(t) = \tfrac{1}{2}B_1(2t) + B_1(2t-1) + \tfrac{1}{2}B_1(2t-2)\ , \tag{1.10}$$

and by integrating (1.10) from $t-1$ to t and using (1.4) we set

$$B_2(t) = \tfrac{1}{4}B_2(2t) + \tfrac{3}{4}B_2(2t-1) + \tfrac{3}{4}B_2(2t-2) + \tfrac{1}{4}B_2(2t-3)\ . \tag{1.11}$$

In view of (1.10), for B_1-curves the subdivision rule (1.8) has the form

$$p^{k+1}_{2i} = \tfrac{1}{2}p^k_{i-1} + \tfrac{1}{2}p^k_i\ , \qquad p^k_{2i+1} = p^k_i\ , \tag{1.12}$$

and the control points at stage $k+1$ stay on the control polygon of stage k and hence on the initial one, namely on the limit curve. This scheme is interpolatory: all the points at all stages are on the limit curve.

For B_2-curves, by (1.11) the mask is $a_{0,2} = a_{3,2} = 1/4$, $a_{1,2} = a_{2,2} = 3/4$, and the subdivision scheme (1.8) has the form

$$p^{k+1}_{2i} = \tfrac{3}{4}p^k_{i-1} + \tfrac{1}{4}p^k_i\ , \qquad p^{k+1}_{2i+1} = \tfrac{1}{4}p^k_{i-1} + \tfrac{3}{4}p^k_i\ . \tag{1.13}$$

This scheme is known as Chaikin's algorithm [C]. One step of this shceme is depicted in Figure 1.

Figure 1. Corner cutting with the Chaikin's algorithm

The Chaikin's algorithm is a geometric process of "corner cutting": the control points are cut away at each stage.

Equation (1.8) consists of two rules, one for i odd involving the odd coefficients of the mask and one for i even involving the even coefficients of the mask. It is easy to verify that (1.7) together with (1.4) yields

$$a_{j,m+1} = \tfrac{1}{2}(a_{j-1,m} + a_{j,m}) \ , \qquad j \in \mathbb{Z} \ , \tag{1.14}$$

and since $a_{j,1} \neq 0$ only for $j \in \{0,1,2\} = \overline{\operatorname{supp} B_1} \cap \mathbb{Z}$, we get

$$a_{j,m} \neq 0 \quad \text{only for} \quad j \in \{0,1,\ldots,m+1\} = \overline{\operatorname{supp} B_m} \cap \mathbb{Z} \tag{1.15}$$

Also by (1.14) and the initial values $a_{0,1} = 1/2$, $a_{1,1} = 1$, $a_{2,1} = 1/2$, we conclude that

$$a_{j,m} = 2^{-m} \binom{m+1}{j} \ , \qquad j = 0,1,\ldots,m+1 \ . \tag{1.16}$$

The subdivision shemes (s.s.) for B-spline curves given by (1.8) and (1.16) are prototypes for general s.s. for curves determined by masks of compact support $\mathbf{a} = \{a_j : j \in \mathbb{Z}\}$. In this paper we analyze s.s. for curves and surfaces given by a rule of the form (1.8) with a general mask of compact support $\mathbf{a}$. Our aim is to give conditions on the mask which guarantee the convergence of the s.s. to a limit curve/surface and to analyze the smoothness of this limit.

In Section 2 we discuss the relation between the convergence of a s.s. and the existence of a related compactly supported function (the analogue of the B-spline), satisfying a functional equation of the form (1.7). This is done in the multivariate setting which applies to curves and surfaces.

The analysis of the convergence and smoothness is first done for the curve case (univariate case), since it is simpler conceptually and closer to being complete. In Section 3 sufficient conditions for convergence to curves with C^ν components are given, in terms of a polynomial formalism. In Section 4, a matrix formalism is introduced and necessary conditions for convergence to curves with C^ν components are given, together with refined sufficient conditions. Section 5 deals with interpolatory s.s., and shows the necessity of the sufficient conditions of Section 3 for such schemes. Examples of s.s. with their concrete analysis are given in Sections 3 and 4.

Subdivision schemes for surfaces, determined by control points with the topology of a regular grid, are discussed in Sections 6-8. The convergence is analyzed in Section 6 and the smoothness of the components of the limit surfaces in Section 7, with special results for interpolatory s.s., extending those of Section 5. The analysis of the general multivariate case requires the introduction of s.s with matrix masks and the extension of the analysis to such schemes.

The tools developed for the multivariate setting are much harder to apply to concrete examples, due to non-unicity in certain reductions as well as the order of magnitude of complicated involved algebraic manipulations required. Yet one example of an interpolatory s.s. is presented and analysed in Section 8.

This paper is mainly a review paper, which tries to present the results taken from several sources in a unified and easy to follow way. Each section ends with bibliographical notes, pointing to references for the material in the section and for related material. Some of the results and proofs in Sections 6 and 7 appear here for the first time.

Bibliographical notes.

General methods in CAGD are reviewed in [BFK]. Regularity conditions for B-spline curves/surfaces are derived in [DLY]. Computation methods for B-spline representations are given in [B]. Subdivision techniques for B-spline curves/surfaces are first discussed in [CLR1] and [LR]. Subdivision schemes with non-standard limit functions were first analyzed in [R] and in recent years in many works, e.g. [CDM]. [DL1], [D2], [DD], [DGL1-3], [DL2,3], [DLL], [DLM], [MP1-3], [W]. The main sources for the material in this paper are [CDM], [DGL1-3], [DHL], [DL3], [DLM].

2. The Subdivision Mask and the Functional Equation

A subdivision scheme is defined in terms of a mask consisting of a finite set of non-zero coefficients $\mathbf{a} = \{a_\alpha : \alpha \in \mathbb{Z}^s\}$, where $s = 1$ in the curve case, and $s = 2$ in the case of surfaces defined by control nets with the topology of a regular grid. Other topologies require a different approach.

The scheme is given by

$$p_\alpha^{k+1} = \sum_{\beta\in\mathbb{Z}^s} a_{\alpha-2\beta} p_\beta^k \ , \qquad \alpha \in \mathbb{Z}^s \ . \tag{2.1}$$

Here we use the multi-index notation $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}^s$ and $x = (x_1, \ldots, x_s) \in \mathbb{R}^s$. We assume for the sake of simplicity that the control points are defined for all $\mathbb{Z}^s$. In practice the set of initial control points $\{p_\alpha^0 : \alpha \in J_0\}$ is finite, and the set of control points at level $k+1$, denoted by $\{p_\alpha^{k+1} : \alpha \in J_{k+1}\}$, is the maximal set for which the rule

$$p_\alpha^{k+1} = \sum_{\beta\in J_k} a_{\alpha-2\beta} p_\beta^k \ ,$$

coincides with (2.1). Thus

$$J_{k+1} = 2J_k + \mathrm{supp}(\mathbf{a}) = \big\{\alpha \in \mathbb{Z}^s : \alpha = 2\beta + \gamma \ , \ \beta \in J_k \ , \ \gamma \in \mathrm{supp}(\mathbf{a})\big\} \ .$$

Where $s = 1$ there are two rules for defining new control points

$$p_{2\alpha}^{k+1} = \sum_{\beta\in\mathbb{Z}} a_{2\alpha-2\beta} p_\beta^k = \sum_{\gamma\in\mathbb{Z}} a_{2\gamma} p_{\alpha-\gamma}^k \ , \qquad \alpha \in \mathbb{Z} \ , \tag{2.2}$$

and

$$p_{2\alpha+1}^{k+1} = \sum_{\beta\in\mathbb{Z}} a_{2\alpha+1-2\beta} p_\beta^k = \sum_{\gamma\in\mathbb{Z}} a_{2\gamma+1} p_{\alpha-\gamma}^k \ , \qquad \alpha \in \mathbb{Z} \ . \tag{2.3}$$

In the surface case $s = 2$, there are 4 rules depending on the parity of each component of the vector $\alpha \in \mathbb{Z}^2$. Thus, defining $E_2 = \big\{\gamma : \gamma_i \in \{0,1\} \ , \ i = 1,2\big\}$, namely E_2 consists of the extreme points of $[0,1]^2$, we get for each $\gamma \in E_2$ a different rule:

$$p_{\gamma+2\alpha}^{k+1} = \sum_{\beta\in\mathbb{Z}^2} a_{\gamma+2\alpha-2\beta} p_\beta^k = \sum_{\beta\in\mathbb{Z}^2} a_{\gamma+2\beta} p_{\alpha-\beta}^k \ , \qquad \gamma \in E_2 \ , \ \alpha \in \mathbb{Z}^s \ . \tag{2.4}$$

For general s the number of rules is 2^s, as the cardinality of the set E_s of the extreme points of $[0,1]^s$. As in the case of the B-spline functions, the control point p_α^k is related to the parameter value $2^{-k}\alpha$. The analysis of the s.s. determines the smoothness of each component of the generated curve/surface. Since each component is a scalar function generated by the same s.s., it is sufficient to analyse control points in $\mathbb{R}^1$.

Definition. A subdivision scheme S is a *convergent subdivision scheme* if for every set of control point $f^0 = \{f^0_\alpha \in \mathbb{R} \mid \alpha \in \mathbb{Z}^s\}$, there exists a continuous function $f \in C(\mathbb{R}^s)$ such that

$$\lim_{\ell<k\to\infty} \left|(S^k f^0)_{2^{k-\ell}\alpha} - f(2^{-\ell}\alpha)\right| = 0 , \qquad \alpha \in \mathbb{Z}^s \ , \ \ell \in \mathbb{Z}_+ \ , \tag{2.5}$$

and such that for some initial data the above function $f \not\equiv 0$. The function f is denoted by $S^\infty f^0$. S is a *uniformly convergent s.s.*, if for any bounded domain $\Omega \subset \mathbb{R}^s$ and $\varepsilon > 0$ there exists $K(\varepsilon, \Omega)$ such that

$$\left|(S^k f^0)_\alpha - f(2^{-k}\alpha)\right| < \varepsilon \ , \quad , k > K(\varepsilon,\Omega) \ , \qquad \alpha \in \mathbb{Z}^s \cap 2^k\Omega \ . \tag{2.6}$$

This is equivalent to the requirement that for all $f^0 \in \ell^\infty(\mathbb{Z}^s)$,

$$\lim_{k\to\infty} \left\|S^k f^0 - f\left(\frac{\cdot}{2^k}\right)\right\|_\infty = 0 \ ,$$

where $f\left(\frac{\cdot}{2^k}\right)$ denotes the sequence $\left\{f\left(\frac{\alpha}{2^k}\right) : \alpha \in \mathbb{Z}^s\right\}$.

A simple necessary condition for S to be uniformly convergent is the following:

Proposition 2.1. *Suppose S is a uniformly converent s.s., then*

$$\sum_{\alpha\in\mathbb{Z}^s} a_{\gamma-2\alpha} = 1 \ , \qquad \gamma \in E_s \ . \tag{2.7}$$

Proof: Let f^0 be such that $S^\infty f^0 \not\equiv 0$. By the continuity of f there exist $\ell \in \mathbb{Z}_+$ and $\alpha \in \mathbb{Z}^s$, such that $f(2^{-\ell}\alpha) \neq 0$.

Now for $k > \ell$,

$$f^k_{2^{k-\ell}\alpha+\gamma} = \sum_{\beta\in\mathbb{Z}^s} a_{2^{k-\ell}\alpha+\gamma-2\beta} f^{k-1}_\beta = \sum_{\beta\in\mathbb{Z}^s} a_{\gamma+2\beta} f^{k-1}_{2^{k-1-\ell}\alpha-\beta} \ . \tag{2.8}$$

Let Ω be a neighborhood of $2^{-\ell}\alpha$. For k large enough $2^{-\ell}\alpha + 2^{-k}\gamma \in \Omega$ for $\gamma \in E_s$ and $2^{-\ell}\alpha - 2^{-k+1}\beta \in \Omega$ for $\beta \in \Lambda$ where $\Lambda = \{\beta : a_{\gamma+2\beta} \neq 0, \gamma \in E_s\}$.

Hence by the uniform convergence of S, given $\varepsilon > 0$ there exists $K = K(\varepsilon, \Omega)$, such that

$$\begin{aligned} f^k_{2^{k-\ell}\alpha+\gamma} &= f(2^{-\ell}\alpha + 2^{-k}\gamma) + \eta_k \ , \qquad \gamma \in E_s \ , \\ f^{k-1}_{2^{k-1-\ell}\alpha-\beta} &= f(2^{-\ell}\alpha - 2^{-k+1}\beta) + \theta_{k,\beta} \ , \qquad \beta \in \Lambda \ , \end{aligned} \tag{2.9}$$

with $|\eta_k| \le \varepsilon$, $|\theta_{k,\beta}| \le \varepsilon$, $k > K(\varepsilon, \Omega)$. Substitution of (2.9) into (2.8) yields

$$f(2^{-\ell}\alpha + 2^{-k}\gamma) + \eta_k = \sum_{\beta \in \mathbb{Z}^s} a_{\gamma+2\beta} f(2^{-\ell}\alpha - 2^{-k+1}\beta) + \sum_{\beta \in \mathbb{Z}^s} a_{\gamma+2\beta}\theta_{k,\beta} \ .$$

Taking $k \to \infty$ we get

$$\Big| f(2^{-\ell}\alpha) - \sum_{\beta \in \mathbb{Z}^s} a_{\gamma+2\beta} f(2^{-\ell}\alpha)\Big| \le \Big(1 + \sum_{\beta \in \mathbb{Z}^s} |a_{\gamma+2\beta}|\Big)\varepsilon \ .$$

Since ε can be chosen arbitrarily small, and since $f(2^{-\ell}\alpha) \ne 0$, we conclude that

$$\sum_{\beta \in \mathbb{Z}^s} a_{\gamma+2\beta} = 1 \ , \qquad \gamma \in E_s \ . \qquad \square$$

The next lemma gives necessary and sufficient conditions for the uniform convergence of a s.s.

Lemma 2.2. *Let $\psi \in C(\mathbb{R}^s)$ be of compact support and satisfy*

$$\sum_{\alpha \in \mathbb{Z}^s} \psi(x - \alpha) = 1 \ , \qquad x \in \mathbb{R}^s \ , \tag{2.10}$$

and let S be a s.s. with a mask satisfying (2.7). If S is a uniformly convergent s.s., then

$$\lim_{k \to \infty} \sum_{\alpha \in \mathbb{Z}^s} (S^k f^0)_\alpha \psi(2^k \cdot - \alpha) = S^\infty f^0 \ .$$

Moreover, if ψ satisfies the stability condition

$$c_1 \|f\|_\infty \le \Big\| \sum_{\alpha \in \mathbb{Z}^s} f_\alpha \psi(x - \alpha) \Big\|_\infty \ , \qquad f \in \ell^\infty(\mathbb{Z}^s) \ ,$$

then uniform convergence of the sequence

$$\Big\{ \sum_{\alpha \in \mathbb{Z}^s} (S^k f^0)_\alpha \psi(2^k x - \alpha) : k \in \mathbb{Z}_+ \Big\} \ ,$$

for any initial data f^0, implies the uniform convergence of S.

Proof: To prove the first claim, observe that

$$\begin{aligned}
e_k(x) &= \Big| \sum_{\alpha\in\mathbb{Z}^s} (S^k f^0)_\alpha \psi(2^k x-\alpha) - S^\infty f^0(x) \Big| = \\
&= \Big| \sum_{\alpha\in\mathbb{Z}^s} \big[(S^k f^0)_\alpha - (S^\infty f^0)(x)\big]\psi(2^k x-\alpha)\Big| \\
&\le \sum_{\alpha\in\Gamma_{2^k x}} \big|(S^k f^0)_\alpha - (S^\infty f^0)(2^{-k}\alpha)\,\big|\,\big|\psi(2^k x-\alpha)\big| \\
&\quad + \sum_{\alpha\in\Gamma_{2^k x}} \big|S^\infty f^0(2^{-k}\alpha) - S^\infty f^0(x)\big|\big|\psi(2^k x-\alpha)\big| \ ,
\end{aligned}$$

where $\Gamma_{2^k x} = \big\{\alpha : 2^k x - \alpha \in \mathrm{supp}(\psi)\big\} \cap \mathbb{Z}^s$. The cardinality of $\Gamma_{2^k x}$, $|\Gamma_{2^k x}|$, satisfies $|\Gamma_{2^k x}| \le \sup_{x\in[0,1]^s} \big|\big\{[x+\mathrm{supp}(\psi)] \cap Z^s\big\}\big| = M$, and for $\alpha \in \Gamma_{2^k x}$, $x - 2^{-k}\alpha = y \in 2^{-k}\,\mathrm{supp}(\psi)$.

By the uniform convergence of S in

$$\Omega_{x,\rho} = \big\{y : \|x-y\|_\infty < \rho\big\} \ ,$$

and by the continuity of $S^\infty f^0$, for any $\varepsilon > 0$ there exists $K(\varepsilon, \Omega_{x,\rho})$ such that

$$e_k(x) \le 2\varepsilon M \|\psi\|_\infty \ , \qquad k > K(\varepsilon, \Omega_{x,\rho}) \ .$$

To prove the converse direction, denote

$$f(x) = \lim_{k\to\infty} \sum_{\alpha\in\mathbb{Z}^s} (S^k f^0)_\alpha \psi(2^k x-\alpha) \ .$$

Then for $k > K(\varepsilon, \Omega)$ and $x \in \Omega$

$$\begin{aligned}
\varepsilon &> \Big\| f(x) - \sum_{\alpha\in\mathbb{Z}^s} (S^k f^0)_\alpha \psi(2^k x-\alpha)\Big\|_\infty = \Big\| \sum_{\alpha\in\Gamma_{2^k x}} \big(f(x) - (S^k f^0)_\alpha\big)\psi(2^k x-\alpha)\Big\|_\infty \\
&\ge \Big\| \sum_{\alpha\in\Gamma_{2^k x}} \big(f(2^{-k}\alpha) - (S^k f^0)_\alpha\big)\psi(2^k x-\alpha)\Big\|_\infty - \Big\| \sum_{\alpha\in\Gamma_{2^k x}} \big(f(x) - f(2^{-k}\alpha)\big)\psi(2^k x-\alpha)\Big\|_\infty \ .
\end{aligned}$$

Since by continuity of $f(x)$, $\big|f(x) - f(2^{-k}\alpha)\big| < \varepsilon/\|\psi\|_\infty$ for $\alpha \in \Gamma_{2^k x}$, $k > \widetilde{K}(\varepsilon, \Omega)$, we obtain

$$\Big\| \sum_{\alpha\in\Gamma_{2^k x}} \big(f(2^{-k}\alpha) - (S^k f^0)_\alpha\big)\psi(2^k x-\alpha)\|_\infty \le (M+1)\varepsilon \ , \qquad x \in \Omega \ ,$$

and from the stability of ψ we conclude that $\left|f(2^{-k}\alpha) - (S^k f^0)_\alpha\right| < c_2\varepsilon$ for $k > \max\left\{K(\varepsilon,\Omega), \widetilde{K}(\varepsilon,\Omega)\right\}$ and $\alpha \in \bigcup_{x\in\Omega} \Gamma_{2^k x}$, proving the uniform convergence of S for any initial data f^0. Condition (2.7) guarantees that $S^\infty f^0 \equiv 1$ for $\{f^0_\alpha = 1 \ : \ \alpha \in \mathbb{Z}^s\}$. Hence S is a uniformly convergent s.s. □

In most applications we use the above sufficient condition with the symmetric hat-function $\psi(x) = \prod_{i=1}^s B_1(1+x_i)$.

Lemma 2.3. *Let S be a s.s with a mask* $\mathbf{a}$, *and define the linear operator*

$$T\psi = \sum_{\alpha\in\mathbb{Z}^s} a_\alpha \psi(2\cdot - \alpha) \ . \tag{2.11}$$

Then

$$\sum_{\alpha\in\mathbb{Z}^s} (S^k f^0)_\alpha \psi(2^k x - \alpha) = \sum_{\alpha\in\mathbb{Z}^s} f^0_\alpha (T^k\psi)(x-\alpha) \ . \tag{2.12}$$

Proof: It is sufficient to prove (2.12) for the case $k = 1$, and apply it repeatedly.

Writing $(Sf^0)_\alpha$ in terms of the mask $\mathbf{a}$ and changing the order of summation in the lefthand sum we obtain

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^s} (Sf^0)_\alpha \psi(2x-\alpha) &= \sum_{\beta\in\mathbb{Z}^s} f^0_\beta \sum_{\alpha\in\mathbb{Z}^s} a_{\alpha-2\beta}\psi(2x-\alpha) = \\ &= \sum_{\beta\in\mathbb{Z}^s} f^0_\beta \sum_{\gamma\in\mathbb{Z}^s} a_\gamma \psi(2x - 2\beta - \gamma) = \sum_{\beta\in\mathbb{Z}^s} f^0_\beta (T\psi)(x-\beta) \ . \end{aligned}$$ □

We can now prove a theorem relating the s.s. with a unique function satisfying a corresponding functional equation.

Theorem 2.4. *Let S be a uniformly convergent s.s. Then its mask* $\mathbf{a} = \{a_\alpha : \alpha \in \mathbb{Z}^s\}$ *determines a unique compactly supported continuous function φ with the following properties*

$$\varphi(x) = T\varphi(x) = \sum_{\alpha\in\mathbb{Z}^s} a_\alpha \varphi(2x-\alpha) \ , \qquad x \in \mathbb{R}^s \ , \tag{2.13}$$

$$\sum_{\alpha\in\mathbb{Z}^s} \varphi(x-\alpha) = 1 \ , \qquad x \in \mathbb{R}^s \ . \tag{2.14}$$

Moreover, for any f^0

$$S^\infty f^0 \equiv \sum_{\alpha\in\mathbb{Z}^s} f^0_\alpha \varphi(\cdot - \alpha) \ . \tag{2.15}$$

Proof: Choose $f^0 = \delta$, namely, $f^0_\alpha = \delta_{\alpha,0}$, $\alpha \in \mathbb{Z}^s$, and denote $S^\infty f^0 = \varphi$. By assumption on S, $\varphi \in C(\mathbb{R}^s)$. By the relation $(S\delta)_\alpha = \sum\limits_\beta a_{\alpha-2\beta}\delta_{\beta,0}$ we conclude that $(S\delta)_\alpha = a_\alpha$. Thus $\mathrm{supp}(S\delta) = \mathrm{supp}(\mathbf{a})$.

Now

$$(S^k\delta)_\alpha = \sum_\beta a_{\alpha-2\beta}(S^{k-1}\delta)_\beta \ ,$$

hence

$$\begin{aligned}\mathrm{supp}(S^k\delta) &= \{\alpha : \beta \in \mathrm{supp}(S^{k-1}\delta)\ ,\ \alpha - 2\beta \in \mathrm{supp}(\mathbf{a})\} \\ &= \{\alpha : \alpha \in \mathrm{supp}(\mathbf{a}) \oplus 2\,\mathrm{supp}(S^{k-1}\delta)\} \ .\end{aligned}$$

Since $\mathrm{supp}(S\delta) = \mathrm{supp}(\mathbf{a})$, we conclude that if $\mathrm{supp}(\mathbf{a})$ is convex then $\mathrm{supp}(S^k\delta) = (2^k-1)\,\mathrm{supp}(\mathbf{a})$. Otherwise $\mathrm{supp}(S^k\delta) \subset (2^k-1)\,\langle\mathrm{supp}(\mathbf{a})\rangle$. The values $S^k\delta$ are attached to the parameter values $2^{-k}\,\mathrm{supp}(S^k\delta) \subset (1-2^{-k})\,\langle\mathrm{supp}(\mathbf{a})\rangle$. Hence the limit function φ satifies $\mathrm{supp}(\varphi) \subset \langle\mathrm{supp}(\mathbf{a})\rangle^0$.

Since φ is of compact support and S is a linear operator

$$S^\infty f^0 = \sum_{\alpha\in\mathbb{Z}^s} f^0_\alpha\varphi(\cdot - \alpha) \ .$$

Similarly, we get $S^\infty f^0 = \sum_{\alpha\in\mathbb{Z}^s} f^1_\alpha \varphi(2\cdot - \alpha) = \sum_{\alpha\in\mathbb{Z}^s} f^k_\alpha\varphi(2^k\cdot - \alpha)$. Specializing to $f^0 = \delta$ and recalling that $(S\delta)_\alpha = a_\alpha$, $\alpha \in \mathbb{Z}^s$, we obtain

$$(S^\infty\delta)(x) = \varphi(x) = \sum_{\alpha\in\mathbb{Z}^s} a_\alpha\varphi(2x-\alpha) \ .$$

Thus φ is a solution of the functional equation. Now for $f^0_\alpha = 1$, $\alpha \in \mathbb{Z}^s$, $S^\infty f^0 \equiv 1$ by (2.7), and (2.14) follows from (2.15).

To conclude the proof it remains to show that φ is the unique continuous, compactly supported solution of (2.13) satisfying (2.14). Suppose ψ is a continuous compactly supported function satisfying (2.13) and (2.14). Then by (2.13) and (2.12), for all $k \geq 1$,

$$\sum_{\alpha\in\mathbb{Z}^s} f^0_\alpha\psi(x-\alpha) = \sum_{\alpha\in\mathbb{Z}^s}(S^kf)_\alpha\psi(2^kx-\alpha) \ .$$

This together with the first part of Lemma 2.2 yields

$$\sum_{\alpha\in\mathbb{Z}^s} f^0_\alpha \psi(x-\alpha) = S^\infty f^0(x) \ .$$

In particular for $f^0 = \delta$ we get

$$\psi(x) = S^\infty \delta = \varphi(x) \ ,$$

and the unicity of φ is established. $\square$

Corollary 2.5. *Let $\varphi_0 \in C(\mathbb{R}^s)$ be of compact support and satisfy $\sum_{\alpha\in\mathbb{Z}^s} \varphi_0(\cdot - \alpha) = 1$, and let* **a** *be a mask of a converging s.s. Then*

$$\lim_{m\to\infty} T^m \varphi_0 = \varphi \ ,$$

where φ is the unique solution of $\varphi = T\varphi$ guaranteed by Theorem 2.4.

Proof: By Lemma 2.2 applied to $f^0 = \delta$, and by Lemma 2.3,

$$\begin{aligned}\varphi(x) &= \lim_{k\to\infty} \sum_{\alpha\in\mathbb{Z}^s} (S^k f^0)_\alpha \varphi_0(2^k x - \alpha) = \lim_{k\to\infty} \sum_{\alpha\in\mathbb{Z}^s} \delta_{0,\alpha} (T^k \varphi_0)(x-\alpha) \\ &= \lim_{k\to\infty} T^k \varphi_0(x) \ . \end{aligned}$$

$\square$

The function φ defined by S is called the S-refinable function. It can be computed from the mask in several ways.

Methods to compute φ.

(i) Apply repeatedly S starting with $f^0 = \delta$.

(ii) Choose φ_0 a continuous compactly supported function satisfying $\sum_{\alpha\in\mathbb{Z}^s} \varphi_0(\cdot - \alpha) = 1$, and apply T repeatedly (the cascade algorithm).

This method is equivalent to (i) since by (2.12)

$$T^k \varphi_0(x) = \sum_\alpha (S^k\delta)_\alpha \varphi_0(2^k x - \alpha) \ .$$

(iii) Compute the values of φ at the diadic points recursively by the relation $\varphi = T\varphi$, namely,

$$\varphi(2^{-m}\alpha) = \sum_{\beta\in\mathbb{Z}^s} a_\beta \varphi(2^{-m+1}\alpha - \beta) \ .$$

In order to start the process, the values $\{(S^\infty \delta)(\alpha) : \alpha \in \mathbb{Z}^s\}$ are needed. Corollary 4.3 gives a simple formula for the computation of $\{S^\infty f^0(\alpha) : \alpha \in \mathbb{Z}^s\}$ from the initial data f^0.

Note that if $\varphi(\alpha) = 0$, $\alpha \in \mathbb{Z}^s$ then $\varphi(2^{-k}\alpha) = 0$ for all $k \in \mathbb{Z}_+$ and by continuity $\varphi \equiv 0$. Thus $\varphi(\alpha) \neq 0$ for at least one $\alpha \in \mathbb{Z}^s$.

(iv) Starting with arbitrary y^0, $\sum_\alpha y^0_\alpha = 1$, compute

$$y^k_\alpha = \sum_{\beta \in \mathbb{Z}^s} a_\beta y^{k-1}_{\alpha - 2^{k-1}\beta} , \qquad \alpha \in \mathbb{Z}^s ,$$

and define $\varphi_k(x) = \sum_{\alpha \in \mathbb{Z}^s} y^k_\alpha \varphi_0(2^k x - \alpha)$ for any φ_0 satisfying the requirements of Corollary 2.5. Then $\lim_{k \to \infty} \varphi_k(x) = \varphi(x)$.

Method (iv) works since

$$\varphi_k(x) = T\varphi_{k-1}(x) = \cdots = T^k \varphi_0(x) ,$$

and hence by Corollary 2.5 $\varphi(x) = \lim_{k \to \infty} \varphi_k(x)$.

Next, we relate the existence of $\varphi \in C(\mathbb{R}^s)$ solving the functional equation with the convergence of the corresponding s.s.

Theorem 2.6. *Let $\varphi \in C(\mathbb{R}^s)$ be of compact support and satisfy the functional equation (2.13) corresponding to a mask* **a** *of compact support, with property (2.7). If the integer translates of φ satisfy the stability condition*

$$c_1 \|f\|_\infty \leq \Big\| \sum_{\alpha \in \mathbb{Z}^s} f_\alpha \varphi(\cdot - \alpha) \Big\|_\infty, \; f \in \ell^\infty(\mathbb{Z}^s) , \tag{2.16}$$

then the s.s. associated with the above mask is uniformly convergent.

Proof: Consider the one-periodic function

$$\phi(x) = \sum_{\alpha \in \mathbb{Z}^s} \varphi(x - \alpha) . \tag{2.17}$$

We claim that $\phi(x)$ is constant. Let $g^0_\alpha = 1$, $\alpha \in \mathbb{Z}^s$. Then by (2.7) $S^k g^0 = g^0$, while by (2.12) and (2.13)

$$\phi(2^k x) = \sum_{\alpha \in \mathbb{Z}^s} \varphi(2^k x - \alpha) = \sum_{\alpha \in \mathbb{Z}^s} \left(S^k g^0\right)_\alpha \varphi(2^k x - \alpha) = \sum_{\alpha \in \mathbb{Z}^s} g^0_\alpha (T^k \varphi)(x - \alpha) = \phi(x) .$$

Hence $\phi(x)$ is constant on diadic points. By continuity $\phi(x)$ is constant. The stability condition guarantees that $\phi(x) \not\equiv 0$ and hence we get after normalization

$$\sum_{\alpha\in\mathbb{Z}^s} \varphi(x-\alpha) = 1 \; . \tag{2.18}$$

Applying Lemma 2.3 with $\psi = \varphi$, we conclude that

$$\sum_{\alpha\in\mathbb{Z}^s} (S^k f^0)_\alpha \varphi(2^k x - \alpha) = \sum_{\alpha\in\mathbb{Z}^s} f^0_\alpha T^k \varphi(x-\alpha) = \sum_{\alpha\in\mathbb{Z}^s} f^0_\alpha \varphi(x-\alpha) \; .$$

Hence

$$\lim_{k\to\infty} \sum_{\alpha\in\mathbb{Z}^s} (S^k f^0)_\alpha \varphi(2^k x - \alpha) = \sum_{\alpha\in\mathbb{Z}^s} f^0_\alpha \varphi(x-\alpha) \; ,$$

and it follows from the second part of Lemma 2.2 with $\psi = \varphi$ that S is a uniformly convergent s.s. □

Remark. The stability condition (2.16) for φ of compact support implies the $\ell^\infty(\mathbb{Z}^s)$-linear independence of $\{\varphi(x-\alpha) : \alpha \in \mathbb{Z}^s\}$, namely $\sum_{\alpha\in\mathbb{Z}^s} b_\alpha \varphi(\cdot - \alpha) = 0$, $b \in \ell^\infty(\mathbb{Z}^s) \Rightarrow b \equiv 0$. In the reverse direction we have the implication that the stability condition (2.16) follows from the local linear independence of $\{\varphi(\cdot - \alpha) : \alpha \in \mathbb{Z}^s\}$, namely $\sum_{\alpha\in\mathbb{Z}^s} b_\alpha \varphi(x - \alpha) = 0$, $x \in \Omega \Rightarrow b_\alpha = 0$, $\alpha \in \Gamma_\Omega$, where Ω is any bounded open domain in $\mathbb{R}^s$ and $\Gamma_\Omega = \{\alpha \in \mathbb{Z}^s : supp(\varphi(\cdot - \alpha)) \cap \Omega \neq \emptyset\}$. To see this , define

$$c = \inf \big\{ \big\| \sum_{\alpha\in\mathbb{Z}^s} b_\alpha \varphi(x-\alpha) \big\|_\infty : \|b\|_\infty = 1 \big\} \; .$$

If $c \neq 0$ then the stability condition holds. Suppose $c = 0$, then there exists a sequence $\{b^k : k \in \mathbb{Z}_+\} \in \ell^\infty(\mathbb{Z}^s)$, $\|b^k\|_\infty = 1$, such that

$$\lim_{k\to\infty} \big\| \sum_{\alpha\in\mathbb{Z}^s} b^k_\alpha \varphi(x-\alpha) \big\|_\infty = 0 \; .$$

For each k there exists α_k such that $|b^k_{\alpha_k}| > \frac{1}{2}$. Consider

$$\phi_k(x) = \sum_{\alpha\in\mathbb{Z}^s} b^k_{\alpha+\alpha_k} \varphi(x-\alpha) = \sum_{\alpha\in\mathbb{Z}^s} d^k_\alpha \varphi(x-\alpha) \; ,$$

where $d^k_\alpha = b^k_{\alpha+\alpha_k}$ and $|d^k_0| > \frac{1}{2}$ for all k. Since $\phi_k = \sum\limits_{\alpha\in\mathbb{Z}^s} b^k_\alpha \varphi(\cdot + \alpha_k - \alpha)$, $\lim\limits_{k\to\infty} \|\phi_k\|_\infty = 0$. Now let Ω_0 be a bounded open domain in $\mathbb{R}^s$, such that $supp(\varphi) \subset \Omega_0$. Since $\|d^k\|_\infty = 1$ for all k, there exists a subsequence $\{d^{k'}\}$ satisfying $\lim\limits_{k'\to\infty} d^{k'}_\alpha = d^\infty_\alpha$, $\alpha \in \Gamma_{\Omega_0}$. Thus by the compact support of φ

$$\sum_{\alpha\in\Gamma_{\Omega_0}} d^\infty_\alpha \varphi(x-\alpha) = 0 \ , \qquad x \in \Omega_0 \ , \ |d^\infty_0| \geq \tfrac{1}{2} \ ,$$

in contradiction to the assumption of local linear independence. In fact it is sufficient to require only "weak-local linear independence", namely for each $\alpha \in \mathbb{Z}^s$, there exists an open bounded domain $\Omega_\alpha \subset \mathbb{R}^s$, such that $\sum_{\alpha\in\mathbb{Z}^s} b_\alpha \varphi(x-\alpha) = 0$, $x \in \Omega_\alpha \ \Rightarrow b_\alpha = 0$.

Theorem 2.6 suggests an indirect way for the analysis of the convergence of a given s.s., namely analysis of the functional equation and the nature of its solutions. Here we present a direct analysis of the s.s., and of the convergence of the control points generated by it. We start with the curve case $s = 1$, where the analysis of convergence and of the smoothness of the limit functions generated by s.s. is simpler. Then extensions to the case $s > 1$ will be given.

Bibliographical notes.

Most of the results in this section are taken from [CDM], with considerable changes in the proofs. Methods (ii) and (iii) for the computation of φ are suggested in [DL1]. Analysis of the solutions of the functional equations is done in several papers, see e.g. [CDM],[D2],[DD],[DL1]. It is also done in the contex of orthonormal wavelets [D1].

3. Analysis of Convergence and Smoothness – the case $s = 1$

It has been observed in Proposition 2.1, that a necessary condition for the uniform convergence of a s.s. given by a compactly supported mask $\mathbf{a} = \{a_\alpha : \ \alpha \in \mathbb{Z}\}$ is

$$\sum_{\alpha\in\mathbb{Z}} a_{2\alpha} = 1 \ , \qquad \sum_{\alpha\in\mathbb{Z}} a_{2\alpha+1} = 1 \ . \tag{3.1}$$

This condition guarantees the existence of a related s.s. for the divided differences of the original control points.

Proposition 3.1. *Let S be a s.s. defined by a mask satisfying (3.1). Then there exists a s.s. S_1 with the property*

$$df^k = S_1 df^{k-1} \ , \tag{3.2}$$

where $f^k = S^k f^0$, and $(df^k)_\alpha = 2^k \left(f^k_{\alpha+1} - f^k_\alpha\right)$.

Proof: Let $\mathcal{L}$ denote the set of all Laurent polynomials and define the characteristic $\mathcal{L}$-polynomial of S by $a(z) = \sum_\alpha a_\alpha z^\alpha \in \mathcal{L}$. Then by (3.1) $a(-1) = 0$, and therefore $a^{(1)}(z) = \frac{2za(z)}{z+1} \in \mathcal{L}$. We now show that the mask determined by $a^{(1)}(z)$ defines a s.s. S_1 with the required properties. Let $F_k(z) = \sum_{\alpha\in\mathbb{Z}} f^k_\alpha z^\alpha$ be a formal generating function associated with the control points f^k. The relation

$$f^{k+1}_\alpha = \sum_{\beta\in\mathbb{Z}^s} a_{\alpha-2\beta} f^k_\beta \ , \tag{3.3}$$

is written formally in terms of the generating functions by

$$F_{k+1}(z) = a(z) F_k(z^2) \ . \tag{3.4}$$

Indeed comparing the coefficients of the same power of z on both sides we get

$$f^{k+1}_{2\alpha} = \sum_{\beta\in\mathbb{Z}^s} a_{2\beta} f^k_{\alpha-\beta} \ , \quad f^{k+1}_{2\alpha+1} = \sum_{\beta\in\mathbb{Z}^s} a_{2\beta+1} f^k_{\alpha-\beta} \ ,$$

which is equivalent to (3.3). Now observe that

$$H_k(z) = \sum_{\alpha\in\mathbb{Z}^s} (df^k)_\alpha z^\alpha = 2^k \sum_{\alpha\in\mathbb{Z}^s} \left(f^k_{\alpha+1} - f^k_\alpha\right) z^\alpha = 2^k \left(z^{-1}F_k(z) - F_k(z)\right) \ .$$

Hence

$$H_k(z) = 2^k F_k(z) \frac{1-z}{z} \ , \tag{3.5}$$

and by application of (3.4) one gets

$$H_{k+1}(z) = 2^{k+1} F_{k+1}(z) \frac{1-z}{z} = 2^{k+1} a(z) F_k(z^2) \frac{1-z}{z} \ .$$

Thus, by (3.5)

$$H_{k+1}(z) = 2a(z)\frac{z}{1+z}H_k(z^2) = a^{(1)}(z)H_k(z^2) \ ,$$

a relation similar in form to (3.4). Recalling the definition of $H_k(z)$, we conclude the existence of a s.s. S_1 satisfying (3.2) with a mask determined by the characteristic $\mathcal{L}$-polynomial

$$a^{(1)}(z) = 2z(1+z)^{-1}a(z) \ . \tag{3.6}$$

□

Remark. Property (3.2) of the s.s. S_1 corresponding to the characteristic $\mathcal{L}$-polynomial $a^{(1)}(z) = 2a(z)z(1+z)^{-1}$, can be written as $\Delta S = \frac{1}{2}S_1\Delta$ where Δ is the operator defined by $(\Delta f)_\alpha = f_{\alpha+1} - f_\alpha$.

We can now determine the convergence of S by analyzing the s.s. $\frac{1}{2}S_1$.

Theorem 3.2. *S is a uniformly convergent s.s., if and only if $\frac{1}{2}S_1$ converges uniformly to the zero function for all initial data f^0.*

Proof: Suppose S converges uniformly, and let $f^k = S^k f^0$. Then for $\alpha_0 \in \mathbb{Z}$,

$$\begin{aligned}\left|(\Delta f^k)_{\alpha_0}\right| =& \mid f^k_{\alpha_0+1} - f^k_{\alpha_0}\mid \leq \mid f^k_{\alpha_0+1} - S^\infty f^0\left(2^{-k}\alpha_0 + \frac{1}{2^k}\right)\mid \\ &+ \left|f^k_{\alpha_0} - S^\infty f^0(2^{-k}\alpha_0)\mid + \right|S^\infty f^0\left(2^{-k}\alpha_0 + \frac{1}{2^k}\right) - S^\infty f^0(2^{-k}\alpha_0)\right| \ .\end{aligned}$$

Let I_1 be an open interval strictly containing the open interval I. For any point $2^{-k}\alpha_0 \in I$ and $k > K_0$, $2^{-k}\alpha_0 + \frac{1}{2^k} \in I_1$. Hence by the uniform convergence of S and by the continuity of $S^\infty f^0$, there exists for any $\varepsilon > 0$, $K = K(\varepsilon, I) > K_0$, such that for any $2^{-k}\alpha_0 \in I$,

$$\left|(\Delta f^k)_{\alpha_0}\right| \leq 3\varepsilon \ , \qquad k > K(\varepsilon, I) \ ,$$

proving the uniform convergence to zero of $\frac{1}{2}S_1$.

The proof of the converse direction is more involved. First observe that if $\frac{1}{2}S_1$ converges uniformly to zero then $\lim\limits_{k\to\infty} \| \left(\frac{1}{2}S_1\right)^k \|_\infty = 0$. Indeed, for $f^0 \in \ell^\infty(\mathbb{Z})$, $\|f^0\|_\infty = 1$, and $k > K(\varepsilon)$

$$\begin{aligned}\| \left(\tfrac{1}{2}S_1\right)^k f^0\|_\infty &= \| \sum_{\alpha\in\mathbb{Z}} f^0_\alpha \left(\left(\tfrac{1}{2}S_1\right)^k \delta\right)_{\cdot - 2^k\alpha} \|_\infty \\ &\leq \| \sum_{\alpha\in\mathbb{Z}} \left(\left(\tfrac{1}{2}S_1\right)^k \delta\right)_{\cdot - 2^k\alpha} \|_\infty \leq M\| \left(\tfrac{1}{2}S_1\right)^k \delta\|_\infty < \varepsilon \ ,\end{aligned}$$

where the last two inequalities follow from

$$\operatorname{supp}\left(\tfrac{1}{2}S_1\right)^k \delta \subset (2^k-1)\left\langle \operatorname{supp}(\mathbf{a}^{(1)})\right\rangle ,$$

and from the uniform convergence to zero of $\left(\frac{1}{2}S_1\right)^k\delta$. Thus there exists a positive integer L, and $0<\mu<1$, such that for all $f^0 \in \ell^\infty(\mathbb{Z})$,

$$\left\| \left(\tfrac{1}{2}S_1\right)^L f^0 \right\|_\infty < \mu \|f^0\|_\infty . \tag{3.7}$$

Consider now the sequence of control points $f^k = S^k f^0$, and the piecewise linear functions interpolating these control points

$$f^k(x) = \sum_{\alpha\in\mathbb{Z}^s} f^k_\alpha \psi(2^k x-\alpha) , \qquad x\in\mathbb{R} ,$$

where $\psi = B_1(\cdot+1)$ with B_1 as defined in the Introduction. We now show that $\{f^k(x) : k\in\mathbb{Z}_+\}$ is a Cauchy sequence, and hence converges uniformly to a continuous function. Then, since $\psi\in C(\mathbb{R})$ satisfies (2.10) and the stability condition

$$\left\| \sum_{\alpha\in\mathbb{Z}} f^0_\alpha \psi(x-\alpha)\right\|_\infty = \|f^0\|_\infty ,$$

the second part of Lemma 2.2 implies that S is uniformly convergent.

Consider the differences $f^{k+1}(x)-f^k(x)$, and denote by U the s.s. corresponding to the function ψ. Then by (2.12) applied to U and ψ we get

$$f^{k+1}(x)-f^k(x) = \sum_{\alpha\in\mathbb{Z}} (Sf^k - Uf^k)_\alpha \psi(2^{k+1}x-\alpha) . \tag{3.8}$$

Now $S-U$ is a s.s with a characteristic $\mathcal{L}$-polynomial $d(x) = a(z) - \left(\frac{1}{2}z^{-1}+1+\frac{1}{2}z\right)$, which by (3.1) satifies $d(-1)=d(1)=0$. Hence

$$d(z) = \left(\frac{1-z^2}{z^2}\right) e(z) , \qquad e(z) = \sum_{i\in I} e_i z^i , \ |I|<\infty .$$

One application of $S-U$ can be described in terms of generating functions as in (3.4), by

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}} \left((S-U)f^k\right)_\alpha z^\alpha &= d(z)F_k(z^2) = e(z)\left(\frac{1-z^2}{z^2}\right)F_k(z^2)\\ &= e(z)\sum_{\alpha\in\mathbb{Z}} (f^k_{\alpha+1}-f^k_\alpha) z^{2\alpha} .\end{aligned}$$

Comparing equal powers of z we obtain

$$((S-U)f^k)_\alpha = \sum_{\beta\in\mathbb{Z}} e_{\alpha-2\beta}(\Delta f^k)_\beta \ , \tag{3.9}$$

yielding the bound

$$\left\|(S-U)f^k\right\|_\infty \le E\|\Delta f^k\|_\infty \ , \qquad E = \max\left\{\sum_{2\alpha\in I}|e_{2\alpha}| \ , \sum_{2\alpha+1\in I}|e_{2\alpha+1}|\right\} .$$

Combining this with (3.7), (3.8) and $0 \le \psi(x) \le 1$, $\sum_{\alpha\in\mathbb{Z}}\psi(\cdot-\alpha) = 1$, we finally get

$$\begin{aligned} \left|f^{k+1}(x) - f^k(x)\right| &\le \left\|(S-U)f^k\right\|_\infty \le E\|\Delta f^k\|_\infty \\ &\le E \max_{0\le j<L}\|\Delta f^j\|_\infty \quad \mu^{[\frac{k}{L}]} \ , \end{aligned} \tag{3.10}$$

a relation which implies that $\left\{f^k(x) : k\in\mathbb{Z}_+\right\}$ is a Cauchy sequence. □

A repeated use of (3.10) yields an estimate for the deviation of $f^k(\cdot)$ from $S^\infty f^0$.

Corollary 3.3. *Let S be a uniformly convergent s.s., and let μ and L be defined by (3.7). Then*

$$\left\|S^\infty f^0 - f^k(\cdot)\right\|_\infty \le C\mu^{[\frac{k}{L}]} \ ,$$

with

$$C = EL(1-\mu)^{-1}\|\Delta f^0\|_\infty \max_{0\le j<L}\left\|\left(\tfrac{1}{2}S_1\right)^j\right\|_\infty .$$

Moreover

$$|S^\infty f(x) - S^\infty f(y)| \le \widetilde{C}|x-y|^\nu \ , \quad x,y\in\mathbb{R} \ , |x-y| < 1 \ ,$$

where $\nu = -\frac{1}{L}log_2\mu$ and $\widetilde{C} = 2(C + max_{0\le j<L}\|S^j\|_\infty\|\Delta f^0\|_\infty)\mu^{-(1+\frac{1}{L})}$.

Proof: Using the notations in the proof of Theorem 3.2 and the estimate (3.10), we get in view of Lemma 2.2,

$$\begin{aligned} \left|S^\infty f^0(x) - f^k(x)\right| &= \lim_{\ell\to\infty}\left|f^\ell(x) - f^k(x)\right| \le \sum_{j=k}^\infty \left|f^{j+1}(x) - f^j(x)\right| \\ &\le E\max_{0\le j<L}\|\Delta f^j\|_\infty \frac{L}{1-\mu}\mu^{[\frac{k}{L}]} \le E\max_{0\le j<L}\left\|\left(\tfrac{1}{2}S_1\right)^j\right\|_\infty\|\Delta f^0\|_\infty\frac{L}{1-\mu}\mu^{[\frac{k}{L}]} \ , \end{aligned}$$

which proves the first claim, and gives an explicit form of the constant C. Using this result and (3.7), we get for $x, y \in \mathbb{R}$, $|x-y| < 1$, with $k \in \mathbb{Z}_+$, defined by $2^{-k-1} < |x-y| \leq 2^{-k}$,

$$|f(x) - f(y)| \leq |f(x) - f^k(x)| + |f(y) - f^k(y)| + |f^k(x) - f^k(y)|$$

$$\leq 2C\mu^{[k/L]} + 2\|\Delta f^k\|_\infty \leq 2(C + max_{0\leq j<L}\|S^j\|_\infty\|\Delta f^0\|_\infty)\mu^{[k/L]}$$

$$\leq \widetilde{C}\mu^{(k+1)/L} = \widetilde{C}2^{-(k+1)\nu} \leq \widetilde{C}|x-y|^\nu \ .$$

$\square$

Theorem 3.2 indicates that for any given s.s., S, with a mask **a** satisfying (3.1), we can prove the uniform convergence of S by first deriving the mask of $\frac{1}{2}S_1$ and then computing $\left\|(\frac{1}{2}S_1)^k\right\|_\infty$ for $k = 1, 2, 3, \ldots L$, where L is the first integer for which $\left\|(\frac{1}{2}S_1)^L\right\|_\infty < 1$. If S converges uniformly, such an L exists. ¿From the practical point of view, if $L > 10$ no convergence occurs in the actual performance of the scheme, since only a small number of steps ($k < 10$) are carried out in practice.

The formalism of the generating functions allows us to compute the masks of the schemes S^k, $k > 1$, given the mask of S. ¿From (3.4) it follows that

$$F_{k+L}(z) = a(z)a(z^2)\ldots a(z^{2^{L-1}})F_k(z^{2^L}) = a^{[L]}(z)F_k(z^{2^L}) \ .$$

Comparing coefficients of equal powers of z on both sides we get 2^L different rules mapping f^k to f^{k+L}, determined by the coefficients of

$$a^{[L]}(z) = \prod_{j=0}^{L-1} a(z^{2^j}) = \sum_\alpha a^{[L]}_\alpha z^\alpha \ . \tag{3.11}$$

These rules have the form

$$f^{k+L}_{\gamma+2^L\alpha} = \sum_{\beta\in\mathbb{Z}} a^{[L]}_{\gamma+2^L\beta} f^k_{\alpha-\beta} \ , \qquad \gamma = 0, 1, \ldots, 2^L - 1 \ , \tag{3.12}$$

and hence the norm of S^L is given by

$$\|S^L\|_\infty = \max\left\{ \sum_{\beta\in\mathbb{Z}} |a^{[L]}_{\gamma+2^L\beta}| \ : \ \gamma = 0, 1, \ldots, 2^L - 1 \right\} . \tag{3.13}$$

In particular

$$\|S\|_\infty = \max\left\{ \sum_{\alpha\in\mathbb{Z}} |a_{2\alpha}| \ , \ \sum_{\alpha\in\mathbb{Z}} |a_{2\alpha+1}| \right\} . \tag{3.14}$$

Later we discuss several examples and use this approach to establish convergence.

Once the uniform convergence of S is established, we are then interested in determining the smoothness of the limit function $S^\infty f^0$, which is equivalent by Theorem 2.4 to the smoothness of $\varphi = S^\infty \delta$.

Here we state sufficient conditions for $\varphi \in C^\nu$, $\nu \geq 1$. These conditions are also necessary in the case that φ satisfies the stability condition (2.16).

Theorem 3.4. *Let S be a subdivision scheme with a characteristic $\mathcal{L}$-polynomial*

$$a(z) = \big((1+z)/2z\big)^\nu q(z) \ , \qquad q \in \mathcal{L} \ . \tag{3.15}$$

If the s.s S_ν, corresponding to the $\mathcal{L}$-polynomial $q(z)$ converges uniformly, then $S^\infty\delta = \varphi \in C^\nu$ and for any initial control points f^0

$$\frac{d^\nu}{dx^\nu} S^\infty f^0 = S_\nu^\infty \Delta^\nu f^0 \ , \tag{3.16}$$

where $\Delta^\nu f^0 = \sum_{j=0}^{\nu} \binom{\nu}{j} (-1)^{\nu-j} f^0_{\cdot+j}$. Moreover, for $j = 1, \ldots, \nu$, the s.s S_j with characteristic $\mathcal{L}$-polynomial $a(z)\big(2z/(z+1)\big)^j$, satisfies

$$S_j d^j(S^k f^0) = d^j(S^{k+1} f^0) \ , \qquad S_j^\infty d^j f^0 = \frac{d^j}{dx^j} S^\infty f^0 \ , \tag{3.17}$$

where $d^j(S^k f^0) = 2^{kj} \Delta^j (S^k f^0)$.

Proof: Firstly, we prove the theorem for $\nu = 1$. Let $f^k = S^k \delta$. By (3.2) $df^{k+1} = S_1 df^k$ and since S_1 is uniformly convergent, then by the first part of Lemma 2.2, the sequence of functions

$$g^k(x) = \sum_{\alpha\in\mathbb{Z}} (df^k)_\alpha \psi(2^k x - \alpha) \ , \quad \psi = B_1(\cdot + 1) \ , \quad k \geq 1 \ ,$$

converges uniformly to a limit function $g = S_1^\infty \Delta\delta \in C(\mathbb{R})$. Moreover, by the proof of Theorem 3.2,

$$\|\Delta df^k\|_\infty < \widetilde{A} \mu^{[\frac{k}{L}]} \ , \tag{3.18}$$

where $\mu \in (0,1)$, L is a positive integer and $\tilde{A}$ is a constant independent of k. Also, since S_1 converges uniformly, $\frac{1}{2}S_1$ converges uniformly to zero, and by Theorem 3.2, S converges uniformly. Hence $\varphi = S^\infty \delta \in C(\mathbb{R})$.

Let us denote by $h^k(x)$ the piecewise constant function

$$h^k(x) = (df^k)_\alpha, \quad 2^{-k}\alpha \le x < 2^{-k}\alpha + 2^{-k} \ , \quad \alpha \in \mathbb{Z} \ .$$

It is clear that

$$| \, g^k(x) - h^k(x) \, | \le \|\Delta df^k\|_\infty \ ,$$

and by (3.18), $h^k(x)$ converges uniformly to g. Noting that all functions considered here are of compact support, and that

$$| \int_{-\infty}^x g(t)dt - \int_{-\infty}^x h^k(t)dt \, | \le \int_{-\infty}^x | \, g(t) - h^k(t) \, | \, dt \ ,$$

we conclude that the sequence $\{\int_0^x h^k(t)dt : k \in \mathbb{Z}_+\}$ converges uniformly to the function $\int_{-\infty}^x g(t)dt$. But by definition of $h^k(x)$,

$$\int_{-\infty}^x h^k(t)dt = \sum_\alpha \left(S^k\delta\right)_\alpha \psi\left(2^k x - \alpha\right) \ ,$$

and since S is uniformly convergent, Lemma 2.2 implies that $\int_{-\infty}^x h^k(t)dt$ converges uniformly to $\varphi(x)$, and therefore $\varphi(x) = \int_{-\infty}^x g(t)dt$, $\frac{d}{dx}\varphi = g = S_1^\infty \Delta\delta \in C(\mathbb{R})$. Thus $\varphi \in C^1(\mathbb{R})$, and we conclude from (2.15) that for all initial data f^0, (3.16) holds with $\nu = 1$. This concludes the proof of the case $\nu = 1$.

If S_ν converges uniformly with $\nu > 1$, then by the claim of the Theorem for $\nu = 1$, $S_{\nu-1}$ converges uniformly for all initial data f^0, and

$$\frac{d}{dx} S_{\nu-1}^\infty f^0 = S_\nu^\infty \Delta f^0 \ .$$

Repeating this argument $\nu - 1$ times we get

$$S_\nu^\infty \Delta^\nu f^0 = \frac{d}{dx} S_{\nu-1}^\infty \Delta^{\nu-1} f^0 = \ldots = \frac{d^\nu}{dx^\nu} S^\infty f^0 \in C(\mathbb{R}) \ ,$$

which yields (3.16) and shows that $\varphi \in C^\nu$. The proof of (3.17) is similar to the proof of Proposition 3.1, observing that $d^\nu(S^k f^0) = d(d^{\nu-1} S^k f^0)$. □

Remark: For interpolatory subdivision schemes we show in Section 5 that if $\varphi \in C^\nu(\mathbb{R})$ then S_ν exists and converges uniformly. We will not show the necessity of the conditions in Theorem 3.4 in the more general case of φ satisfying the stability condition (2.16).

Examples.

1. General uniform "corner cutting"

$$\begin{aligned} f_{2\alpha}^{k+1} &= r f_\alpha^k + (1-r) f_{\alpha+1}^k \ , \\ f_{2\alpha+1}^{k+1} &= s f_\alpha^k + (1-s) f_{\alpha+1}^k \ , \end{aligned} \qquad 0 \le s < r \le 1 \ , \ \alpha \in \mathbb{Z} \ .$$

This scheme satisfies (3.1) for all r, s.

The characteristic $\mathcal{L}$-polynomial of this scheme is

$$\begin{aligned} a(z) &= (1-r)z^{-2} + (1-s)z^{-1} + r + sz \\ &= z^{-2}\left[r(z-1)(z+1) + sz(z-1)(z+1) + (1+z)\right] \\ &= z^{-2}(1+z)\left[(1-r) + (r-s)z + sz^2\right] \ . \end{aligned}$$

Hence the characteristic $\mathcal{L}$-polynomial of S_1 is

$$a^{(1)}(z) = \frac{2za(z)}{z+1} = 2(1-r)z^{-1} + 2(r-s) + 2sz \ ,$$

and the scheme $\frac{1}{2}S_1$ for Δf^k is given by

$$g_{2\alpha}^{k+1} = (r-s)g_\alpha^k \ , \ g_{2\alpha+1}^{k+1} = sg_\alpha^k + (1-r)g_{\alpha+1}^k \ , \ \alpha \in \mathbb{Z} \ .$$

By (3.14), $\|\frac{1}{2}S_1\|_\infty = \max\{r-s, \ 1-(r-s)\} < 1$, hence the corner cutting algorithm converges uniformly to a continuous limit function.

To analyse the smoothness of the limit function observe that $a^{(1)}(z) = 2(1-r)z^{-1} + 2(r-s)+2sz$, and hence this mask satisfies the necessary condition for uniform convergence (3.1) if and only if $r - s = \frac{1}{2}$. In particular $s < \frac{1}{2}$. Under this additional condition $a^{(1)}(z) = (1-2s)z^{-1} + 1 + 2sz$, and the *s.s.* $\frac{1}{2}S_2$ exists with a characteristic $\mathcal{L}$-polynomial

$$\tfrac{1}{2}a^{(2)}(z) = \frac{z}{1+z}a^{(1)}(z) = 1 - 2s + 2sz$$

and norm $\|\frac{1}{2}S_2\|_\infty = \max\{1-2s, 2s\} < 1$. By Theorem 3.2, S_1 converges uniformly, and by Theorem 3.4 $S^\infty f^0 \in C^1(\mathbb{R})$, for all initial control points f^0. The scheme S_2 satisfies (3.1) if and only if $s = \frac{1}{4}$, $r = \frac{3}{4}$, namely the case of the Chaikin's algorithm. For these values of s and r, $a^{(2)}(z) = 1+z$, $\frac{1}{2}a^{(3)}(z) = z$ and $\|\frac{1}{2}S_3\|_\infty = 1$. In fact $\|\left(\frac{1}{2}S_3\right)^k\|_\infty = 1$ for all $k \in \mathbb{Z}_+$, and S_2 does not converge uniformly. As is expected, the Chaikin's algorithm does not produce $C^2(\mathbb{R})$ functions.

2. Uniform B-spline subdivision

For B_1 we saw that $\sum_\alpha f_\alpha^k B_1\left(2^k x - \alpha\right)$ is the piecewise linear function connecting the points $\{(\alpha, f_\alpha^0) : \alpha \in \mathbb{Z}\}$, and that the control points f^k are becoming dense on it as $k \to \infty$. Here $a(z) = \frac{1}{2}(1+z)^2$, $\frac{1}{2}a^{(1)}(z) = \frac{1}{2}(1+z)$ and $\|\frac{1}{2}S_1\|_\infty = \frac{1}{2}$, verifying the convergence of the scheme.

For B_2-curves the $s.s.$ derived in the Introduction has the form

$$f_{2\alpha}^{k+1} = \frac{3}{4}f_{\alpha-1}^k + \frac{1}{4}f_\alpha^k\,, \qquad f_{2\alpha+1}^{k+1} = \frac{1}{4}f_{\alpha-1}^k + \frac{3}{4}f_\alpha^k\,.$$

This is a corner cutting scheme with a different enumeration of the points. The characteristic $\mathcal{L}$-polynomial of this scheme is

$$a(z) = \frac{1}{4}\left(1 + 3z + 3z^2 + z^3\right)\,,$$

which is just the $\mathcal{L}$-polynomial of corner cutting multiplied by z^2. Hence the convergence result holds.

It is shown in the Introduction that the coefficients of the $s.s.$ for B_m-curves are given by (1.16). Thus the characteristic $\mathcal{L}$-polynomial of the $s.s$ for B_m-curves $a_m(z) = \sum_\alpha a_{\alpha,m} z^\alpha$ has the form

$$a_m(z) = 2^{-m}(1+z)^{m+1}\,, \tag{3.19}$$

and

$$\tfrac{1}{2}a_m^{(1)}(z) = \left(\tfrac{1}{2}\right)^m (1+z)^m\,.$$

Hence by (3.14),

$$\|\tfrac{1}{2}S_1\|_\infty = \left(\tfrac{1}{2}\right)^m \max\left\{\sum_{\alpha=0}^{[\frac{m}{2}]}\binom{m}{2\alpha},\ \sum_{\alpha=0}^{[\frac{m}{2}]-1}\binom{m}{2\alpha+1}\right\} = \tfrac{1}{2}\,,$$

proving the uniform convergence of the *s.s.* for B_m-curves.

The explicit formula (3.19) for the characteristic $\mathcal{L}$-polynomial of a B_m-curve s.s., S, allows us to conclude the existence of the scheme S_m with the characteristic $\mathcal{L}$-polynomial

$$a_m^{(m)}(z) = \left(\frac{2z}{1+z}\right)^m a_m(z) = (1+z) \ .$$

Hence $\|\frac{1}{2}S_m\|_\infty = \frac{1}{2}$ and by Theorems 3.2 and 3.4, $S^\infty f^0 \in C^{m-1}(\mathbb{R})$ as is well known from the smoothness of the B-spline functions.

3. 4-point interpolatory subdivision scheme

Interpolatory subdivision schemes retain the points of stage k as a subset of the points of stage $k+1$. Thus the general form of an interpolatory *s.s.* is

$$\begin{aligned} f^k_{2\alpha} &= f^k_\alpha \ , \\ f^k_{2\alpha+1} &= \sum_{\beta\in\mathbb{Z}} a_{1+2\beta} f^k_{\alpha-\beta} \ . \end{aligned} \tag{3.20}$$

The example we consider is a one parameter family of schemes given by the non-zero odd coefficients

$$a_{-3} = a_3 = -w, \quad a_{-1} = a_1 = \tfrac{1}{2} + w \ . \tag{3.21}$$

For $w = 0$ this is the symmetric s.s. for B_1-curves. Questions of interest on this scheme are: What is the range of values of the parameter w corresponding to convergent schemes? Can such a scheme produce C^1 functions? Note that for interpolatory schemes, convergence implies uniform convergence, since the values $\{f^k_\alpha\}$ are on the limit function.

The characteristic $\mathcal{L}$-polynomial of (3.20) with mask coefficients (3.21) is

$$\begin{aligned} a(z) &= -wz^{-3} + \left(\tfrac{1}{2} + w\right) z^{-1} + 1 + \left(\tfrac{1}{2} + w\right) z - wz^3 \\ &= z^{-3}(1+z)^2 \left(\tfrac{1}{2}z^2 - w(z-1)^2\left(1+z^2\right)\right) \ , \end{aligned} \tag{3.22}$$

and

$$\tfrac{1}{2}a^{(1)}(z) = -wz^{-2} + wz^{-1} + \tfrac{1}{2} + \tfrac{1}{2}z + wz^2 - wz^3 \ . \tag{3.23}$$

Hence

$$\|\tfrac{1}{2}S_1\|_\infty = \tfrac{1}{2} + 2 \mid w \mid \ ,$$

and the range of w which guarantees uniform convergence to zero of $\frac{1}{2}S_1$ is $\mid w \mid < \frac{1}{4}$. This range is not the best possible. By considering the scheme $\left(\frac{1}{2}S_1\right)^2$ with the characteristic $\mathcal{L}$-polynomial

$$\tfrac{1}{4}a^{(1)}(z)a^{(1)}\left(z^2\right) =$$

$$\left(-wz^{-2}+wz^{-1}+\tfrac{1}{2}+\tfrac{1}{2}z+wz^2-wz^3\right)\left(-wz^{-4}+wz^{-2}+\tfrac{1}{2}+\tfrac{1}{2}z^2+wz^4-wz^6\right)$$
$$= w^2z^{-6}-w^2z^{-5}-\left(\tfrac{1}{2}w+w^2\right)z^{-4}+\left(w^2-\tfrac{1}{2}w\right)z^{-3}-w^2z^{-2}+\left(w+w^2\right)z^{-1}$$
$$+\left(\tfrac{1}{4}+w^2-\tfrac{1}{2}w\right)+\left(\tfrac{1}{2}w+\tfrac{1}{4}-w^2\right)z+\left(\tfrac{1}{2}w+\tfrac{1}{4}-w^2\right)z^2+\left(\tfrac{1}{4}+w^2-\tfrac{1}{2}w\right)z^3$$
$$+\left(w+w^2\right)z^4-w^2z^5+\left(w^2-\tfrac{1}{2}w\right)z^6-\left(\frac{w}{2}+w^2\right)z^7-w^2z^8+w^2z^9\ ,$$

we obtain in view of (3.13)

$$\|(\tfrac{1}{2}S_1)^2\|_\infty = \max\left\{\mid \tfrac{1}{2}+w \mid |w| + \mid \tfrac{1}{4}+w^2-\tfrac{1}{2}w \mid + |w| \mid 1+w \mid + w^2\ ,\right.$$
$$\left. |w| \mid w-\tfrac{1}{2} \mid + \mid \tfrac{1}{4}+\tfrac{1}{2}w-w^2 \mid + 2w^2\right\}$$

Suppose $w \geq 0$, then the conditions on w become

$$\tfrac{3}{2}w+3w^2+\mid \tfrac{1}{4}-\tfrac{1}{2}w+w^2 \mid < 1,\ w|\tfrac{1}{2}-w| + \mid \tfrac{1}{4}+\tfrac{1}{2}w-w^2 \mid + 2w^2 < 1\ , \tag{3.24}$$

Now, $\frac{1}{4} \mp \frac{1}{2}w \pm w^2 > 0$ for real w, hence the first inequality is $4w^2+w-\frac{3}{4} < 0$, implying $0 \leq w < \frac{-1+\sqrt{13}}{8} < \frac{1}{2}$. For w in this range the second inequality is also valid. Suppose now that $w < 0$, then the condition on w determined by the second inequality in (3.24) is

$$\tfrac{1}{4}+\tfrac{1}{2}w+w^2-w\left(\tfrac{1}{2}-w\right) < 1 \Rightarrow -\frac{\sqrt{3}}{2} < w < 0\ .$$

¿From the first inequality in (3.24), assuming $-\frac{1}{2} < w < 0$, we get

$$-w\left(\tfrac{1}{2}+w\right)+\tfrac{1}{4}+w^2-\tfrac{1}{2}w-w(1+w)+w^2 < 1$$

hence $-\frac{3}{8} < w < 0$. No valid range exists when $w < -\frac{1}{2}$. In fact the range $|w| < \frac{1}{2}$ is forced by a necessary condition derived in Section 4. Thus we conclude that the s.s. (3.20), (3.21) converges for $-\frac{3}{8} < w < \frac{-1+\sqrt{13}}{8}$. The range $-\frac{1}{2} < w \leq 0$ can be obtained by considering results on schemes with non-negative coefficients in the mask. Computations of $\| \left(\frac{1}{2}S_1\right)^L \|_\infty$ show that the range of convergence is indeed $|w| < \frac{1}{2}$.

Using the second line of (3.22) we conclude that for the scheme (3.20), (3.21)

$$\begin{aligned}\tfrac{1}{2}a^{(2)}(z) &= \frac{2z^2}{(1+z)^2}a(z) = z^{-1}\left(z^2 - 2w(z-1)^2\left(1+z^2\right)\right)\\ &= z^{-1}\left(z^2(1-4w) - 2wz^4 + 4wz^3 + 4wz - 2w\right) \ .\end{aligned}$$

Hence $\|\frac{1}{2}S_2\|_\infty = 1$. Calculating $\|\left(\frac{1}{2}S_2\right)^2\|_\infty$ we obtain that for $0 < w < \frac{-1+\sqrt{5}}{8} \cong 0.154$, $\|\left(\frac{1}{2}S_2\right)^2\|_\infty < 1$, and the scheme (3.20), (3.21) generates $C^1(\mathbb{R})$-functions. The scheme S_2 satisfies (3.1) only for $w = 1/16$. In this case

$$a^{(2)}(z) = \left(z^{-1}/4\right)\left(-1 + 2z + 6z^2 + 2z^3 - z^4\right) = \left(z^{-1}/4\right)(1+z)\left(z^3 - 3z^2 - 3z + 1\right) \ ,$$

and $\frac{1}{2}a^{(3)}(z) = \frac{1}{4}\left(z^3 - 3z^2 - 3z + 1\right)$, implying that $\|\frac{1}{2}S_3\|_\infty = 1$. Indeed $\|\left(\frac{1}{2}S_3\right)^k\|_\infty = 1$ for all k, and $S^\infty f^0$ is not in $C^2(\mathbb{R})$ even for $w = 1/16$. Yet, it is shown in [DL1], using the formalism of Section 4, that $(S^\infty f^0)''(x)$ exists for $x \neq \alpha 2^{-k}$, $\alpha \in \mathbb{Z}$, $k \in \mathbb{Z}_+$, and that $(S^\infty f^0)'$ is Holder of order $1 - \epsilon$ for arbitrary small $\epsilon > 0$.

Figure 2. Curves generated by the 4-point and 6-point schemes

4. 6-point interolatory subdivision scheme.

In order to achieve C^2-limit functions we extend the support of the mask of the 4-point scheme, and consider the scheme

$$\begin{aligned}f^{k+1}_{2\alpha} &= f^k_\alpha\\ f^{k+1}_{2\alpha+1} &= \left(\frac{9}{16} + 2\theta\right)\left(f^k_\alpha + f^k_{\alpha+1}\right) - \left(\frac{1}{16} + 3\theta\right)\left(f^k_{\alpha-1} + f^k_{\alpha+2}\right) + \theta\left(f^k_{\alpha-2} + f^k_{\alpha+3}\right) \ .\end{aligned}$$

For $\theta = 0$ it is the 4-point scheme with $w = 1/16$. Algebraic manipulations show that

$$\left\| \left(\tfrac{1}{2} S_3\right)^2 \right\|_\infty < 1 \ , \qquad 0 < \theta < 0.02 \ ,$$

proving that this scheme generates C^2-limit functions in the above range of θ.

The algebraic manipulations are too involved to be performed without the help of a computer program such as Mathematica.

Figures 2 and 3 depict the performance of the 4-point and 6-point schemes, and exhibit the flexibility of the generated curves due to the range of values of the tension paremeters ω and θ respectively.

In Figure 2 the closer curve to the control polygon is generated by the 4-point scheme with $w = .054$, and the outer curve by the 6-point scheme with $\theta = .02$. In Figure 3 the curves, from the closer to the control polygon and outward, correspond to the values of the tension parameters $w = .022,\ .04$, and $\theta = .002,\ .014,\ .02$.

Figure 3. The effect of the tension parameters in the 4-point and 6-point schems

Bibliographical notes.

The results of this section and the examples are taken from [DGL2]. The proofs use the formalism of characteristic $\mathcal{L}$-polynomials introduced in [CDM] and used further in [DLM]. The generating functions technique, which faciliates the derivation of the various $\mathcal{L}$-polynomials related to a given scheme, is taken from [L]. The proofs given here of the case $\nu = 1$ in Theorem 3.4 and of the second claim in Corollary 3.3 are due to I. Yad-Shalom. The 4-point scheme is introduced in [DGL1] and further analysed in [DGL2] and [DLM]. The range of convergence $\frac{-1+\sqrt{13}}{2} < w < \frac{1}{2}$ was computed by M.J.D. Powell. The range $\frac{1}{2} < w \leq 0$ can be concluded from the analysis in [MP3], of schemes with non-negative masks. The 6-point scheme is analysed in detail in [W]. A simpler analysis based on perturbation arguments in done in [DLM]. The necessity of the conditions in Theorem 3.3 for φ satisfying (2.16) is proved in [CDM].

4. The Matrix Formalism –the case $s = 1$

Necessary and sufficient conditions for the convergence of subdivision schemes can be formulated in terms of properties of two matrices which are sections of the infinite matrix $\{a_{\alpha-2\beta}\}_{\alpha,\beta\in\mathbb{Z}}$, where $\mathbf{a} = \{a_\alpha : \alpha \in \mathbb{Z}\}$ is the mask of the s.s.

Due to the finiteness of supp($\mathbf{a}$), there is a unique finite set of control points at level k, F_ℓ^k, which determines all the control points corresponding to diadic values in the interval $\left[\ell 2^{-k}, (\ell+1)2^{-k}\right]$ at levels above k. By the uniformity of the *s.s.*, it follows that F_ℓ^k determines the two sets of control points at stage $k+1$, $F_{2\ell}^{k+1}$ and $F_{2\ell+1}^{k+1}$, corresponding to the two half intervals $\left[\ell 2^{-k}, (2\ell+1)2^{-k-1}\right], \left[(2\ell+1)2^{-k-1}, (\ell+1)2^{-k}\right]$. Thus there are two matrices A_0, A_1 termed refinement matrices, which transform the set F_ℓ^k into the sets $F_{2\ell}^{k+1}$ and $F_{2\ell+1}^{k+1}$ respectively. The matrices A_0, A_1 are of order $N \times N$ where N is the cardinality of F_ℓ^k. The set F_ℓ^k depends on supp($\mathbf{a}$).

Without loss of generality we may assume that supp($\mathbf{a}$) $\subset \{-n, -n+1, \ldots, 1\}$, since a translation of the support corresponds to an extra monomial factor in the characteristic $\mathcal{L}$-polynomial of the scheme.

Proposition 4.1 Let supp($\mathbf{a}$) $\subset \{-n, -n+1, \ldots, 1\}$, $a_1 \neq 0$, $a_{-n} \neq 0$. Then

$$F_\ell^k = \left\{ f_{\ell+\alpha}^k : \alpha \in I_n \right\}, \quad I_n = \{0, 1, \ldots, n\}, \qquad f^k = S^k f^0 . \tag{4.1}$$

The refinement matrices A_0, A_1 are of the form

$$\begin{aligned} (A_0)_{\alpha\beta} &= a_{\alpha-2\beta}, \qquad \alpha, \beta \in I_n, \\ (A_1)_{\alpha\beta} &= a_{1+\alpha-2\beta}, \quad \alpha, \beta \in I_n, \end{aligned} \tag{4.2}$$

and satisfy the relations

$$F_{2\ell}^{k+1} = A_0 F_\ell^k \ , \qquad F_{2\ell+1}^{k+1} = A_1 F_\ell^k \ , \tag{4.3}$$

where F_ℓ^k is regarded as the vector $\left(F_\ell^k\right)_\alpha = f_{\ell+\alpha}^k$, $\alpha \in I_n$.

Proof: Relations (4.3) in view of (4.2) become

$$f_{2\ell+\alpha}^{k+1} = \sum_{\beta \in I_n} a_{\alpha-2\beta} f_{\ell+\beta}^k = \sum_{\beta=\ell}^{n+\ell} a_{2\ell+\alpha-2\beta} f_\beta^k \ , \quad \alpha \in I_n \ , \tag{4.4}$$

$$f_{2\ell+1+\alpha}^{k+1} = \sum_{\beta \in I_n} a_{1+\alpha-2\beta} f_{\ell+\beta}^k = \sum_{\beta=\ell}^{n+\ell} a_{2\ell+1+\alpha-2\beta} f_\beta^k \ , \quad \alpha \in I_n \ , \tag{4.5}$$

which is exactly the way points at stage $k+1$ are generated from points at stage k by the *s.s.*, if the summations in (4.4) and (4.5) include all the non-zero coefficients of the mask. Indeed, in each sum of (4.4) the indices of the mask coefficients corresponding to $\alpha \in I_n$ are $\{\alpha, \alpha-2, \ldots, \alpha-2n\}$, while in (4.5) these indices are $\{\alpha+1, \alpha-1, \ldots, \alpha+1-2n\}$. The assumption on supp($\mathbf{a}$) guarantees that these sums consist of the full mask of the s.s., and that there is no smaller set than I_n for which (4.3) can hold. Hence F_ℓ^k in (4.1) is the set of control points determining all control points $S^m f^0, m > k$, in $\left[\ell 2^{-k}, (\ell+1)2^{-k}\right]$. □

Remark : In case S is a uniformly convergent s.s., the limit function $S^\infty f^0$, is given by $\sum_{\alpha \in \mathbb{Z}} f_\alpha^k \varphi\left(2^k x - \alpha\right)$, where φ is the S-refinable function of Section 2. Since $\text{supp}(\varphi) \subset \langle \text{supp}(\mathbf{a}) \rangle^0 = (-n, 1)$,

$$\sum_{\alpha \in \mathbb{Z}} f_\alpha^k \varphi\left(2^k x - \alpha\right) = \sum_{\alpha=\ell}^{\ell+n} f_\alpha^k \varphi\left(2^k x - \alpha\right), \quad x \in \left[\ell 2^{-k}, (\ell+1)2^{-k}\right] \ ,$$

and F_ℓ^k is easily seen to be (4.1). Also note that condition (3.1) on the mask $\mathbf{a}$ translates easily to

$$A_i e = e, \quad i = 0, 1, \text{ where } \ e_\alpha = 1, \quad \alpha \in I_n \ . \tag{4.6}$$

Theorem 4.2. *Let S be a uniformly convergent s.s. with φ its S-refinable function. Then all eigenvalues of the refinement matrices A_0, A_1 except for the eigenvalue $\lambda_0 = 1$ implied* *by (4.6), are of modulus less than 1. The left eigenvector of A_ν corresponding to the eigenvalue 1, with components summing to 1, is $\{\varphi(\nu-\alpha) : \alpha \in I_n\}$, $\nu = 0, 1$.*

Proof: By (4.3), for all $k \in \mathbb{Z}_+$, $F^k_{2^k\ell} = A_0^k F^0_\ell$. Now, since S is uniformly convergent

$$\lim_{k\to\infty} F^k_{2^k\ell} = S^\infty f^0(\ell)e = \lim_{k\to\infty} A_0^k F^0_\ell \ , \tag{4.7}$$

and (4.7) holds for arbitrary vectors F^0_ℓ. Thus all eigenvalues of A_0, except $\lambda_0 = 1$, have modulus less than 1, and

$$\lim_{k\to\infty} A_0^k = A_0^\infty \ , \qquad (A_0^\infty)_{\alpha\beta} = \upsilon_\beta, \ \ \alpha, \beta \in I_n \ , \tag{4.8}$$

where υ is a left eigenvector of A_0 satisfying $\upsilon A_0 = \upsilon$, $\upsilon e = 1$.

With $f^0 = \delta$ and $\ell \in I_n$, $F^k_{-2^k\ell} = A_0^k F^0_{-\ell}$ is the ℓ'th column of A_0^k, and therefore by (4.8)

$$\lim_{k\to\infty} F^k_{-2^k\ell} = \upsilon_\ell e \ . \tag{4.9}$$

On the otherhand, since $\varphi = S^\infty\delta$, we conclude from (4.7) that $\lim_{k\to\infty} F^k_{-2^k\ell} = \varphi(-\ell)e$. Hence $\upsilon_\ell = \varphi(-\ell)$, $\ell \in I_n$. Note that $\upsilon_n = \varphi(-n) = 0$, since $\mathrm{supp}(\varphi) = (-n, 1)$. This observation is consistent with the structure of the last column of A_0, given by $(0, 0, \ldots, 0, a_{-n})^T$. The proof of the analogous statement for A_1 is similar. □

Theorem 4.2 allows us to determine the values of $\left\{S^\infty f^0(2^{-k}\alpha) : \alpha \in \mathbb{Z}\right\}$ from the computed values $\{f^k_\alpha : \alpha \in \mathbb{Z}\}$.

Corollary 4.3. *Let S be a uniformly convergent s.s., and let υ satisfy $\upsilon A_0 = \upsilon$, $\upsilon e = 1$. Then*

$$S^\infty f^0\left(\ell 2^{-k}\right) = \sum_{\beta\in I_n} \upsilon_\beta f^k_{\ell+\beta} \ , \quad \ell \in \mathbb{Z}, \ k \in \mathbb{Z}_+ \ . \tag{4.10}$$

Proof: Let φ be the S-refinable function. Then

$$S^\infty f^0 = \sum_{\alpha\in\mathbb{Z}} \left(S^k f^0\right)_\alpha \varphi\left(2^k \cdot - \alpha\right) \ ,$$

and in particular $S^\infty f^0\left(\ell 2^{-k}\right) = \sum_{\alpha\in\mathbb{Z}}\left(S^k f^0\right)_\alpha \varphi(\ell-\alpha)$. Since $\mathrm{supp}(\varphi)\subset(-n,1)$ we conclude (4.10) from Theorem 4.2. $\square$

Consider a diadic point $\ell 2^{-k} = \sum_{m=0}^{k} \ell_m 2^{-m}$, with $\ell_0\in\mathbb{Z}$ and $\ell_m\in\{0,1\}$ for $1\le m\le k$. Then by (4.3)

$$F_\ell^k = A_{\ell_k}\dots A_{\ell_2}A_{\ell_1}F_{\ell_0}^0 \ .$$

Hence if S is uniformly convergent

$$S^\infty f^0\left(\ell 2^{-k}\right)e = \lim_{m\to\infty} A_0^m A_{\ell_k}\dots A_{\ell_1}F_{\ell_0}^0 \ .$$

For $x=\sum_{m=0}^{\infty}\ell_m 2^{-m}$, with $\ell_0\in\mathbb{Z}$ and $\ell_m\in\{0,1\}$, we get a similar relation

$$S^\infty f^0(x)e = \lim_{m\to\infty} A_{\ell_m}A_{\ell_{m-1}}\dots A_{\ell_1}F_{\ell_0}^0 \ . \tag{4.11}$$

Let S be a uniformly convergent *s.s.* and let $A_0^{(1)}, A_1^{(1)}$ be the refinement matrices of the scheme S_1. It is not difficult to observe that

$$\tfrac{1}{2}A_i^{(1)} = \left\{\left(EA_iE^{-1}\right)_{\alpha\beta} : \alpha,\beta\in I_{n-1}\right\}\ , \quad (EA_iE^{-1})_{\alpha n} = \delta_{\alpha,n}\ , \quad \alpha\in I_n\ , \tag{4.12}$$

where $E=\{\delta_{\alpha,\beta-1}-\delta_{\alpha,\beta} : \alpha,\beta\in I_n\}$. Indeed

$$\left\{(df^k)_{\ell+\alpha} : \alpha\in I_{n-1}\right\} = \left\{2(EF_\ell^k)_\alpha : \alpha\in I_{n-1}\right\}\ ,$$

and (4.12) follows directly from (4.3), after multiplication by E from the left. Also, since $(EA_iE^{-1})_{\alpha n}=\delta_{\alpha,n}$, $\alpha\in I_n$, all the eigenvalues of $\frac{1}{2}A_i^{(1)}$ are the eigenvalues of A_i with modulus less than 1.

The uniform convergence to zero of $\frac{1}{2}S_1$, which is necessary and sufficient for the uniform convergence of S, can be reformulated in terms of the refinement matrices of the scheme S_1.

Proposition 4.4. *Let S be a s.s. with a mask satisfying (3.1), and let $A_0^{(1)}, A_1^{(1)}$ be the refinement matrices of S_1. A necessary and sufficient condition for the uniform convergence of S is the existence of a positive integer L, such that*

$$\left(\tfrac{1}{2}\right)^L \|A_{i_1}^{(1)}A_{i_2}^{(1)}\dots A_{i_L}^{(1)}\|_\infty < 1\ , \qquad \forall\,(i_1,\dots,i_L)\in\{0,1\}^L\ . \tag{4.13}$$

Proof: It is easy to verify that condition (4.13) in view of (4.11), is equivalent to the condition $\|(\frac{1}{2}S_1)^L\|_\infty < 1$, which is necessary and sufficient for the uniform convergence of S. $\square$

Remark. The norm in (4.13) can be any matrix norm induced by a vector norm, since by the equivalence of norms on the space of all matrices of a prescribed order,

$$c_2\|A_{i_1}\ldots A_{i_m}\|_\infty \le \|A_{i_1}\ldots A_{i_m}\| \le c_1\|A_{i_1}\ldots A_{i_m}\|_\infty \ ,$$

and by choosing m large enough we can guarantee $\|A_{i_1}\ldots A_{i_m}\| < 1$ whenever $\|A_{i_1}\ldots A_{i_L}\|_\infty < 1$ and vice versa.

With the above observations, we can simplify the analysis of convergence of schemes with characteristic $\mathcal{L}$-polynomial of the form (3.15), with $\nu > 1$, if the schemes satisfy the necessary condition of Theorem 4.2. Here we use the scheme $S_{\Delta^\nu} = 2^{-\nu}S_\nu$ instead of $S_\Delta = \frac{1}{2}S_1$, and hence have to analyze a simpler $\mathcal{L}$-polynomial.

Theorem 4.5. *Let S be a s.s. with a characteristic $\mathcal{L}$-polynomial*

$$a(z) = (z^{-1}+1)^\nu q(z) \ , \quad \nu > 1 \ ,$$

such that its refinement matrices A_0, A_1 have eigenvalues of modulus less than 1, except for their eigenvalue 1. Then S is uniformly convergent if and only if the scheme S_{Δ^ν} corresponding to the $\mathcal{L}$-polynomial $q(z)$ converges uniformly to zero.

Proof: Suppose S is uniformly convergent. Then by Theorem 3.2, $S_\Delta = \frac{1}{2}S_1$ converges uniformly to zero. Since any initial data g^0 can be expressed as $\Delta^\nu f^0$, and since

$$S_{\Delta^\nu}\Delta^\nu f^k = \Delta^{\nu-1}S_\Delta f^k \ , \qquad f^k = S^k f^0 \ ,$$

S_{Δ^ν} converges uniformly to zero.

To prove the converse direction, observe that the refinement matrices of $S_{\Delta^j} = 2^j S_j$, corresponding to $a(z)(z^{-1}+1)^{-j}$, $j = 1,\ldots,\nu$, have spectral radius less than 1, since by (4.12) their eigenvalues are a subset of the eigenvalues of A_0, A_1, which are different from 1. Now by Theorem 3.2 and by the uniform convergence to zero of S_{Δ^ν} the scheme $S_{\Delta^{\nu-1}}$ is uniformly convergent, and since the refinement matrices of $S_{\Delta^{\nu-1}}$ have spectral radius less than 1, (4.11) implies that $S^\infty_{\Delta^{\nu-1}} f^0(x) = 0$ for $x \in 2^{-\ell}\mathbb{Z}$, $\ell \in \mathbb{Z}_+$. This and the continuity of $S^\infty_{\Delta^{\nu-1}} f^0$ prove that $S^\infty_{\Delta^{\nu-1}}$ converges uniformly to zero. Repeating this argument $\nu - 2$ times we conclude that S_Δ converges uniformly to zero which, by Theorem 3.2, guarantees the uniform convergence of S. □

Combining Theorem 3.4 with Corollary 4.3 we get

Corollary 4.6. *Let S be a s.s., with corresponding derived schemes S_j, satisfying (3.17) for $j = 1, \ldots, \nu$. Let $A_0^{(j)}, A_1^{(j)}$ be the refinement matrices for the scheme $S_j, j = 1, \ldots, \nu$. If S_ν converges uniformly then for $k \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$,*

$$\frac{d^j}{dx^j}(S^\infty f^0)\left(\ell 2^{-k}\right) = \sum_{\alpha=0}^{n-j} v_\alpha^{(j)} \left(d^j S^k f^0\right)_{\ell+\alpha} \ , \quad j = 1, \ldots, \nu \ , \tag{4.14}$$

where $v^{(j)}$ satisfies $v^{(j)} = v^{(j)} A_0^{(j)}$, $\sum_{\alpha=0}^{n-j} v_\alpha^{(j)} = 1$, $j = 1, \ldots, \nu$.

Corollary 4.6 indicates a way to construct a good approximation to $S^\infty f^0$ from the final computed values $S^k f^0$, by interpolating to the values of the limit function and its derivatives, which can be computed for all points in $2^{-k}\mathbb{Z}$, according to (4.10) and (4.14).

Proposition 4.7. *Under the conditions of Corollary 4.6, with $\nu = 2m$, let $Q^k \in C^{m-1}(\mathbb{R})$ be the unique function satisfying for $\ell \in \mathbb{Z}$*

$$\begin{aligned} &Q^k\big|_{[\ell 2^{-k},(\ell+1)2^{-k}]} \in \pi_{2m-1} \ , \\ &\frac{d^j}{dx^j} Q^k(\ell 2^{-k}) = \frac{d^j}{dx^j}(S^\infty f^0)(\ell 2^{-k}) \ , \quad j = 0, 1, \ldots, m-1 \ . \end{aligned} \tag{4.15}$$

Then

$$\|S^\infty f^0 - Q^k\|_{\infty,[\ell,\ell+1]} \le C_\ell (1/2^\nu)^k \ , \qquad \ell \in \mathbb{Z} \ , \tag{4.16}$$

where C_ℓ is a constant depending on $\{f^0_{\ell+\alpha} \ , \ \alpha \in I_n\}$ and S, but not on k.

The estimate in (4.16) is the classical error bound for Hermite interpolation. C_ℓ is a constant multiple of $\|\frac{d^\nu}{dx^\nu} S^\infty f^0\|_{\infty,[\ell,\ell+1]}$, which is bounded by the convergence of S_ν, and depends on the initial data F_ℓ^0. Comparting (4.16) with the error bound given in Corollary 3.3, we see the advantage in using $Q^k(x)$ instead of $f^k(x)$, in case $\mu \gg (1/2)^{\nu L}$.

The matrix formalism extends straightforwardly to the multivariate case where there are 2^s refinement matrices.

Examples.

1. General Uniform "corner cutting".

The matrics A_0, A_1 for the uniform corner cutting with mask $a_1 = s$, $a_0 = r$, $a_{-1} = 1-s$, $a_{-2} = 1-r$, $0 \le s < r \le 1$, are

$$A_0 = \begin{pmatrix} r & 1-r & 0 \\ s & 1-s & 0 \\ 0 & r & 1-r \end{pmatrix} , \qquad A_1 = \begin{pmatrix} s & 1-s & 0 \\ 0 & r & 1-r \\ 0 & s & 1-s \end{pmatrix} .$$

To compute the left eigenvector v of A_0, observe that

$$\sum_{\alpha=0}^{2} v_\alpha = 1 , \quad vA_0 = v \ \Rightarrow \ v_2 = 0 , \quad rv_0 + sv_1 = v_0 ,$$

hence,

$$v = (1-r, s, 0)/\big(1-(r-s)\big) .$$

By Corollary 4.3, $S^\infty f^0(\ell 2^{-k}) = \frac{1-r}{1-(r-s)} f_\ell^k + \frac{s}{1-(r-s)} f_{\ell+1}^k$. Note that for $r = \frac{3}{4}$, $s = \frac{1}{4}$, $S^\infty f^0(\ell 2^{-k}) = \frac{1}{2}(f_\ell^k + f_{\ell+1}^k)$, i.e. the limit curve generated by Chaikin's algorithm contains the midpoints of all the segments of all control polygons produced by the scheme. Using the mask of S_1 derived in Section 3, we get

$$A_0^{(1)} = 2 \begin{pmatrix} r-s & 0 \\ s & 1-r \end{pmatrix} , \qquad A_1^{(1)} = 2 \begin{pmatrix} s & 1-r \\ 0 & r-s \end{pmatrix} .$$

S_1 converges uniformly only if $r - s = \frac{1}{2}$. In this case

$$\sum_{\alpha=0}^{1} v_\alpha^{(1)} = 1 , \quad v^{(1)} A_0^{(1)} = v^{(1)} \ \Rightarrow \ v_1^{(1)} = 0 , \quad v_0^{(1)} = 1 .$$

Hence $\frac{d}{dx}(S^\infty f^0)(\ell 2^{-k}) = df_\ell^k = 2^k(f_{\ell+1}^k - f_\ell^k)$, and the limit curve touches the control polygon at their common points.

2. 4-point interpolatory subdivision scheme.

The Scheme defined by (3.20) and (3.21) with the convention of this section has the mask

$$a_{-5} = -w \ , \quad a_{-3} = \tfrac{1}{2} + w \ , \quad a_{-2} = 1 \ , \quad a_{-1} = \tfrac{1}{2} + w \ , \quad a_1 = -w \ ,$$

with all other coefficients equal to zero. This shift of the support of the mask causes a shift in the limit function, namely with the above mask $S^\infty f^0 = f^\infty(\cdot - 2)$ where f^∞ is the limit function corresponding to the mask (3.20),(3.21) and the initial data f^0.

By (4.2)

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -w & \frac{1}{2}+w & \frac{1}{2}+w & -w & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -w & \frac{1}{2}+w & \frac{1}{2}+w & -w \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} ,$$

and hence $v = (0,0,1,0,0)$, while by (4.10), $S^\infty f^0(\ell 2^{-k}) = f^k_{\ell+2}$ indicating again the interpolatory nature of the scheme. The matrix $A_0^{(1)}$, according to (3.23), has the form

$$A_0^{(1)} = \begin{pmatrix} 2w & 1 & -2w & 0 \\ -2w & 1 & 2w & 0 \\ 0 & 2w & 1 & -2w \\ 0 & -2w & 1 & 2w \end{pmatrix} ,$$

and the left eigenvector $v^{(1)}$ is given by

$$v^{(1)} = \big(-w, \tfrac{1}{2} - w, \tfrac{1}{2} - w, -w \big)/(1 - 4w) \ .$$

This together with (4.14), yields the following formula for the derivative of the limit function $f = S^\infty f^0$ at diadic points,

$$\begin{aligned} \frac{d^j}{dx^j} f(\ell 2^{-k}) &= \frac{2^k}{1-4w} \big\{ -w(df^k)_\ell + \big(\tfrac{1}{2} - w\big)(df^k)_{\ell+1} + \big(\tfrac{1}{2} - w\big)(df^k)_{\ell+2} - w(df^k)_{\ell+3} \big\} \\ &= \frac{2^k}{1-4w} \big\{ w f^k_\ell - \tfrac{1}{2} f^k_{\ell+1} + \tfrac{1}{2} f^k_{\ell+3} - w f^k_{\ell+4} \big\} \\ &= \frac{2^k}{1-4w} \big\{ w f\big((\ell-2)2^{-k}\big) - \tfrac{1}{2} f\big((\ell-1)2^{-k}\big) + \tfrac{1}{2} f\big((\ell+1)2^{-k}\big) - w f\big((\ell+2)2^{-k}\big) \big\} \ . \end{aligned}$$

By Theorem 4.2 and the observation following (4.12), a necessary condition for convergence of the scheme to C^0-limit functions is that the spectral radius of $\frac{1}{2}A_0^{(1)}$ should be less than 1. The eigenvalues of $\frac{1}{2}A_0^{(1)}$ are

$$\lambda_1 = \tfrac{1}{2} \ , \quad \lambda_2 = 2w \ , \quad \lambda_3 = \tfrac{1}{4}\big(1 + \sqrt{1-16w}\big) \ , \quad \lambda_4 = \tfrac{1}{4}\big(1 - \sqrt{1-16w}\big) \ .$$

Hence $|w| < \frac{1}{2}$ is a necessary condition for convergence to C^0 limit functions.

By (4.12) and the remark following it, all eigenvalues of $\frac{1}{2}A_0^{(2)}$ are those of $A_0^{(1)}$ which are less than 1 in module. Thus the eigenvalues of $\frac{1}{2}A_0^{(2)}$ are $4w$, $\frac{1}{2}(1+\sqrt{1-16w})$, $\frac{1}{2}(1-\sqrt{1-16w})$, and a necessary condition for the convergence of the scheme to C^1 limit functions is $0 < w < \frac{1}{4}$.

Bibliographical notes.

The matrix formalism as stated in Proposition 4.1 and (4.11) was introduced in [MP3], and developed further in [DGL2] in the analysis of convergence and smoothness. The derivation of the necessary conditions for convergence and the expressions for the values of the limit function and its derivatives follows the derivation in [DGL1], where the example of the 4-point interpolatory dubdivision scheme is analyzed. A similar analysis of the 6-point interpolatory subdivision scheme is done in [W]. The use of second differences in the convergence analysis is taken from [DGL2]. See [CDM] for generalizations. The matrix formalism for multivariate s.s. is discussed in [CDM].

5. Interpolatory Subdivision Schemes – the case $s = 1$

An interolatory s.s. has a mask **a** with the property $a_{2\alpha} = \delta_{\alpha,\beta}$, $\alpha \in \mathbb{Z}$, for some $\beta \in \mathbb{Z}$. Without loss of generality we assume that $\beta = 0$. The condition $\alpha_{2\alpha} = \delta_{\alpha,0}$, $\alpha \in \mathbb{Z}$, guarantees that for all $k \in \mathbb{Z}_+$ the points $\left\{(\alpha 2^{-k}, f_\alpha^k) : \alpha \in \mathbb{Z}\right\}$, with $f^k = S^k f^0$, belong to the graph of the limit function when it exists. This property implies that an interpolatory s.s. which converges in the weak sense of (2.5) converges uniformly. The integer translates of the refinable function φ, corresponding to a converging interpolatory s.s., are $\ell^\infty(\mathbb{Z})$-linearly independent as well as weakly-local linearly independent, since $\varphi(\alpha) = \delta_{\alpha,0}$, $\alpha \in \mathbb{Z}$. Hence φ satisfies the stability condition (2.16).

There is a simple necessary condition for convergence to C^ν-limit functions, for interpolatory s.s.

Theorem 5.1. *An interpolatory s.s. S, converges to C^ν-limit functions only if it repro-*

duces π_ν *namely if*

$$(5.1)\qquad (k+\tfrac{1}{2})^\ell = \sum_{\alpha\in\mathbb{Z}} a_{1+2\alpha}(k-\alpha)^\ell\ , \qquad \ell = 0,1,\dots,\nu\ , \qquad k\in\mathbb{Z}\ .$$

Proof: Consider the n'th order divided difference of the limit function $f = S^\infty f^0$ given by

$$(5.2)\qquad \delta_\epsilon^n f(x) = [x+\epsilon, x+2^{-1}\epsilon,\dots,x+2^{-n}\epsilon]f = \epsilon^{-n}\sum_{i=0}^{n} b_i f(x+2^{-i}\epsilon)\ .$$

where $b_i^{-1} = \prod_{j=0,j\neq i}^{n}(2^{-i}-2^{-j})$, $i=0,\dots,n$. For fixed $x\in 2^{-k}\mathbb{Z}$ and $\epsilon = 2^{-\ell}$, $\ell > k$, we get from (5.2), after expressing $f(x+2^{-i-\ell}) = f^{\ell+i}_{2^{\ell+i}x+1}$, in terms of the mask and the values of $f^{\ell+i-1}$, the following relation:

$$\begin{aligned}\delta^n_{2^{-\ell}} f(x) &= 2^{n\ell}\sum_{i=0}^{n} b_i \sum_{\alpha\in\mathbb{Z}} a_{1-2\alpha} f(x+\alpha 2^{-\ell-i+1}) \\ &= 2^n \sum_{\alpha\in\mathbb{Z}} a_{1-2\alpha}\alpha^n \delta^n_{\alpha 2^{1-\ell}} f(x)\ .\end{aligned}$$

Taking the limit as $\ell\to\infty$, and recalling the assumption $f = S^\infty f^0 \in C^\nu(\mathbb{R})$, we get

$$(5.3)\qquad f^{(n)}(x) = 2^n\sum_{\alpha\in\mathbb{Z}} a_{1-2\alpha}\alpha^n f^{(n)}(x)\ , \qquad n\le\nu\ .$$

Since (5.3) holds for all points $x\in 2^{-k}\mathbb{Z}$, $k\in\mathbb{Z}_+$, which are dense in $\mathbb{R}$, and since $f^{(n)}$ is continuous for $n\le\nu$, (5.3) holds for $x\in\mathbb{R}$. But $f^{(n)}(x)\equiv 0$ cannot hold for all arbitrary initial data f^0. Hence

$$\left(\tfrac{1}{2}\right)^n = \sum_{\alpha\in\mathbb{Z}} a_{1-2\alpha}\alpha^n\ , \qquad 0\le n\le\nu\ ,$$

which is equivalent to (5.1) and hence to reproduction of π_ν by S. □

This easy to verify property of S, guarantees the existence of the schemes $S_1,\dots,S_{\nu+1}$, which are used in Theorem 3.4 to prove the smoothness of the limit functions generated by S.

Theorem 5.2. *Let S be an interpolatory s.s. If S generates C^ν-limit functions then there exists for each n, $1 \le n \le \nu+1$, a s.s., S_n, with the property*

$$d^n(S^{k+1}f^0) = S_n d^n(S^k f^0) \ , \qquad k \in \mathbb{Z}_+ \ . \tag{5.4}$$

Moreover, S_n converges uniformly to $C^{\nu-n}$-limit functions for $1 \le n \le \nu$.

Proof: Since (5.4) holds for $n = 0$ with $S_0 = S$, in order to prove the first part of the claim it is sufficient to show that if (5.4) holds for some $n \le \nu$ then it holds for $n+1$.

Starting with $f^0_\alpha = \alpha^n$, $\alpha \in \mathbb{Z}$, and recalling that by Theorem 5.1, $f(x) = x^n$ is reproduced by S, we get from (5.4) that S_n reproduces π_0, since $d^n f^0_\alpha = d^n(S^k f^0)_\alpha = n!$. Hence the mask $\mathbf{a}^{(n)}$ of S_n has property (3.1), which together with Proposition 3.1 guarantees the existence of S_{n+1}.

To prove the second claim, observe that any initial data g^0 can be represented as $d^n f^0$ for some data f^0. Let $f^k = S^k f^0$, $f = S^\infty f^0$, $g^k = S^k_n g^0$. Then by the interpolation property of S, $f^k_\alpha = f(\alpha 2^{-k})$, and it follows from (5.4) and the continuity of $f^{(n)}$ that

$$g^k_\alpha = (d^n f^k)_\alpha = f^{(n)}(\xi) \ , \qquad 2^{-k}\alpha < \xi < 2^{-k}(\alpha+n) \ .$$

Thus

$$\left|g^k_\alpha - f^{(n)}(\alpha 2^{-k})\right| = \left|f^{(n)}(\xi) - f^{(n)}(\alpha 2^{-k})\right| \ ,$$

with $|\xi - \alpha 2^{-k}| \le n2^{-k}$. The uniform continuity of $f^{(n)}$ on any closed interval implies the uniform convergence of $S^k_n g^0$ to the limit function $f^{(n)} \in C^{\nu-n}(\mathbb{R})$. □

Combining Theorem 5.2 with Theorems 3.2 and 3.4, we get

Theorem 5.3. *Let S be an interpolatory s.s. which reproduces π_ν. Then the following conditions are equivalent*

(a) S converges uniformly to C^ν-limit functions.

(b) S_n converges uniformly to $C^{n-\nu}$-limit functions for $n = 1, 2, \ldots, \nu$.

(c) $\frac{1}{2}S_{\nu+1}$ converges uniformly to zero for all initial control points.

The extension of Theorems 5.1 and 5.2 to multivariate interpolatory s.s. is discussed in Section 7.

Bibligraphical notes.

The results of this section are taken from [DL3]. The necessary condition of Theorem 5.1 appears in [DL2] in terms of the matrix formalism. The results of this section can be concluded also from those in [CDM], concerning s.s. with a refinable function φ satisfying (2.16). Yet the direct analysis of the interpolatory case is much simpler.

6. Analysis of Convergence – $s > 1$

Two different extensions of the analysis of convergence of univariate s.s. are presented here for the analysis of multivariate schemes of the form

$$f_\alpha^{k+1} = \sum_{\beta \in \mathbb{Z}^s} a_{\alpha - 2\beta} f_\beta^k \quad , \quad \alpha \in \mathbb{Z}^s \ , \tag{6.1}$$

with masks satisfying

$$\sum_{\beta \in \mathbb{Z}^s} a_{\gamma - 2\beta} = 1 \quad , \quad \gamma \in E_s \ . \tag{6.2}$$

Both methods are based on a "contractivity principle".

Definition 6.1. A s.s. S is contractive relative to a non-negative, non-trivial function D, defined on all sets of control points, if for $f \in \ell^\infty(\mathbb{Z}^s)$, $D(f) < \infty$ and there exists $\mu \in (0,1)$ and $L \in \mathbb{Z}_+$ such that for all $f \in \ell^\infty(\mathbb{Z}^s)$

$$D\left(S^L f\right) \le \mu D(f) \ . \tag{6.3}$$

Theorem 6.2. *Let S be contractive relative to D, with a mask* $\mathbf{a}$ *satisfying (6.2). Let $\widetilde{S}$ be a uniformly convergent s.s. with a mask* $\widetilde{\mathbf{a}}$ *and $\widetilde{S}$- refinable function, $\psi \in C(\mathbb{R}^s)$ of compact support, satisfying*

$$\sum_{\alpha \in \mathbb{Z}^s} \psi(x - \alpha) = 1 \ , \qquad x \in \mathbb{R}^s \ , \tag{6.4}$$

$$c_1 \|f\|_\infty \le \| \sum_{\alpha \in \mathbb{Z}^s} f_\alpha \psi(\cdot - \alpha) \|_\infty \le c_2 \|f\|_\infty \quad , \quad f \in \ell^\infty(\mathbb{Z}^s) \ . \tag{6.5}$$

If

$$\|(S-\widetilde{S})f\|_\infty \le cD(f) \quad , \quad f\in \ell^\infty(\mathbb{Z}^s)\ , \tag{6.6}$$

then S *converges uniformly.*

Proof: For any initial control points $f^0\in\ell^\infty(\mathbb{Z}^s)$, let

$$f^k(x)=\sum_\alpha \left(S^kf^0\right)_\alpha \psi\left(2^kx-\alpha\right)=\sum_\alpha \left(\widetilde{S}S^kf^0\right)_\alpha \psi\left(2^{k+1}x-\alpha\right)\ , \tag{6.7}$$

where for the last equality we used (2.12) with $k=1$. As has been observed in Lemma 2.2, it is sufficient to show that the sequence of functions $\left\{f^k(\cdot):k\in\mathbb{Z}_+\right\}$ is a Cauchy sequence to conclude the uniform convergence of $\left\{S^kf^0:k\in\mathbb{Z}_+\right\}$ to a continuous limit function. By (6.7), (6.5) and (6.6)

$$\begin{aligned}\|f^{k+1}(\cdot)-f^k(\cdot)\|_\infty &\le \left\|\sum_\alpha\left(SS^kf^0-\widetilde{S}S^kf^0\right)_\alpha\psi\left(2^{k+1}x-\alpha\right)\right\|_\infty\\ &\le c_2\|\left(S-\widetilde{S}\right)S^kf^0\|_\infty\le(c_2c)D\left(S^kf^0\right)\ .\end{aligned}$$

Application of the contractivity property (6.3) yields

$$\|f^{k+1}(\cdot)-f^k(\cdot)\|_\infty\le(c_2c)\mu^{\left[\frac{k}{L}\right]}\max_{0\le j<L}D\left(S^jf^0\right)\ . \tag{6.8}$$

Thus $\left\{f^k(\cdot):k\in\mathbb{Z}_+\right\}$ is a Cauchy sequence, and S is uniformly convergent. □

Remark. Theorem 6.2 is an extension of the sufficiency part of Theorem 3.2, which claims that S is uniformly convergent if $\frac{1}{2}S_1$ converges uniformly to zero. In this case (6.3) holds with $D(f^0)=\|\Delta f^0\|_\infty$, $\mu=\|\left(\frac{1}{2}S_1\right)^L\|_\infty<1$ and $\psi=B_1(\cdot+1)$, the symmetric univariate B-spline of degree 1.

We apply Theorem 6.2 in two different ways, to the analysis of convergence of multivariate schemes. First we use it to prove convergence of schemes with mask **a** consisting of positive coefficients on a rectangular support. Then we prove a multivariate analogue of Theorem 3.2.

In the following we use the multi-index notation. In particular $x=(x_1,\ldots,x_s)$, $x^\alpha=\prod_{i=1}^s x_i^{\alpha_i}$, $|\alpha|=\sum_{i=1}^s|\alpha_i|$, $x^n=\prod_{i=1}^s x_i^n$, $n\in\mathbb{Z}$. Also $x<y$ stands for $x_i<y_i$,

$i = 1, \ldots, s$, and $e = (1, \ldots, 1) \in \mathbb{Z}^s$. We note that the formalism of generating functions introduced for the case $s = 1$ in Section 2 is valid also in the multivariate setting. With

$$F_k(z) = \sum_{\alpha \in \mathbb{Z}^s} f_\alpha^k z^\alpha , \qquad a(z) = \sum_{\alpha \in \mathbb{Z}^s} a_\alpha z^\alpha , \quad z = (z_1, \ldots, z_s) ,$$

relation (6.1) can be written as

$$F_{k+1}(z) = a(z) F_k(z^2) , \tag{6.9}$$

and

$$F_{k+L}(z) = \prod_{j=0}^{L-1} a(z^{2^j}) F_k(z^{2^L}) = a^{[L]}(z) F_k(z^{2^L}) . \tag{6.10}$$

The following proposition, which has interest of its own, is required in our analysis.

Proposition 6.3. *Let S be a uniformly convergent s.s. defined on $\mathbb{Z}$, with a mask $\mathbf{a} = \{a_\alpha : \alpha \in \operatorname{supp}(\mathbf{a})\}$, and S-refinable function φ. Then the tensor-produce s.s. $\mathcal{S}$ with the mask $\mathfrak{a} = \{\mathfrak{a}_\alpha = \prod_{i=1}^{s} a_{\alpha_i} : \alpha \in \mathbb{Z}^s\}$, is uniformly convergent and its refinable function ϕ has the form*

$$\phi(x) = \prod_{i=1}^{s} \varphi(x_i) , \qquad x \in \mathbb{R}^s . \tag{6.11}$$

Proof: The characteristic $\mathcal{L}$-polynomial of $\mathcal{S}$ is

$$\mathfrak{a}(z) = \sum_{\alpha \in \mathbb{Z}^s} \mathfrak{a}_\alpha z^\alpha = \sum_{\alpha \in \mathbb{Z}^s} \prod_{i=1}^{s} a_{\alpha_i} z_i^{\alpha_i} = \prod_{i=1}^{s} a(z_i) .$$

where $a(z) = \sum_{\alpha \in \mathbb{Z}} a_\alpha z^\alpha$. Then by (6.10), the characteristic $\mathcal{L}$-polynomial of $\mathcal{S}^L$ has the form

$$\mathfrak{a}^{[L]}(z) = \sum_{\alpha \in \mathbb{Z}^s} \mathfrak{a}_\alpha^{[L]} z^\alpha = \prod_{j=0}^{L-1} \mathfrak{a}(z^{2^j}) = \prod_{i=1}^{s} \prod_{j=0}^{L-1} a(z_i^{2^j}) = \prod_{i=1}^{s} a^{[L]}(z_i) .$$

Thus

$$(\mathcal{S}^L \delta)_\alpha = \mathfrak{a}_\alpha^{[L]} = \prod_{i=1}^{s} a_{\alpha_i}^{[L]} = \prod_{i=1}^{s} (S^L \delta)_{\alpha_i} ,$$

and by taking the limit $L \to \infty$, and recalling that $S^\infty \delta = \varphi$, we conclude that the sequence $\{\mathcal{S}^k \delta : k \geq 0\}$ is uniformly convergent to a limit function ϕ satisfying (6.11).

Since ϕ is of compact support and $\mathcal{S}$ is linear, we conclude that $\mathcal{S}$ is uniformly convergent and

$$\mathcal{S}^\infty f^0 = \sum_{\alpha \in \mathbb{Z}^s} f^0_\alpha \phi(\cdot - \alpha) \ . \qquad \square$$

Theorem 6.4. *Let* $\mathbf{a}$ *be a mask satisfying (6.2) and the conditions*

$$\mathrm{supp}(\mathbf{a}) = \prod_{i=1}^{s} [\ell_i, u_i] \cap \mathbb{Z}^s \quad , \quad a_\alpha > 0 \quad , \quad \alpha \in \mathrm{supp}(\mathbf{a}) \ ,$$

where $\ell = (\ell_1, \ldots, \ell_s)$, $u = (u_1, \ldots, u_s) \in \mathbb{Z}^s$ *satisfy* $\ell < 0 \cdot e < u$. *Then the corresponding s.s.* S *converges uniformly.*

Proof: Define for $f \in \ell^\infty(\mathbb{Z}^s)$

$$D(f) = \sup\{|f_\alpha - f_\beta| : (\alpha, \beta) \in \Gamma\} \ , \tag{6.12}$$

where $\Gamma = \{(\alpha, \beta) \in (\mathbb{Z}^s)^2 : \ell - u < \alpha - \beta < u - \ell\}$. Also observe that for any constant c and fixed α, β, by (6.2) and the hypothesis on the mask

$$(Sf)_\alpha - (Sf)_\beta = \sum_{\gamma \in \Gamma_{\alpha,\beta}} (f_\gamma - c)(a_{\alpha - 2\gamma} - a_{\beta - 2\gamma}) \ , \tag{6.13}$$

where $\Gamma_{\alpha,\beta} = \Gamma_\alpha \cup \Gamma_\beta$, with

$$\Gamma_\alpha = \{\gamma : \ell \leq \alpha - 2\gamma \leq u\} \ .$$

Now for $(\alpha, \beta) \in \Gamma$, and $\gamma, \delta \in \Gamma_{\alpha,\beta}$, we get $(\gamma, \delta) \in \Gamma$, since

$$\begin{aligned}
&\ell \leq \alpha - 2\gamma \leq u \ , \quad \ell \leq \alpha - 2\delta \leq u &&\Longrightarrow \ell - u \leq 2(\delta - \gamma) \leq u - \ell \ , \\
&\ell \leq \alpha - 2\gamma \leq u \quad \ell \leq \beta - 2\delta \leq u &&\Longrightarrow \ell - u \leq \alpha - \beta - 2(\gamma - \delta) \leq u - \ell \ , \\
& &&\Longrightarrow \ell - u < \gamma - \delta < u - \ell \ ,
\end{aligned}$$

and similarly for the two other possibilities. Hence it is possible to choose c in (6.13) so that $|f_\gamma - c| \leq \frac{1}{2} D(f)$ for $\gamma \in \Gamma_{\alpha,\beta}$, and to conclude that for $f \in \ell^\infty(\mathbb{Z}^s)$

$$D(Sf) = \sup\left\{\left| \sum_{\gamma \in \Gamma_{\alpha,\beta}} (f_\gamma - c)(a_{\alpha-2\gamma} - a_{\beta-2\gamma}) \right| : (\alpha, \beta) \in \Gamma \right\}$$

$$\leq \frac{1}{2} D(f) \sup\left\{ \sum_{\gamma \in \Gamma_{\alpha,\beta}} |a_{\alpha-2\gamma} - a_{\beta-2\gamma}| : (\alpha, \beta) \in \Gamma \right\} \ . \tag{6.14}$$

Now, if for each $(\alpha, \beta) \in \Gamma$, there exists $\gamma^* \in \Gamma_{\alpha,\beta}$ such that

$$\ell \le \alpha - 2\gamma^* \le u \quad \text{and} \quad \ell \le \beta - 2\gamma^* \le u \ , \tag{6.15}$$

then

$$|a_{\alpha-2\gamma^*} - a_{\beta-2\gamma^*}| < a_{\alpha-2\gamma^*} + a_{\beta-2\gamma^*} \ ,$$

and for all $(\alpha, \beta) \in \Gamma$, by (6.2)

$$\sum_{\gamma\in\Gamma_{\alpha,\beta}} |a_{\alpha-2\gamma} - \alpha_{\beta-2\gamma}| < 2 \ .$$

The contractivity relation

$$D(Sf) = \mu D(f) \ , \tag{6.16}$$

now follows from (6.14) with

$$\mu = \frac{1}{2} \sup \left\{ \sum_{\gamma\in\Gamma_{\alpha,\beta}} |a_{\alpha-2\gamma} - a_{\beta-2\gamma}| : (\alpha,\beta) \in \Gamma \right\} < 1 \ . \tag{6.17}$$

To prove the existence of γ^* satisfying (6.15) for $(\alpha, \beta) \in \Gamma$, assume without loss of generality that $\alpha_i \le \beta_i$ for some $i \in \{1, 2, \ldots, s\}$. Then

$$0 \le \beta_i - \alpha_i < u_i - \ell_i \ . \tag{6.18}$$

Rearranging (6.15), we get

$$\alpha_i - u_i \le 2\gamma_i^* \le \alpha_i - \ell_i \quad \text{and} \quad \beta_i - u_i \le 2\gamma_i^* \le \beta_i - \ell_i \ ,$$

but since $\alpha_i \le \beta_i$, $\beta_i - u_i \le 2\gamma_i^* \le \alpha_i - \ell_i$, and by (6.18) $\alpha_i - \ell_i > \beta_i - u_i$, a condition which guarantees the existence of an even integer in the interval $[\beta_i - u_i, \alpha_i - \ell_i]$. Thus $2\gamma_i^*$ is chosen to be that even integer. The argument is repeated for all $i = 1, \ldots, s$. To complete the convergence proof choose ψ in Theorem 6.2 to be

$$\psi(x) = \prod_{i=1}^{s} B_1(x_i + 1) \ . \tag{6.19}$$

Since the mask defining $B_1(\cdot+1)$ consists of three positive coefficients u_{-1}, u_0, u_1, the mask defining ψ, by Proposition 6.3, has the form $\widetilde{a}_\alpha = u_{\alpha_1}\cdots u_{\alpha_s}$, $\alpha \in \mathbb{Z}^s$, so that $\mathrm{supp}(\widetilde{\mathbf{a}}) \subset \mathrm{supp}(\mathbf{a})$. As both masks $\mathbf{a}$ and $\widetilde{\mathbf{a}}$ satisfy (6.2),

$$(Sf)_\alpha - (\widetilde{S}f)_\alpha = \sum_{\beta\in\Gamma_\alpha} (f_\beta - c)\,(a_{\alpha-2\beta} - \widetilde{a}_{\alpha-2\beta}) \tag{6.20}$$

To see that c can be chosen so that

$$|f_\beta - c| \le \frac{1}{2}D(f) \quad , \quad \beta \in \Gamma_\alpha \ ,$$

observe that for $\gamma, \beta \in \Gamma_\alpha$, $\ell \le \alpha - 2\beta \le u$ and $\ell \le \alpha - 2\gamma \le u$, hence $(\beta,\gamma) \in \Gamma$. With this observation we get from (6.20)

$$|(Sf)_\alpha - (\widetilde{S}f)_\alpha| \le \frac{1}{2}D(f) \sum_{\beta\in\Gamma_\alpha} (a_{\alpha-2\beta} + \widetilde{a}_{\alpha-2\beta}) \le D(f) \ , \tag{6.21}$$

and (6.6) of Theorem 6.2 holds. Now (6.16) and (6.21) together with Theorem 6.2 yield the uniform convergence of S. □

A direct consequence of Theorem 6.4 is

Corollary 6.5. *Let* $\mathbf{b}$ *be a univariate mask with positive coefficients on its support. Let* $q(z) = \sum_{\alpha\in\mathbb{Z}} b_\alpha z^\alpha$ *and consider the s.s with a characteristic* $\mathcal{L}$*-polynomial*

$$a(z) = \left(\frac{1+z}{2z}\right)^\nu q(z) = \sum_{\alpha\in\mathbb{Z}} a_\alpha z^\alpha \ .$$

Then the s.s. with the tensor-product mask

$$\underline{\mathfrak{a}} = \left\{\mathfrak{a}_\alpha = a_{\alpha_1}\cdots a_{\alpha_s} : \alpha \in \mathbb{Z}^s\right\} \ , \tag{6.22}$$

generates limit functions which are in $C^\nu(\mathbb{R}^s)$.

Proof: By Theorem 6.4 the s.s. with mask $\mathbf{b}$ is uniformly convergent. Hence by Theorem 3.4 the s.s S with the mask $\mathbf{a} = \{a_\alpha : \alpha \in \mathbb{Z}\}$ converges uniformly to $C^\nu(\mathbb{R})$-limit functions. Let $\varphi \in C^\nu(\mathbb{R})$ be its S-refinable function. By Proposition 6.3 the s.s $\mathcal{S}$ with the mask (6.22) is uniformly convergent, and the $\mathcal{S}$-refinable function has the form

$$\phi(x) = \varphi(x_1)\cdots\varphi(x_s) \ , \qquad x \in \mathbb{R}^s \ , \tag{6.23}$$

which in view of Theorem 2.4 proves that this scheme generates $C^{\nu}(\mathbb{R}^s)$-limit functions. □

Corollary 6.5 applies to tensor-product schemes obtained from uniform B-spline s.s. These schemes have positive masks on rectangular supports. (Recall that $(1+z)^{n+1}2^{-n}$ is the characteristic $\mathcal{L}$-polynomial corresponding to B-spline curves of degree n.)

A main tool for the analysis of convergence of multivariate s.s. is an extension of Theorem 3.2 to the multivariate setting. This extension requires the introduction of a vector of differences in the coordinate directions at each $\alpha \in \mathbb{Z}^s$:

$$\Delta f_\alpha = \{f_{\alpha+e^{(i)}} - f_\alpha : 1 \le i \le s\} \ , \tag{6.24}$$

where $e^{(i)} \in \mathbb{Z}_+^s$ is defined as $e_j^{(i)} = \delta_{ij}$, $j = 1, \dots, s$. The use of the contractivity principle is relative to the function

$$D(f) = \sup_{\alpha \in \mathbb{Z}^s} |\Delta f_\alpha|_\infty \ , \qquad f \in \ell^\infty(\mathbb{Z}^s) \ , \tag{6.25}$$

where $|v|_\infty = \max\limits_{1 \le i \le s} |v_i|$, $v \in \mathbb{R}^s$.

In order to prove property (6.6) of D we need the following lemma.

Lemma 6.6. *Let $p(z) \in \mathcal{L}$ satisfy*

$$p\big((-e)^\gamma\big) = 0 \ , \qquad \gamma \in E_s \ , \qquad (-e)^\gamma = \prod_{i=1}^{s} (-1)^{\gamma_i} \ . \tag{6.26}$$

Then there exist $p_1, \dots, p_s \in \mathcal{L}$ such that

$$p(z) = \sum_{i=1}^{s} p_i(z)(z_i^{-2} - 1) \ . \tag{6.27}$$

Proof: The claim for $s = 1$ is obvious. We prove it by induction on s. Suppose the claim holds for $s-1$, then

$$\begin{aligned} p(z_1, \dots, z_{s-1}, +1) &= \sum_{i=1}^{s-1} p_{1,i}(z_1, \dots, z_{s-1})(z_i^{-2} - 1) \ , \\ p(z_1, \dots, z_{s-1}, -1) &= \sum_{i=1}^{s-1} p_{-1,i}(z_1, \dots, z_{s-1})(z_i^{-2} - 1) \ . \end{aligned} \tag{6.28}$$

Now, the polynomial $q(z) = \frac{1}{2}\big[(1+z_s)p(z_1, \dots, z_{s-1}, 1) + (1-z_s)p(z_1, \dots, z_{s-1}, -1)\big]$ satisfies $q(z) = p(z)$ for $z_s = \pm 1$, hence there exists $p_s \in \mathcal{L}$ such that $p(z) - q(z) = (z_s^{-2} - 1)p_s(z)$, which in view of the definition of $q(z)$ and (6.28) implies (6.27). □

Remark. The representation (6.27) is non-unique for $s > 1$.

Proposition 6.7. *Let S and $\widetilde{S}$ be two s.s. of the form (6.1), with masks satisfying (6.2). Then for $f \in \ell^\infty(\mathbb{Z}^s)$*

$$\left\|(S-\widetilde{S})f\right\|_\infty \le ED(f) \ , \tag{6.29}$$

where E is a constant depending on the masks of S and $\widetilde{S}$.

Proof: Let $a(z)$ and $\widetilde{a}(z)$ denote the characteristic $\mathcal{L}$-polynomials of S and $\widetilde{S}$ respectively. Then by (6.2), $d(z) = a(z) - \widetilde{a}(z)$ satisfies (6.26), and therefore by Lemma 6.6,

$$d(z) = a(z) - \widetilde{a}(z) = \sum_{i=1}^{s} e_i(z)(z_i^{-2} - 1) \ , \tag{6.30}$$

where $e_i(z) = \sum_{\alpha\in\mathbb{Z}^s} e_{i,\alpha} z^\alpha \in \mathcal{L}$, $i = 1, \ldots, s$.

Using the generating functions formalism, we conclude from (6.30) that

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^s} (Sf^k - \widetilde{S}f^k)_\alpha z^\alpha = d(z)F_k(z^2) &= \sum_{i=1}^{s} e_i(z)(z_i^{-2}-1)F_k(z^2) \\ &= \sum_{i=1}^{s} e_i(z) \sum_{\alpha\in\mathbb{Z}^s} (f^k_{\alpha+e^{(i)}} - f^k_\alpha) z^{2\alpha} \ .\end{aligned}$$

Hence

$$(Sf^k - \widetilde{S}f^k)_\alpha = \sum_{i=1}^{s}\sum_{\beta\in\mathbb{Z}^s} e_{i,\alpha-2\beta}(f^k_{\beta+e^{(i)}} - f^k_\beta) \ , \tag{6.31}$$

and the claim (6.29) follows from (6.31) and (6.25), with

$$E = \max{}_{\gamma\in E_s} \sum_{i=1}^{s}\sum_{\alpha\in\mathbb{Z}^s} |e_{i,\gamma-2\alpha}| \ . \tag{6.32}$$

□

In the univariate case, conditions (6.2) guarantee the existence of the scheme $\frac{1}{2}S_1$ for Δf. In the multivariate case we get an analogous result.

Proposition 6.8. *Let S be a s.s. with a mask* **a** *satisfying (6.2). Then there exists a matrix mask*

$$\mathbf{A} = \{A_\alpha : \alpha \in \mathbb{Z}^s\} \subset R^s \times R^s \ , \tag{6.33}$$

with the property

$$(\Delta Sf)_\alpha = \sum_{\beta\in\mathbb{Z}^s} A_{\alpha-2\beta}(\Delta f)_\beta\ , \qquad \alpha\in\mathbb{Z}^s\ . \tag{6.34}$$

If we denote by S_Δ the s.s. with the matrix mask (6.33) then (6.34) can be written as

$$\Delta S = S_\Delta \Delta\ , \tag{6.35}$$

where Δ maps a set of control points defined on $\mathbb{Z}^s$ into a set of control vectors $(\Delta f)_\alpha \in \mathbb{R}^s$ defined on $\mathbb{Z}^s$.

Proof: Consider the $\mathcal{L}$-polynomial $a(z) = \sum_{\alpha\in\mathbb{Z}^s} a_\alpha z^\alpha$ of the mask $\mathbf{a}$. By (6.2)

$$\begin{aligned} &a\big((-e)^\gamma\big) = 0\ , \qquad \gamma\in E_s\backslash\{e\}\ ,\\ &a(e) = 2^s\ . \end{aligned} \tag{6.36}$$

Since $a_i(z) = a(z)(z_i^{-1}-1), i=1,\dots,s$ satisfies $a_i\big((-e)^\gamma\big)=0$, $\gamma\in E_s$, we conclude from Lemma 6.6 that

$$\left(z_i^{-1}-1\right) a(z) = \sum_{j=1}^{s}\left(z_j^{-2}-1\right) q_{ij}(z) \quad , \quad q_{ij}\in\mathcal{L}\ . \tag{6.37}$$

Consider the vector

$$(z^{-1}-e)F_k(z) = \left(z^{-1}-e\right)\sum_{\alpha\in\mathbb{Z}^s} f_\alpha^k z^\alpha = \left\{\sum_{\alpha\in\mathbb{Z}^s}\left(f^k_{\alpha+e^{(i)}} - f^k_\alpha\right) z^\alpha\ ,\ i=1,\dots,s\right\}\ .$$

Then by (6.9) and (6.37)

$$(z^{-1}-e)F_{k+1}(z) = a(z)\left(\mathbf{z}^{-1}-e\right)F_k(z^2) = Q(z)\left(\mathbf{z}^{-2}-e\right)F_k(z^2)\ , \tag{6.38}$$

where $Q(z)$ is a matrix with elements in $\mathcal{L}$ of the form

$$Q(z) = \{q_{ij}(z)\}_{i,j=1}^s = \sum_{\alpha\in\mathbb{Z}^s} A_\alpha z^\alpha\ , \qquad A_\alpha\in R^s\times R^s\ .$$

The vector equation (6.38), when equating equal powers of z, yields

$$(\Delta f^{k+1})_\alpha = \sum_\beta A_{\alpha-2\beta}(\Delta f^k)_\beta\ , \qquad \alpha\in\mathbb{Z}^s\ , \tag{6.39}$$

which proves the claim of the proposition. □

Combining Propositions 6.7 and 6.8 with Theorem 6.2, we obtain an extenion to the multivariate case of Theorem 3.2.

Theorem 6.9. *Under the conditions of Proposition 6.8, S is uniformly convergent only if $S_\Delta^k \Delta f^0$ converges uniformly to zero for all initial data $f^0 = \{f_\alpha^0 \in \mathbb{R} : \alpha \in \mathbb{Z}^s\}$. Moreover, S is uniformly convergent if $S_\Delta^k g^0$ converges uniformly to zero for all initial control vectors*

$$g^0 = \{g_\alpha^0 \in \mathbb{R}^s : \alpha \in \mathbb{Z}^s\} .$$

Proof: The proof is analogous to that of Theorem 3.2. Suppose S converges uniformly, and let $f^k = S^k f^0$ and $f^\infty = S^\infty f^0$. Then for any bounded domain $\Omega \subset \mathbb{R}^s$ and $\epsilon > 0$, there exists $K(\epsilon, \Omega)$ such that for all $2^{-k}\alpha_0 \in \Omega$, $\alpha_0 \in \mathbb{Z}^s$, and $k > K(\epsilon, \Omega)$

$$\begin{aligned}
\big|(S_\Delta^k \Delta f^0)_{\alpha_0}|_\infty &= \max_{1\le i\le s} |f^k_{\alpha_0+e^{(i)}} - f^k_{\alpha_0}| \\
&\le \max_{1\le i\le s} \big\{ \big|f^k_{\alpha_0+e^{(i)}} - f^\infty(2^{-k}\alpha_0 + 2^{-k}e^{(i)})\big| \\
&\quad + \big|f^k_{\alpha_0} - f^\infty(2^{-k}\alpha_0)\big| + \big|f^\infty(2^{-k}\alpha_0 + 2^{-k}e^{(i)}) - f^\infty(2^{-k}\alpha_0)\big| \big\} \\
&\le 3\epsilon .
\end{aligned}$$

Here we used the uniform continuity of f^∞ and the uniform convergence of S on the domain $\widetilde{\Omega} = \Omega + 2^{-K(\epsilon,\Omega)}\{x \in \mathbb{R}^s : |x|_\infty = 1\}$. Thus $S_\Delta^k \Delta f^0$ converges uniformly to zero, proving the necessity part of the theorem.

To prove the sufficiency part, observe that by the assumption on S_Δ, for $k > K(\epsilon)$, $f^0 \in \ell^\infty(\mathbb{Z}^s)$, and $\delta = \{\delta_{\alpha,0} : \alpha \in \mathbb{Z}^s\}$

$$\tag{6.40} \begin{aligned}
\sup_{\alpha\in\mathbb{Z}^s} \big|(S_\Delta^k \Delta f^0)_\alpha\big|_\infty &= \sup_{\alpha\in\mathbb{Z}^s} \left| \sum_{i=1}^s (f^0_{\beta+e^{(i)}} - f^0_\beta)(S_\Delta^k \delta e^{(i)})_{\alpha-2^k\beta} \right|_\infty \\
&\le M \sup_{\alpha\in\mathbb{Z}^s} \big\{ \big|(\Delta f^0)_\alpha\big|_\infty \big\} \left| \sum_{i=1}^s S_\Delta^k \delta e^{(i)} \right|_\infty < \epsilon \sup_{\alpha\in\mathbb{Z}^s} \big|(\Delta f^0)_\alpha\big|_\infty ,
\end{aligned}$$

where the last inequality follows from the uniform convergence to zero of the sequences $\{S_\Delta^k \delta e^{(i)}\}$, $i = 1, \ldots, s$, and the one before it from the relation

$$\tag{6.41} \operatorname{supp}(S_\Delta^k \delta v) \subset (2^k - 1)\big\langle \operatorname{supp}(\mathbf{A})\big\rangle , \qquad v \in \mathbb{R}^s .$$

Relation (6.41) is derived in a way similar to the derivation of the scalar case (see the proof of Theorem 2.4).

Inequality (6.40) guarantees the existence of $\mu \in (0,1)$ and a positive integer L such that for all $f^0 \in \ell^\infty(\mathbb{Z}^s)$

$$D(S^L f^0) = \sup_{\alpha\in\mathbb{Z}^s} \left|((S_\Delta)^L \Delta f^0)_\alpha\right|_\infty \le \mu D(f^0) \ , \tag{6.42}$$

where $D(f)$ is defined in (6.25). Thus S is contractive relative to D, while by Proposition 6.7, S satisfies (6.6). These together with Theorem 6.2 imply the uniform convergence of S. □

Remark. Let $\|S_\Delta^L\|_\infty$ be the operator norm of S_Δ^L relative to the norm

$$\|g^0\|_\infty = \sup_{\alpha\in\mathbb{Z}^s} |g_\alpha^0|_\infty \ ,$$

defined on control vectors of the form $g^0 = \{g_\alpha^0 \in \mathbb{R}^s : \alpha \in \mathbb{Z}^s\}$. Replacing Δf^0 by g^0 in (6.40), we conclude the existence of $L \in \mathbb{Z}_+$ and $\mu \in (0,1)$, both independent of g^0, such that

$$\|S_\Delta^L g^0\|_\infty \le \mu \|g^0\|_\infty \ . \tag{6.43}$$

Thus $\|S_\Delta^L\|_\infty = \mu < 1$, is equivalent to the uniform convergence to zero of all sequences of the form $\{S_\Delta^k g^0\}$. This condition which is sufficient for the uniform convergence of S might be too strong, since in the proof of Theorem 6.9 we use (6.43) with $g^0 = \Delta f^0$, and the space $\{\Delta f : f \in \ell^\infty(\mathbb{Z}^s)\}$ is a proper subspace of

$$\left(\ell^\infty(\mathbb{Z}^s)\right)^s = \{g : g_\alpha \in \mathbb{R}^s \ , \ \alpha \in \mathbb{Z}^s \ , \ \|g\|_\infty < \infty\} \ .$$

That this is the case, can be concluded from the following relations satisfied by all control vectors of the form Δf:

$$e^{(j)} \cdot \left[(\Delta f)_{\alpha+e^{(i)}} - (\Delta f)_\alpha\right] = e^{(i)} \cdot \left[(\Delta f)_{\alpha+e^{(j)}} - (\Delta f)_\alpha\right] \ , \qquad \alpha \in \mathbb{Z}^s \ , \tag{6.44}$$

where $i, j \in \{1, \ldots, s\}$. It can be shown by using the matrix formalism of Section 4, and in particular the approach of Proposition 4.4, extended to the multivariate setting and to matrix masks, that if $\{S_\Delta^k \Delta f\}$ converges to zero for all Δf, then there exist $L \in \mathbb{Z}_+$ and $\widetilde{\mu} \in (0,1)$ such that

$$\|S_\Delta^L \Delta f\|_\infty \le \widetilde{\mu} \|\Delta f\|_\infty \ , \qquad f \in \ell^\infty(\mathbb{Z}^s) \ . \tag{6.45}$$

Thus the necessary condition of Theorem 6.9 is also sufficient, but impractical, since the estimation of $\widetilde{\mu}$ is very difficult, in cases where $\widetilde{\mu} < \|S_\Delta^L\|_\infty$.

In the following proposition the form of $\|S_\Delta^L\|_\infty$ is expressed in terms of the mask $\mathbf{A}$. For that we use the notation $E_s^L = \{\alpha \in \mathbb{Z}_+^s : \alpha < 2^L e\}$, $|A| = \big\{|A_{ij}|\big\}_{i,j=1}^s$ and $\|A\|_\infty = \max\big\{\sum_{j=1}^s |A_{ij}| : i = 1,\dots,s\big\}$.

Proposition 6.10. *Under the conditions and notations of Proposition 6.8, let*

$$Q^{[L]}(z) = Q(z)Q(z^2)\cdots Q(z^{2^{L-1}}) = \sum_{\alpha\in\mathbb{Z}^s} A_\alpha^{[L]} z^\alpha \ . \tag{6.46}$$

Then

$$\|S_\Delta^L\|_\infty = \max_{\gamma\in E_s^L} \left\| \left(\sum_{\alpha\in\mathbb{Z}^s} |A_{\gamma-2^L\alpha}^{[L]}| \right) \right\|_\infty \ . \tag{6.47}$$

Proof: Let $G_k(z) = \sum_{\alpha\in\mathbb{Z}^s} g_\alpha^k z^\alpha$ with $g^k = S_\Delta^k g^0$. Then $G_{k+1}(z) = Q(z)G_k(z^2)$ and in view of (6.46)

$$G_{k+L}(z) = Q^{[L]}(z)G_k(z^{2^L}) \ ,$$

or equivalently

$$(S_\Delta^L g^0)_\alpha = \sum_{\beta\in\mathbb{Z}^s} A_{\alpha-2^L\beta}^{[L]} g_\beta^0 \ , \qquad \alpha \in \mathbb{Z}^s \ .$$

Hence

$$|S_\Delta^L g^0|_\infty \le \sum_{\beta\in\mathbb{Z}^s} |A_{\alpha-2^L\beta}^{[L]}||g_\beta^0| \ ,$$

and for $g^0 \in \big(\ell^\infty(\mathbb{Z}^s)\big)^s$

$$\|S_\Delta^L g^0\|_\infty \le \|g^0\|_\infty \max_{\alpha\in\mathbb{Z}^s} \max_{1\le i\le s} \sum_{j=1}^s \left(\sum_{\beta\in\mathbb{Z}^s} |A_{\alpha-2^L\beta}^{[L]}| \right)_{ij} \ . \tag{6.48}$$

Inequality (6.48) implies that

$$\|S_\Delta^L\|_\infty \le \max_{\alpha\in\mathbb{Z}^s} \left\| \left(\sum_{\beta\in\mathbb{Z}^s} |A_{\alpha-2^L\beta}^{[L]}| \right) \right\|_\infty = \max_{\gamma\in E_s^L} \left\| \left(\sum_{\beta\in\mathbb{Z}^s} |A_{\gamma-2^L\beta}^{[L]}| \right) \right\|_\infty \ . \tag{6.49}$$

To complete the proof of (6.47), it is sufficient to exhibit $g^0 \in \big(\ell^\infty(\mathbb{Z}^s)\big)^s$, $\|g^0\|_\infty = 1$, for which

$$\|S_\Delta^L g^0\|_\infty \geq \max_{\gamma\in E_s^L} \left\| \Big(\sum_{\beta\in\mathbb{Z}^s} |A^{[L]}_{\gamma-2^L\beta}| \Big) \right\|_\infty .$$

Let $\gamma^* \in E_s^L$ achieve the maximum in (6.49), and let i^* be such that

$$\left\| \Big(\sum_{\alpha\in\mathbb{Z}^s} |A^{[L]}_{\gamma^*-2^L\alpha}| \Big) \right\|_\infty = \sum_{j=1}^{s} \Big(\sum_{\alpha\in\mathbb{Z}^s} |A^{[L]}|_{\gamma^*-2^L\alpha} \Big)_{i^*,j} .$$

Choosing g^0 of the form

$$g^0_\alpha = \begin{cases} \big\{ \operatorname{sgn}(A^{[L]}_{\gamma^*-2^L\alpha})_{i^*,j} \ , \ j=1,\dots,s \big\} \ , & \gamma^* - 2^L\alpha \in \operatorname{supp}(\mathbf{A}^{[L]}) \ , \\ 0 \ , & \text{otherwise} \ , \end{cases}$$

we get $\|g^0\|_\infty = 1$, and

$$e^{(i^*)} \cdot (S_\Delta^L g^0)_{\gamma^*} = e^{(i^*)} \cdot \sum_{\alpha\in\mathbb{Z}^s} A^{[L]}_{\gamma^*-2^L\alpha} g^0_\alpha = \sum_{\alpha\in\mathbb{Z}^s} \sum_{j=1}^{s} |A^{[L]}_{\gamma^*-2^L\alpha}|_{i^*,j} .$$

Thus by the choice of γ^* and i^*

$$\|S_\Delta^L g^0\|_\infty \geq \Big(\sum_{\alpha\in\mathbb{Z}^s} \sum_{j=1}^{s} |A^{[L]}_{\gamma^*-2^L\alpha}| \Big)_{i^*,j} = \max_{\gamma\in E_s^L} \left\| \Big(\sum_{\alpha\in\mathbb{Z}^s} |A^{[L]}_{\gamma-2^L\alpha}| \Big) \right\|_\infty ,$$

which completes the proof of the proposition. $\square$

Remark. Suppose S have a characteristic $\mathcal{L}$-polynomial of the form

(6.50)
$$a(z) = q(z) \prod_{i=1}^{s} (z_i^{-1} + 1) \ , \qquad q \in \mathcal{L} \ .$$

Then the s.s. S_Δ is "diagonal", namely $Q(z)$ in (6.38) is the diagonal matrix

$$Q(z) = a(z) \operatorname{diag} \big\{ (z_1^{-1}+1)^{-1}, (z_2^{-1}+1)^{-1}, \dots, (z_s^{-1}+1)^{-1} \big\} \ ,$$

and S_Δ decomposes into s scalar s.s., S_{Δ_i}, with the property $S_{\Delta_i}(e^{(i)} \cdot \Delta f^k) = e^{(i)} \cdot \Delta f^{k+1}$, corresponding to the $\mathcal{L}$-polynomials $a(z)(z_i^{-1}+1)^{-1}$, $i = 1, \dots, s$. Also

(6.51)
$$\|S_\Delta^L\|_\infty = \max_{1\leq i\leq s} \|S_{\Delta_i}^L\|_\infty \ , \qquad L \in \mathbb{Z}_+ \ ,$$

and the condition $\|S_{\Delta}^{L}\|_{\infty} < 1$ for some $L \in \mathbb{Z}_+$ is necessary and sufficient for the convergence of S.

This remark holds for more general forms of $a(z)$ in (6.50), namely

$$a(z) = q(z) \prod_{i=1}^{s} (z^{-\theta^{(i)}} + 1) \ , \tag{6.52}$$

where $\theta^{(1)}, \ldots, \theta^{(s)} \in \mathbb{Z}^s$ satisfy

$$\left| \det(\theta^{(1)}, \ldots, \theta^{(s)}) \right| = 1 \ . \tag{6.53}$$

Then there exist s schemes $S_{\theta^{(i)}} \Delta_{\theta^{(i)}} f^k = \Delta_{\theta^{(i)}} S f^k$, with $(\Delta_{\theta^{(i)}} f^k)_\alpha = f^k_{\alpha+\theta^{(i)}} - f^k_\alpha$, $i = 1, \ldots, s$. If all these schemes converge uniformly to zero then S converges uniformly. This follows from the observation that by (6.53) any $v \in \mathbb{Z}^s$ can be expressed as a linear combination with integer coefficients of $\theta^{(1)}, \ldots, \theta^{(s)}$, and therefore there exists a finite set of matrices $\{B_\gamma\}$, with elements in $\{-1, 0, 1\}$, such that

$$(\Delta f^k)_\alpha = \sum_\gamma B_\gamma \{\Delta_{\theta^{(1)}} f^k, \ldots, \Delta_{\theta^{(s)}} f^k\}^T_{\alpha-\gamma} \ ,$$

implying that $\{\Delta f^k\}_{k \geq 0}$ converges uniformly to zero.

In Section 8 we present an example of a one-parameter family of interpolatory s.s. for the design of surfaces and analyze its convergence and smoothness.

Bibliographical notes.

The contractivity principle (Theorem 6.2) and its application to the convergence of schemes with positive masks on rectangular supports follow [CDM]. The convergence analysis based on s.s. with matrix masks appears in [DL3] and [CDM], but not in full details. The generating functions technique for the derivation of the various $\mathcal{L}$-polynomials is based on [L]. Proposition 6.10 is due to S. Hed [H].

7. Analysis of Smoothness – $s > 1$

The analysis of smoothness of schemes which do not exhibit directional factorization of their characteristic $\mathcal{L}$-polynomials such as in (6.50) or (6.52), requires the introduction and analysis of non-degenerate s.s. with matrix masks.

Definition. A s.s. with a matrix mask is called non-degenerate, if the components of the limit vector-valued function generated by the scheme are linearly independent for generic initial data.

Two key observations in the analysis of such s.s. are the analogue of Proposition 2.1 and the necessity part of Theorem 6.9. We state the results without proofs, because the proofs are based on the same arguments.

Proposition 7.1. *Let S be a uniformly convergent s.s. with a matrix mask*

$$\mathbf{A} = \{A_\alpha : \alpha \in \mathbb{Z}^s\} \subset R^\ell \times R^\ell \ . \tag{7.1}$$

Let $f^k = S^k f^0 = \{f^k_\alpha \in \mathbb{R}^\ell : \alpha \in \mathbb{Z}^s\}$, and define

$$\Delta f^k = \left\{(\Delta f^k)_\alpha : \alpha \in \mathbb{Z}^s\right\} \ , \qquad (\Delta f^k)_\alpha = \{f^k_{\alpha+e^{(j)}} - f^k_\alpha : j = 1, \ldots, s\} \ . \tag{7.2}$$

Then $\{\Delta f^k\}$ converges uniformly to zero, namely for $f^0 \in (\ell^\infty(\mathbb{Z}^s))^\ell$ and any $\epsilon > 0$,

$$\|\Delta f^k\|_\infty = \max_{\alpha\in\mathbb{Z}^s} \max_{1\le j\le s} |f^k_{\alpha+e^{(j)}} - f^k_\alpha|_\infty < \epsilon \ , \qquad k > K(\epsilon) \ .$$

Moreover, if S is non-degenerate then

$$\sum_{\alpha\in\mathbb{Z}^s} A_{\gamma-2\alpha} = I_{\ell\times\ell} \ , \qquad \gamma \in E_s \ , \tag{7.3}$$

where $I_{\ell\times\ell}$ is the unit matrix of order ℓ.

Remark. The operator Δ maps sequences of scalars $\{f^k_\alpha \in \mathbb{R} : \alpha \in \mathbb{Z}^s\}$ into sequences of vectors of order s as in Section 6, and it maps sequences of vectors of order ℓ into sequences of matrices of order $\ell \times s$ as in (7.2).

We first discuss the simpler case of a s.s. S with a scalar mask $\mathbf{a}$. In the convergence analysis of such schemes in Section 6, it is shown that the necessary conditions for convergence (6.2) imply the existence of a s.s. S_Δ with a matrix mask (6.33) satisfying (6.34). Let $S_1 = 2S_\Delta$, and $f^k = S^k f^0$, then

$$S_1 df^k = df^{k+1} \ , \qquad df^k = 2^k \Delta f^k \in \mathbb{R}^s \ . \tag{7.4}$$

The following is a direct extension of Theorem 3.4 with $\nu = 1$, and the proof supplied is an extension of the proof of Theorem 3.4.

Theorem 7.2. *Let S be a uniformly convergent s.s. with a corresponding scalar mask* **a** *and a s.s. S_1 satisfying (7.4). If S_1 converges uniformly then $S^\infty f^0 \in C^1(\mathbb{R}^s)$ for all initial data f^0, and S_1 is non-degenerate.*

Proof: Let $f^0 = \delta$, $f^k = S^k \delta$ and define $\psi(x) = \prod_{i=1}^s B_1(1+x_i)$,

$$f^k(x) = \sum_{\alpha \in \mathbb{Z}^s} f^k_\alpha \psi(2^k x - \alpha) \ , \qquad g^k(x) = \sum_{\alpha \in \mathbb{Z}^s} (df^k)_\alpha \psi(2^k x - \alpha) \ .$$

Since $\sum_{\alpha \in \mathbb{Z}^s} \psi(\cdot - \alpha) = 1$, by Lemma 2.2 and its extension to s.s. with matrix masks

$$\lim_{k\to\infty} f^k(x) = S^\infty f^0(x) = \varphi(x) \ , \qquad \lim_{k\to\infty} g^k(x) = S_1^\infty df^0(x) \ , \qquad x \in \mathbb{R}^s \ , \tag{7.5}$$

and the convergence is uniform.

Introducing a sequence of vector valued functions $h^k(x)$ with components

$$h^k_i(x) = \sum_{\alpha \in \mathbb{Z}^s} (d_i f^k)_\alpha \psi_i(2^k x - \alpha) \ , \qquad i = 1, \ldots, s \ ,$$

where $d_i f^k = e^{(i)} \cdot df^k$, and $\psi_i(x) = \psi(x) H(x_i)/B_1(1+x_i)$, with $H(x_i) = 1$ for $x_i \in [0,1]$ and $H(x_i) = 0$ elsewhere, we first show that

$$\left\| g^k_i(x) - h^k_i(x) \right\|_\infty \le c \| e^{(i)} \cdot \Delta d_i f^k \|_\infty \ . \tag{7.6}$$

Now, $g^k_i(x) - h^k_i(x) = \sum_{\alpha \in \mathbb{Z}^s} (d_i f^k)_\alpha \big(\psi(2^k x - \alpha) - \psi_i(2^k x - \alpha)\big)$, and since $\phi_i(x) = \psi(x) - \psi_i(x)$ satisfies $\phi_i(x) = -\phi_i(x - e^{(i)})$ for $x_i \in (0,1)$ and $\phi_i(x) = 0$ for $x_i \notin (-1,1)$,

$$g^k_i(x) - h^k_i(x) = \sum_{\alpha \in \mathbb{Z}^s} \big((d_i f^k)_{\alpha + e^{(i)}} - (d_i f^k)_\alpha\big) \widetilde{\phi}(2^k x - \alpha) \ ,$$

where $\widetilde{\phi}(x) = -\phi(x)$ for $x_i \in (0,1)$ and $\widetilde{\phi}(x) = 0$ elsewhere.

Thus

$$\| g^k_i - h^k_i \|_\infty \le \| e^{(i)} \cdot \Delta d_i f^k \|_\infty \Big\| \sum_{\alpha \in \mathbb{Z}^s} \widetilde{\phi}(2^k \cdot - \alpha) \Big\|_\infty \ ,$$

proving (7.6), in view of the local support of $\widetilde{\phi}$ and the bound $\|\widetilde{\phi}\|_\infty = 1$.

By the uniform convergence of S_1 and by Proposition 7.1, $\{\Delta df^k\}_{k\geq 1}$ converges uniformly to zero, and due to its compact support $\{\|\Delta df^k\|_\infty\}_{k\geq 1}$ converges to zero. Combining this with (7.6), (7.5) and the bound

$$\left\|h^k(x) - S_1^\infty df^0\right\|_\infty \leq \left\|h^k(x) - g^k(x)\right\|_\infty + \left\|g^k(x) - S_1^\infty df^0\right\|_\infty ,$$

we conclude the uniform convergence of $h^k(x)$ to $S_1^\infty df^0$. Here the sup-norm of vector valued functions $u = \{u_i \in C(\mathbb{R}^s) \ , \ i = 1, \ldots, s\}$ is defined by

$$\|u\|_\infty = \max_{1\leq i\leq s} \|u_i\|_\infty = \max_{x\in\mathbb{R}^s} \max_{1\leq i\leq s} \left|u_i(x)\right| .$$

To complete the proof of the theorem we show that $\partial\varphi = S_1^\infty df^0$, with $\partial = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_s}\right)$. Rewriting $h_i^k(x)$ as

$$h_i^k(x) = \sum_{\alpha\in\mathbb{Z}^s} f_\alpha^k 2^k\left(\psi_i(2^k x - \alpha + e^{(i)}) - \psi_i(2^k x - \alpha)\right) ,$$

and observing that

$$\int_{-\infty}^t \left(\psi_i(x + (\tau+1)e^{(i)}) - \psi_i(x + \tau e^{(i)})\right)d\tau = \psi(x + te^{(i)}) ,$$

we conclude that

$$f^k(x + te^{(i)}) = \int_{-\infty}^t h_i^k(x + \tau e^{(i)})d\tau ,$$

which in view of (7.5) implies the uniform convergence of the sequence $\left\{\int_{-\infty}^t h_i^k(x + \tau e^{(i)})d\tau\right\}_{k\geq 0}$ to $\varphi(x + te^{(i)})$. On the other hand this sequence converges uniformly to $\left\{\int_{-\infty}^t e^{(i)} \cdot S_1^\infty df^0(x + \tau e^{(i)})d\tau\right\}$, due to the compact support of the integrated functions. Therefore

$$\frac{\partial}{\partial x_i}\varphi = e^{(i)} \cdot S_1^\infty df^0 \in C(\mathbb{R}^\nu) , \qquad i = 1, \ldots, s , \tag{7.7}$$

proving that $\varphi = S^\infty f^0 \in C^1(\mathbb{R}^s)$. Since φ is of compact support, there is no $v \in \mathbb{R}^s$ such that $v^T\partial\varphi(x) = 0, \quad x \in \mathbb{R}^s$, and therefore S_1 is non-degenerate . □

To analyze smoothness of higher orders, a result as that in Theorem 7.2 for s.s. with matrix masks is needed. Before that the existence of S_1 for non-degenerate schemes should be guaranteed.

Proposition 7.3. *Let S be a s.s. with a matrix mask (7.1) satisfying (7.3). Then there exists a s.s. S_Δ with a matrix mask*

$$\mathbf{Q} = \{Q_\alpha : \alpha \in \mathbb{Z}^s\} \subset R^{\ell s} \times R^{\ell s} ,$$

operating on the matrix Δf^k, defined in (7.2) and rearranged in a vector of length ℓs, such that

$$(\Delta f^{k+1})_\alpha = (S_\Delta \Delta f^k)_\alpha = \sum_{\beta \in \mathbb{Z}^s} Q_{\alpha - 2\beta} (\Delta f^k)_\beta , \qquad f^k = S^k f^0 . \tag{7.8}$$

Proof: By (7.3) the matrix polynomial $A(z) = \sum_{\alpha \in \mathbb{Z}^s} A_\alpha z^\alpha$ satisfies

$$A(z) = 0 , \qquad z \in E_s \backslash \{e\} , \qquad A(e) = 2^s I_{\ell \times \ell}. \tag{7.9}$$

Hence, by Lemma 6.6

$$(z - e)\big(A(z)\big)_{ij} = B^{(i,j)}(z)(z^2 - e) , \qquad i, j = 1, \ldots, \ell , \tag{7.10}$$

with $B^{(i,j)}(z)$ a matrix of order $s \times s$ with $\mathcal{L}$-polynomials as elements.

Let $F_k(z) = \sum_{\alpha \in \mathbb{Z}^s} f^k_\alpha z^\alpha$, where $f^k_\alpha \in \mathbb{R}^\ell$, then

$$F_{k+1}(z) = A(z) F_k(z) , \qquad (z - e) F_{k+1}(z) = \mathcal{B}(z)(z^2 - e) F_k(z^2) , \tag{7.11}$$

with $\mathcal{B}(z) = \sum_{\alpha \in \mathbb{Z}^s} \mathcal{B}_\alpha z^\alpha$, where $\mathcal{B}_\alpha$ is a linear map from matrices of order $s \times \ell$ to matrices of order $s \times \ell$.

Rearranging the matrices $(\Delta f^k)_\alpha$, $\alpha \in \mathbb{Z}^s$, and $(z - e)F_k(z) = \sum_{\alpha \in \mathbb{Z}^s} (\Delta f^k)_\alpha z^\alpha$ into vectors, we finally get (7.8), with $\{Q_\alpha : \alpha \in \mathbb{Z}^s\}$ the matrices obtained from the corresponding rearrangement of $\{\mathcal{B}_\alpha : \alpha \in \mathbb{Z}^s\}$. ◻

Once the existence of S_Δ is established, analogous results to Theorems 6.9 and 7.2, stating sufficient conditions for the convergence of s.s. with matrix masks, and for the smoothness of the generated vector-valued functions are needed, in order to complete the analysis of higher order smoothness of multivariate s.s. with scalar masks. We state these results without the proofs, which are straightforward generalizations of the proofs given to Theorems 6.9 and 7.2, with f^k regarded as a vector rather than a scalar.

Proposition 7.4. *Let S be a s.s. with a matrix mask (7.1) satisfying (7.3), and let S_Δ be the s.s. guaranteed by Proposition 7.3. S is uniformly convergent if S_Δ converges uniformly to zero. S generates C^1-limit vector-valued functions if $S_1 = 2S_\Delta$ converges uniformly. In this case*

$$\partial S^\infty f^0 = S_1^\infty df^0 \ . \tag{7.12}$$

A repeated application of Proposition 7.4 yields the analogue of Theorem 3.4 with $\nu > 1$. For that we introduce the vectors $d^\nu f^k$ consisting of all distinct differences of order ν in the corrdinate directions. First consider the vector

$$\left\{ (d_{i_1} d_{i_2} \cdots d_{i_\nu} f^k)_\alpha : (i_1, \ldots, i_\nu) \in \{1, \ldots, s\}^\nu \right\} \in \mathbb{R}^{s^\nu} \ , \tag{7.13}$$

for all s^ν repeated differences of order ν, where $d_i = e^{(i)} \cdot d$. There are only $m_{s,\nu} = \binom{\nu+s-1}{\nu}$ distinct elements in (7.13), since $(d_{i_1} d_{i_2} \cdots d_{i_\nu} f^k)_\alpha$ is independent of the order of $(i_1, i_2, \ldots, i_\nu)$. We define $(d^\nu f^k)_\alpha$ as the vector consisting of the $m_{s,\nu}$ distinct elements in (7.13).

Theorem 7.5. *Let S be a s.s. with a scalar mask **a**. S converges uniformly to C^ν-limit functions, if there exist s.s. $S_1, \ldots, S_\nu$ with the property*

$$S_i d^i S^k f^0 = d^i S^{k+1} f^0 \ , \qquad i = 1, \ldots, s, \tag{7.14}$$

and S_ν converges uniformly.

Remark. It is sufficient to require the uniform convergence of S_ν for all initial data of the form $d^\nu f^0$. Also note that a necessary condition for the uniform convergence of S_i satisfying (7.14) is that it is non-degenerate, since $S_i^\infty d^i f^0 = \partial^i S^\infty f^0$, and no function of compact support satisfies a homogeneous linear partial differential equation with constant coefficients in $\mathbb{R}^s$. Thus in the analysis of smoothness we consider only s.s. S_ν with masks satisfying (7.3), and determine their uniform convergence by analyzing the uniform convergence to zero of $\frac{1}{2} S_{\nu+1}$.

The existence of S_1 is guaranteed by Proposition 6.8 for all masks **a** satisfying (6.2). Conditions on the mask **a** guaranteeing the existence of S_ν for $\nu > 1$ are much harder to obtain. In the case of interpolatory s.s. such conditions are easily derived, and can be easily checked for a given concrete s.s. These conditions are later shown to be necessary for convergence to C^{ν}-limit functions.

Theorem 7.6. *Let S be an interpolatory s.s. with mask* $\mathbf{a}$. *If S reproduces π_ν, the space of all s-variate polynomials of total degree $\leq \nu$, then there exist s.s. $S_1, \dots, S_{\nu+1}$ satisfying*

$$S_n d^n S^k f^0 = d^n S^{k+1} f^0 \ . \tag{7.15}$$

Proof: The existence of S_1 is guaranteed by Proposition 6.8. We show that if S_n exists for $n \leq \nu$, then S_{n+1} exists. Choose the initial data $f^0 = \{f^0_\alpha = \alpha^\beta : \alpha \in \mathbb{Z}^s\}$ for fixed $\beta \in \mathbb{Z}^n_+$, $|\beta| = n$, and observe that $(d^n f^0)_\alpha = \beta! v(\beta)$ with $v(\beta) = \left\{\delta_{\beta,\gamma} : \gamma \in \mathbb{Z}^s_+ \ , \ |\gamma| = n\right\} \in \mathbb{R}^{m_{s,n}}$, and $\beta! = \prod_{i=1}^s \beta_i!$. Then the reproduction of π_ν by S implies that $(S^k f^0)_\alpha = (2^{-k}\alpha)^\beta$, and $(d^n S^k f^0)_\alpha = \beta! v(\beta)$. Hence by (7.15) S_n reproduces the constant data $f = \left\{f_\alpha = v(\beta) : \alpha \in \mathbb{Z}^s\right\}$, and since the vectors $\left\{v(\beta) : \beta \in \mathbb{Z}^s_+, \ |\beta| = n\right\}$ consititute a basis of $\mathbb{R}^{m_{s,n}}$, S_n reprodces all constant vectors. This property of the scheme is equivalent to property (7.3) of the mask. The existence of a s.s. S_{n+1} satisfying (7.15) is now concluded from Proposition 7.3. □

In the following we state the multivariate analogue of the necessary conditions of Section 5. We omit the proofs.

Theorem 7.7. *Let S be an interpolatory s.s. with mask* $\mathbf{a}$ *which converges uniformly to C^ν-limit functions. Then S reproduces π_ν, and for $n = 1, \dots, \nu$, the scheme S_n, guaranteed by Theorem 7.6, converges unformly to $C^{\nu-n}$-limit vector-valued-functions for all initial data of the form $d^n f^0$. More concretely*

$$S_n^\infty d^n f^0 = \partial^n S^\infty f^0 \ , \qquad 1 \leq n \leq \nu \ , \tag{7.16}$$

where ∂^n consists of all the n-th order partial derivatives of $S^\infty f^0$ ordered in accordance with $d^n f^0$. Also the s.s. $\frac{1}{2} S_{\nu+1}$ converges uniformly to zero for all initial data of the form $d^{\nu+1} f^0$.

The analysis of smoothness of s.s. with scalar mask $\mathbf{a}$ and characteristic $\mathcal{L}$-polynomial of the form

$$a(z) = \prod_{i=1}^{\ell}(z^{-\theta^{(i)}} + 1)^{\nu_i} q(z) , \quad \theta^{(1)}, \ldots, \theta^{(\ell)} \in \mathbb{Z}^s , \quad q \in \mathcal{L} , \tag{7.17}$$

can be done similarly to the analysis in Section 3, without the introduction of s.s. with matrix masks, by showing the smoothness of certain directional derivatives of the limit functions. $\mathcal{L}$-polynomials of the form (7.17) are typical to s.s. for box-splines.

Theorem 7.8. *Let S be a convergent s.s. with a characteristic $\mathcal{L}$-polynomial $a(z) = (z^{-\theta} + 1)^{\nu} 2^{-\nu} q(z)$, $\theta \in \mathbb{Z}^s$, $q \in \mathcal{L}$. If the s.s. S_q with a characteristic $\mathcal{L}$-polynomial $q(z)$ converges uniformly, then for all initial data f^0*

$$\partial_\theta^\nu S^\infty f^0 \in C(\mathbb{R}^s) , \tag{7.18}$$

where $\partial_\theta f = \lim_{t \to 0} \big(f(\cdot + t\theta) - f\big)/t$.

Proof: The proof follows the lines of the proof of Theorem 3.4 (and Theorem 7.2). As in Theorem 3.4, it is sufficient to prove the case $\nu = 1$.

Let $f^k = S^k f^0$ and $(d_\theta f^k)_\alpha = 2^k(f^k_{\alpha+\theta} - f^k_\alpha)$, then by the generating functions formalism and the structure of $a(z)$

$$S_q d_\theta f^k = d_\theta f^{k+1} = d_\theta S f^k . \tag{7.19}$$

Here we show that

$$\partial_\theta S^\infty \delta = S_q^\infty d_\theta \delta , \tag{7.20}$$

proving that the S-refinable function $\varphi_a = S^\infty \delta$ satisfies $\partial_\theta \varphi_a \in C(\mathbb{R}^\nu)$, which is equivalent to (7.18) with $\nu = 1$.

Introducing an orthogonal set of vectors in $\mathbb{Z}^s$, $\theta^{(1)}, \ldots, \theta^{(s)}$, with $\theta^{(1)} = \theta$, and defining the function

$$\widetilde{\psi}(x) = \prod_{i=1}^{s} N_i(x) , \qquad N_i(x) = B_1\big(1 + (\theta^{(i)} \cdot x)/\|\theta^{(i)}\|^2\big) ,$$

we observe that $\operatorname{supp}\widetilde{\psi}(x) = \{x : -1 < x \cdot \theta^{(i)} < 1 \ , \ i = 1, \ldots, s\}$, and

$$\sum_{\alpha \in \mathbb{Z}^s} \widetilde{\psi}\big(\,\cdot - \sum_{j=1}^{s} \alpha_j \theta^{(j)}\big) = 1 \ .$$

Thus $\psi(x) = m^{-1}\widetilde{\psi}(x)$, with $m = \big|\{x \in \mathbb{Z}^s : 0 < x \cdot \theta^{(i)} < 1 \ , \ i = 1, \ldots, s\}\big|$, satisfies $\sum_{\alpha \in \mathbb{Z}^s} \psi(\cdot - \alpha) = 1$, and by Lemma 2.2 with $f^0 = \delta$,

$$\lim_{k \to \infty} f^k(x) = S^\infty f^0 = \varphi_a \ , \qquad \lim_{k \to \infty} g^k(x) = S_q^\infty d_\theta f^0 \ , \tag{7.21}$$

where

$$f^k(x) = \sum_{\alpha \in \mathbb{Z}^s} f_\alpha^k \psi(2^k x - \alpha) \ , \qquad g^k(x) = \sum_{\alpha \in \mathbb{Z}^s} (d_\theta f^k)_\alpha \psi(2^k x - \alpha) \ ,$$

and the convergence in (7.21) is uniform.

As in the proof of Theorem 3.4, we also consider the sequence of functions

$$h^k(x) = \sum_{\alpha \in \mathbb{Z}^s} (d_\theta f^k)_\alpha \psi_\theta(2^k x - \alpha) \ , \qquad \psi_\theta(x) = m^{-1} H\left(\frac{\theta \cdot x}{\|\theta\|^2}\right) \prod_{i=2}^{s} N_i(x) \ ,$$

and show that

$$\big\|g^k(x) - h^k(x)\big\|_\infty \le c \|\Delta_\theta d_\theta f^k\|_\infty \ , \qquad \Delta_\theta f = f_{\cdot + \theta} - f \ , \tag{7.22}$$

with c a constant independent of f^k.

The bound (7.22) guarantees the uniform convergence of $\big\{h^k(x)\big\}$ to $S_q^\infty d_\theta f^0$, since $\|\Delta_\theta d_\theta f^k\|_\infty \le \big(\sum_{i=1}^{s} |\theta_i|\big) \|\Delta d_\theta f^k\|_\infty$, and by Theorem 6.9 and the uniform convergence of S_q, the sequence $\|\Delta d_\theta f^k\|_\infty$ converges uniformly to zero. Observing that

$$h^k(x) = \sum_{\alpha \in \mathbb{Z}^s} f_\alpha^k 2^k \big(-\psi_\theta(2^k x - \alpha) + \psi_\theta(2^k x - \alpha + \theta)\big) \ ,$$

and

$$\int_{-\infty}^{t} \big(\psi_\theta(x + (\tau + 1)\theta) - \psi_\theta(x + \tau\theta)\big) d\tau = \psi(x + t\theta) \ ,$$

we get

$$f^k(x + t\theta) = \int_{-\infty}^{t} h^k(x + \tau\theta) d\tau \ .$$

Hence $\left\{\int_{-\infty}^{t} h^k(x+\tau\theta)d\tau\right\}_{k\geq 0}$ converges uniformly to $\varphi(x+t\theta)$ proving that $\partial_\theta\varphi = \lim_{k\to\infty} h^k = S_q^\infty d_\theta f^0 \in C(\mathbb{R}^s)$.

Thus in order to complete the proof we have to verify (7.22). Now,

$$g^k(x) - h^k(x) = \sum_{\alpha\in\mathbb{Z}^s} (d_\theta f^k)_\alpha \phi(2^k x - \alpha) ,$$

where

$$\phi(x) = m^{-1}\prod_{i=2}^{s} N_i(x)\left\{N_1(x) - H\left(\frac{\theta\cdot x}{\|\theta\|^2}\right)\right\} .$$

Since $\phi(x) = -\phi(x-\theta)$ for $0 \leq \theta\cdot x < \|\theta\|^2$, we finally get

$$g^k(x) - h^k(x) = \sum_{\alpha\in\mathbb{Z}^s} (\Delta_\theta d_\theta f^k)_\alpha \widetilde{\phi}(2^k x - \alpha) \tag{7.23}$$

with $\widetilde{\phi}(x) = -\phi(x)$ for $0 < \theta\cdot x < \|\theta\|^2$ and $\widetilde{\phi}(x) = 0$ elsewhere. Equation (7.23) implies (7.22) with $c = \left\|\sum_{\alpha\in\mathbb{Z}^s}\widetilde{\phi}(2^k x-\alpha)\right\|_\infty < \infty$. □

As a direct consequence of the theorem we obtain a sufficient condition for $S^\infty f^0 \in C^1(\mathbb{R}^s)$.

Corollary 7.9. *Let S be a s.s. with a characteristic $\mathcal{L}$-polynomial.*

$$a(z) = q(z)\prod_{i=1}^{s+1}(z^{-\theta^{(i)}}+1) , \qquad q\in\mathcal{L} , \tag{7.24}$$

with $\theta^{(1)},\ldots,\theta^{(s+1)} \in \mathbb{Z}^s$ and such that any subset of s vectors satisfies condition (6.53). If the s.s. corresponding to the $s(s+1)/2$ $\mathcal{L}$-polynomials

$$a_{ij}(z) = 2a(z)/(z^{-\theta^{(i)}}+1)(z^{-\theta^{(j)}}+1) , \qquad i\neq j , \quad i,j = 1,\ldots,s+1 ,$$

converge uniformly to zero, then S converges uniformly to $C^1(\mathbb{R}^s)$-limit functions for all initial data f^0.

Proof: By Theorem 6.9 and the last Remark of Section 6, each s.s. S_i with a characteristic $\mathcal{L}$-polynomial

$$a_i(z) = 2a(z)/(z^{-\theta^{(i)}}+1) , \qquad i = 1,\ldots,s ,$$

converges uniformly. By the same reasoning S converges uniformly, since $\frac{1}{2}S_i$, $i = 1, \ldots, s$, converge uniformly to zero. Moreover, by Theorem 7.8, $\partial_{\theta^{(i)}} S^\infty f^0 \in C(\mathbb{R}^s)$, $i = 1, \ldots, s$, and since $\theta^{(1)}, \ldots, \theta^{(s)}$ are linearly independent, $S^\infty f^0 \in C^1(\mathbb{R}^s)$.

We use Corollary 7.9 in the analysis of smoothness of the butterfly scheme considered in Section 8.

Bibliographical notes.

The sufficient conditions for smoothness of general s.s. in terms of s.s. for vectors of divided differences are stated without proof in [DL3]. The proofs follow the analysis in [DHL]. The necessary conditions for the smoothness of interpolatory s.s. are stated and proved in [DL3]. The result on smoothness of s.s. corresponding to characteristic $\mathcal{L}$-polynomials with directional factorization, is proved in [CDM] and used in [DLM]. The proof here is different following the approach of Section 3. For a discussion of s.s. for box-splines see, e.g. [CLR2], [DM], [DDL].

8. The Butterfly Subdivision Scheme

The example we present here is an extension of the 4-point interpolatory s.s. to the case of surfaces defined by control points with the topology of general triangulations. This scheme can be analyzed within the setting of the square-grid topology, with the exclusion of a fixed set of irregular points.

Figure 4. The butterfly scheme

Given a set of control points $\{p_i^k\}$ which comprise the vertices of a triangulation T^k, the scheme associates with each edge $e \in T^k$ a new point q_e^k defined according to the rule

$$q_e^k = \tfrac{1}{2}(p_{e,0}^k + p_{e,1}^k) + 2\omega(p_{e,2}^k + p_{e,3}^k) - \omega \sum_{j=4}^{7} p_{e,j}^k \ ,$$

where the locations of the points $p_{e,j}^k$ relative to the edge e in T^k are depicted in Figure 4. The configuration of points in Figure 4. suggests the name "butterfly scheme".

The butterfly scheme defines the control points at stage $k+1$ as

$$\{p_i^{k+1}\} = \{p_i^k\} \cup \{q_e^k : e \in T^k\} \ ,$$

and the triangulation T^{k+1} as the collection of edges

$$\left\{(q_e^k, p_{e,j}^k) \ , \ j = 0,1 \ , \ (q_e^k, q_{e_{ij}}^k) \ , \ i = 0,1 \ , \ j = 2,3 : e \in T^k\right\} \ ,$$

where $e_{ij} = (p_{e,i}^k, p_{e,j}^k)$. With this construction of T^{k+1}, the number of edges having p_i^k as a vertex in T^{k+1} is the same as in T^k, while each new vertex is regular, namely a vertex of six edges in T^{k+1}. Therefore, with the exclusion of the irregular points in T^0, all vertices of T^k are regular.

A triangulation with regular vertices is topologically equivalent to a three direction grid, and thus away from the irregular points, the butterfly scheme has the form (6.1). The explicit expression of the mask $\mathbf{a}_\omega$ for the choise of the three directions $\theta^{(1)} = (1,0)$, $\theta^{(2)} = (0,1)$, $\theta^{(3)} = (1,1)$ is

$$(a_\omega)_{0,0} = 1 \ ,$$

$$(a_\omega)_{1,0} = (a_\omega)_{-1,0} = (a_\omega)_{-1,-1} = (a_\omega)_{1,1} = 1/2 \ ,$$

$$(a_\omega)_{1,-1} = (a_\omega)_{-1,-2} = (a_\omega)_{1,2} = 2\omega \ ,$$

$$(a_\omega)_{1,-2} = (a_\omega)_{-3,-2} = (a_\omega)_{-1,2} = (a_\omega)_{3,2} = (a_\omega)_{-1,-3} = (a_\omega)_{1,3} = -\omega \ ,$$

$$(a_\omega)_{i,j} = (a_\omega)_{j,i} \ , \quad (i,j) \in \mathbb{Z}^2 \ ,$$

and zero otherwise.

The corresponding bivariate $\mathcal{L}$-polynomial can be put in factored form

$$a_\omega(z_1, z_2) = 2^{-1}(1 + z_1^{-1})(1 + z_2^{-1})(1 + z_1^{-1} z_2^{-1}) z_1 z_2 (1 + \omega q(z_1, z_2))$$

where

$$q(z_1,z_2) = 2z_1^{-2}z_2^{-1} + 2z_1^{-1}z_2^{-2} - 4z_1^{-1}z_2^{-1} - 4z_1^{-1} - 4z_2^{-1} + 2z_1^{-1}z_2 + 2z_1z_2^{-1}$$
$$+ 12 - 4z_1 - 4z_2 - 4z_1z_2 + 2z_1^2z_2 + 2z_1z_2^2 \ .$$

It is easy to check that the butterfly scheme maps f^k which is constant in one of the three grid directions to f^{k+1} with the same property. The scheme for such f^k reduces to the 4-point interpolatory s.s. applied along the grid lines of the other two directions. To show that this scheme converges to a surface with $C^1(\mathbb{R}^2)$ components, it suffices to demonstrate, according to Corollary 7.9, that each of the three $\mathcal{L}$-polynomials

$$(1 + z^{-\theta^{(i)}})(1 + \omega q(z))z_1z_2, \ \ i = 1, 2, 3,$$

determines a s.s. which converges uniformly to zero. By the symmetries of $q(z_1, z_2)$:

$$q(z_1, z_2) = q(z_2, z_1) = q(z_1z_2, z_1^{-1}) \ ,$$

it suffices to consider the s.s. S_r corresponding to the $\mathcal{L}$ -polynomial

$$r(z) = (1 + z_1)(1 + \omega q(z)) = \sum_{\alpha\in\mathbb{Z}^2} r_\alpha z^\alpha$$

For $n = 1$

$$\|S_r\|_\infty := \max_{0\le k,\ell\le 1}\left(\sum_{i,j\in\mathbb{Z}} \left|r_{k+2i,\ell+2j}\right|\right)$$

and since

$$\sum_{i,j\in\mathbb{Z}} \left|r_{2i,2j}\right| = |1 - 8\omega| + |8\omega|$$

$\|S_r\|_\infty \ge 1$ for all values of ω.

Considering S_r^2, we find below an interval $\omega \in (0, \omega_0)$ for which $\|S_r^2\|_\infty < 1$. An exact value of ω_0 is not computed since only linear terms in ω are considered. We expand

$$r^{[2]}(z) = r(z)r(z^2) = (1 + z_1 + z_1^2 + z_1^3)(1 - \omega q(z_1, z_2) - \omega q(z_1^2, z_2^2) + O(\omega^2))$$

and find that $r^{[2]}_{i,j} = O(\omega)$ for $j \neq 0$, while $r^{[2]}_{i,0} = 1 + O(\omega)$, $i = 0, 1, 2, 3$. Thus it is sufficient to show that for ω small enough

$$\sum_{i,j\in\mathbb{Z}} \left|r^{[2]}_{\ell+4i,4j}\right| < 1 \ , \ell = 0, 1, 2, 3.$$

For the case $\ell = 0$, all the non-zero coefficients are:

$$r^{[2]}_{0,0} = 1 - 16\omega + O(\omega^2)\ , r^{[2]}_{4,0} = 8\omega + O(\omega^2)\ , r^{[2]}_{4,4} = r^{[2]}_{0,-4} = -2\omega + O(\omega^2)\ .$$

Hence

$$\sum_{i,j\in\mathbb{Z}} \left|r^{[2]}_{4i,4j}\right| = |1 - 16\omega| + 12|\omega| + O(\omega^2) < 1$$

for $\omega > 0$ small enough. For the case $\ell = 1$, the relevant coefficients are:

$$r^{[2]}_{1,0} = 1 - 12\omega + O(\omega^2)\ , r^{[2]}_{5,0} = 4\omega + O(\omega^2)\ , r^{[2]}_{5,4} = r^{[2]}_{1,-4} = -2\omega + O(\omega^2)\ .$$

Hence

$$\sum_{i,j\in\mathbb{Z}} \left|r^{[2]}_{1+4i,4j}\right| = |1 - 12\omega| + 8|\omega| + O(\omega^2) < 1$$

for $\omega > 0$ small enough. The cases $\ell = 2$ and $\ell = 3$ give the same expressions as $\ell = 1$ and $\ell = 0$ respectively. Thus we have shown that S_r converges uniformly to zero, and the butterfly scheme converges uniformly to a surface with $C^1(\mathbb{R}^2)$ components for $\omega > 0$ small enough, away from the irregular control points.

Figure 5. A head-like surface generated by the butterfly scheme.

Local analysis of the convergence near an irregular point, similar to the analysis in [DS], and many numerical experiments with the scheme [DGL3], indicate that the generated surface is C^1 everywhere except at vertices of three edges. Figure 5 depicts an initial surface and the surface generated by four iterations of the butterfly scheme.

Bibliographical notes.

The material in this section is based on [DGL3] and [DLM]. Recently an efficient algorithm for surface-surface intersection, based on the butterfly scheme, has been developed [K].

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