Legendre-Bernstein basis transformations and their applications

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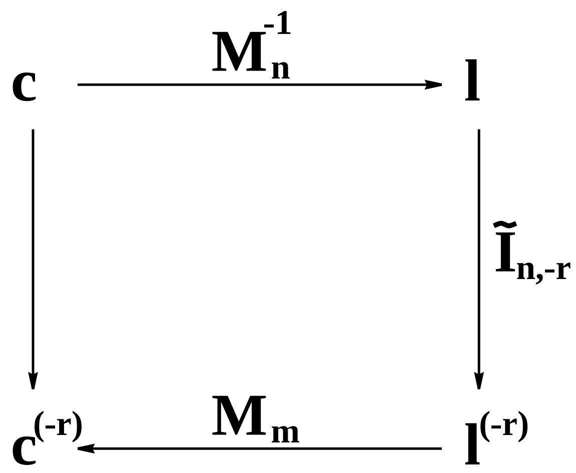
Agenda



1. Bernstein and Legendre polynomials
2. Basis Transformations
3. Relationships
4. Degree elevation and reduction

Bernstein polynomials

$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i$$

Legendre polynomials

$$L_n(t) = \sqrt{2n+1}\sum_{i=0}^{\lfloor n/2 \rfloor}\binom{n}{i,i}(t^2-t)^i(2t-1)^{n-2i}$$

$$\begin{aligned}
L_0(t) &= 1, \\
L_1(t) &= \sqrt{3}(2t-1), \\
L_2(t) &= \sqrt{5}(6t^2-6t+1), \\
L_3(t) &= \sqrt{7}(20t^3-30t^2+12t-1).
\end{aligned}$$

Polynomials of degree n and Basis Transformations

Consider a polynomial $P_n(t)$ of degree n, expressed in the degree n Bernstein and Legendre bases on $t \in [0, 1]$:

$$P_n(t) = \sum_{j=0}^{n} c_j B_j^n(t) = \sum_{k=0}^{n} l_k L_k(t).$$

We are interested in the linear transformation

$$c_j = \sum_{k=0}^{n} M_n(j,k) l_k, \qquad j = 0, \ldots, n,$$

that maps the Legendre coefficients $l_0, l_1, \ldots, l_n$ into the Bernstein coefficients $c_0, c_1, \ldots, c_n$, and its inverse.

THEOREM 1 *The Legendre polynomial* $L_k(t)$ *can be expressed in the Bernstein basis* $B_0^n(t), B_1^n(t), \ldots, B_n^n(t)$ *of degree* n *as*

$$L_k(t) = \sqrt{2k+1}\sum_{i=0}^{k}(-1)^{k+i}\binom{k}{i}B_i^k(t)$$

$$= \sqrt{2k+1}\sum_{i=0}^{k}(-1)^{k+i}\binom{k}{i}\sum_{j=i}^{n-k+i}\frac{\binom{k}{i}\binom{n-k}{j-i}}{\binom{n}{j}}B_j^n(t)$$

$$= \sum_{j=0}^{n}\frac{\sqrt{2k+1}}{\binom{n}{j}}\sum_{i=\max(0,j+k-n)}^{\min(j,k)}(-1)^{k+i}\binom{k}{i}\binom{k}{i}\binom{n-k}{j-i}B_j^n(t).$$

by Farouki, R.T.(2000), Legendre-Bernstein basis transformations, J. Comput. Appl. Math., 119, 145-160.

THEOREM 2 *The elements of the inverse* M_n^{-1} *are given for* $0 \le j, k \le n$ *by*

$$M_n^{-1}(j,k) = \frac{\sqrt{2j+1}}{n+j+1}\binom{n}{k}\sum_{i=0}^{j}(-1)^{j+i}\frac{\binom{j}{i}\binom{j}{i}}{\binom{n+j}{k+i}}$$

$$= \frac{\sqrt{2j+1}}{n+j+1}\frac{1}{\binom{n+j}{n}}\sum_{i=0}^{j}(-1)^{j+i}\binom{j}{i}\binom{k+i}{k}\binom{n-k+j-i}{n-k}.$$

by Farouki, R.T.(2000), Legendre-Bernstein basis transformations, J. Comput. Appl. Math., 119, 145-160.

Example

$$M_1 = \begin{bmatrix} 1 & -\sqrt{3} \\ 1 & \sqrt{3} \end{bmatrix}, M_1^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & -\sqrt{3} & \sqrt{5} \\ 1 & 0 & -2\sqrt{5} \\ 1 & \sqrt{3} & \sqrt{5} \end{bmatrix}, M_2^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{\sqrt{3}}{6} & 0 & \frac{\sqrt{3}}{6} \\ \frac{\sqrt{5}}{30} & -\frac{2\sqrt{5}}{30} & \frac{\sqrt{5}}{30} \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 1 & -\sqrt{3} & \sqrt{5} & -\sqrt{7} \\ 1 & -\frac{\sqrt{3}}{3} & -\sqrt{5} & 3\sqrt{7} \\ 1 & \frac{\sqrt{3}}{3} & -\sqrt{5} & -3\sqrt{7} \\ 1 & \sqrt{3} & \sqrt{5} & \sqrt{7} \end{bmatrix}, M_3^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{3\sqrt{3}}{20} & -\frac{\sqrt{3}}{20} & \frac{\sqrt{3}}{20} & \frac{3\sqrt{3}}{20} \\ \frac{\sqrt{5}}{20} & -\frac{\sqrt{5}}{20} & -\frac{\sqrt{5}}{20} & \frac{\sqrt{5}}{20} \\ -\frac{\sqrt{7}}{140} & \frac{3\sqrt{7}}{140} & -\frac{3\sqrt{7}}{140} & \frac{\sqrt{7}}{140} \end{bmatrix}$$

L_2-norm of the P_n with Bernstein bases

$$
\begin{aligned}
\|P_n\|_2{}^2 &= \int_0^1 \left| \sum_{i=0}^{n} c_i B_i^n(t) \right|^2 dt \\
&= \sum_{i,j} c_i c_j \int_0^1 B_i^n(t) B_j^n(t) dt \\
&= \frac{1}{2n+1} \sum_{i,j} c_i c_j \frac{\binom{n}{i}\binom{n}{j}}{\binom{2n}{i+j}} \\
&= c^t Q_n c.
\end{aligned}
$$

L_2-norm of the polynomial P_n is

$$\|P_n\|_2{}^2 = c^t Q_n c$$

where the elements of the Gram matrix Q_n of Bernstein basis are given by

$$Q_n(i,j) = \frac{1}{2n+1} \frac{\binom{n}{i}\binom{n}{j}}{\binom{2n}{i+j}}, \qquad i,j = 0,1,\ldots,n.$$

Q_n is real symmetric matrix, so it can be diagonalized by a unitary matrix U_n (i.e., $U_n^{-1} = U_n^t$) whose column vectors are orthonormal eigenvectors of Q_n, that is,

$$Q_n = U_n D_n U_n^t$$

where D_n is the diagonal matrix with positive eigenvalues of the matrix Q_n.

THEOREM 3

$$Q_n M_n = M_n D_n$$

where $\lambda_k = \frac{1}{2n+1}\frac{\binom{2n+1}{n-k}}{\binom{2n}{n}}(k = 0, 1, \ldots, n)$ *are eigenvalues of the Gram matrix* Q_n.

by Lyche, T. and Scherer, K.(2000), On the p-norm condition number of the multivariate triangular Bernstein basis, J. Comput. Appl. Math., 119, 259-273.

L_2 norm of the P_n with Legendre bases

$$
\begin{aligned}
{\|P_n\|_2}^2 &= \int_0^1 \left| \sum_{i=0}^{n} l_i L_i(t) \right|^2 dt \\
&= \sum_{i,j} l_i l_j \int_0^1 L_i(t) L_j(t) dt \\
&= \sum_i l_i^2 \\
&= l^t l.
\end{aligned}
$$

THEOREM 4

$$M_n^{-1} = D_n M_n^t.$$

proof.

$$c^t Q_n c = l^t l = c^t (M_n^{-1})^t M_n^{-1} c$$

$$Q_n = (M_n^{-1})^t M_n^{-1} = M_n D_n M_n^{-1}$$

THEOREM 5

$$U_n = M_n \sqrt{D}_n.$$

proof.

$$Q_n = M_n D_n D_n M_n^t = M_n \sqrt{D}_n D_n (M_n \sqrt{D}_n)^t$$

$$(M_n \sqrt{D}_n)^{-1} = \sqrt{D}_n^{\;-1} M_n^{-1} = \sqrt{D}_n M_n^t = (M_n \sqrt{D}_n)^t$$

Example

$$U_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, D_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$$

$$U_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}, D_2 = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{30} \end{bmatrix}$$

$$U_3 = \begin{bmatrix} \frac{1}{2} & -\frac{3}{\sqrt{20}} & \frac{1}{2} & -\frac{1}{\sqrt{20}} \\ \frac{1}{2} & -\frac{1}{\sqrt{20}} & -\frac{1}{2} & \frac{3}{\sqrt{20}} \\ \frac{1}{2} & \frac{1}{\sqrt{20}} & -\frac{1}{2} & -\frac{3}{\sqrt{20}} \\ \frac{1}{2} & \frac{3}{\sqrt{20}} & \frac{1}{2} & \frac{1}{\sqrt{20}} \end{bmatrix}, D_3 = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{3}{20} & 0 & 0 \\ 0 & 0 & \frac{1}{20} & 0 \\ 0 & 0 & 0 & \frac{1}{140} \end{bmatrix}$$

Degree elevation with Bernstein coefficients

$$c^{(1)} = T_n c$$

$$T_n = \frac{1}{n+1} \begin{pmatrix} n+1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & n & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & n-1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n-1 & 2 & 0 \\ 0 & 0 & 0 & \dots & 0 & n & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & n+1 \end{pmatrix}$$

$$c^{(r)} = T_{n,r} c$$

$$T_{n,r} = T_{n+r-1} T_{n+r-2} \dots T_{n+1} T_n$$

$$T_{n,r}(i,j) = \frac{\binom{n}{j}\binom{r}{i-j}}{\binom{n+r}{i}}$$

Degree elevation with Legendre coefficients

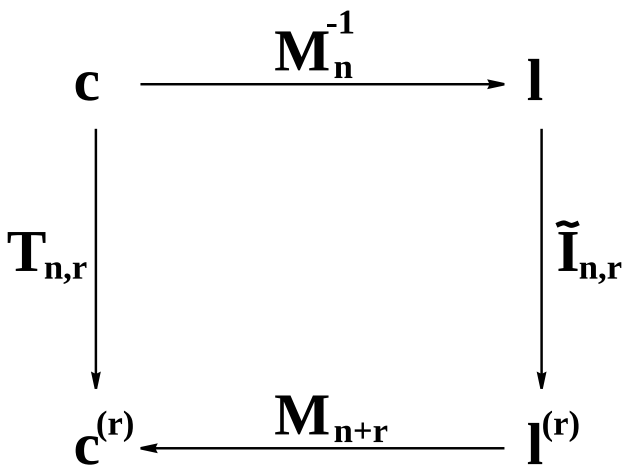
By the orthogonality of Legendre basis, the degree elevation of a polynomial with Legendre basis is given by

$$l = (l_0, l_1, \ldots, l_n)^t,$$

$$l^{(r)} = (l_0, l_1, \ldots, l_n, \underbrace{0, \ldots, 0}_{r})^t.$$

After r degree elevations, we have a linear system $\tilde{I}_{n,r} l = l^{(r)}$, where the $(n+r+1) \times (n+1)$ $\tilde{I}_{n,r}$ matrix has elements

$$\tilde{I}_{n,r}(i,j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$



THEOREM 6 *The degree elevation matrix* $T_{n,r}$ *can be expressed in* M_n^{-1}, $\tilde{I}_{n,r}$ *and* M_{n+r} *as*

$$T_{n,r} = M_{n+r}\tilde{I}_{n,r}M_n^{-1}.$$

Degree Reduction with Bernstein coefficients

PROBLEM 1 (L_2 DEGREE REDUCTION) *Let* $\{c_i\}_{i=0}^n$ *be a given set of control points which define the Bézier curve*

$$c^n(t) = \sum_{i=0}^{n} c_i B_i^n(t)$$

of degree n*. Then find another point set* $\{b_i\}_{i=0}^m$ *defining the approximative Bézier curve*

$$b^m(t) = \sum_{i=0}^{m} b_i B_i^m(t)$$

of lower degree $m < n$ *so that an* L_2*-distance function* $d_2(b^m, c^n)$ *between* b^m *and* c^n *is minimized.*

THEOREM 7 *The L_2-distance between the two Bézier curves b^m and c^n is*

$$d_2^2(b^m, c^n) = d_2^2(b^{(r)}, c^n) = A^t Q_n A$$

where $A = c - T_{m,r}b$, $b = (b_0, b_1, \ldots, b_m)^t$ *and* $c = (c_0, c_1, \ldots, c_n)^t$.

For developing the method, rewrite $d_2^2(b^m, c^n)$.

$$\begin{aligned} d_2^2(b^m, c^n) &= A^t Q_n A \\ &= [c - T_{m,r}b]^t Q_n [c - T_{m,r}b] \\ &= c^t Q_n c - 2b^t T_{m,r}^t Q_n c + b^t T_{m,r}^t Q_n T_{m,r} b. \end{aligned}$$

One method of obtaining the vector b is so-called the method of least squares. This method consists of minimizing $A^t Q_n A$ with respect to b. We choose the vector $\hat{b}$ as that the value of b minimizes $A^t Q_n A$. Equating $\partial(A^t Q_n A)/\partial b$ to zero and writing the resulting equations in terms of $\hat{b}$, we find that these equations are

$$T_{m,r}^t Q_n T_{m,r} \hat{b} = T_{m,r}^t Q_n c.$$

THEOREM 8 *The $(n+1)\times(n+1)$ matrix $T^t_{n-1}Q_nT_{n-1}$ has the following property:*

$$T^t_{n-1}Q_nT_{n-1} = Q_{n-1}.$$

Hence, the real symmetric positive definite matrix $T^t_{m,r}Q_nT_{m,r} = Q_m$ is invertible. Provided $(T^t_{m,r}Q_nT_{m,r})^{-1}$ exists, we have the unique solution for $\hat{b}$,

$$\hat{b} = (T^t_{m,r}Q_nT_{m,r})^{-1}T^t_{m,r}Q_nc.$$

This approximate curve is the best approximation with respect to the L_2-norm.

Lee, B.G. and Park, Y.(1997), Distance for Bézier curves and degree reduction, Bull. Australian Math. Soc., 56, 507-515.

Lutterkort, D., Peters, J. and Reif, U.(1999), Polynomial degree reduction in the L_2-norm equals best Euclidean approximation of Bézier coefficients, Computer Aided Geometric Design, 16, 607-612.

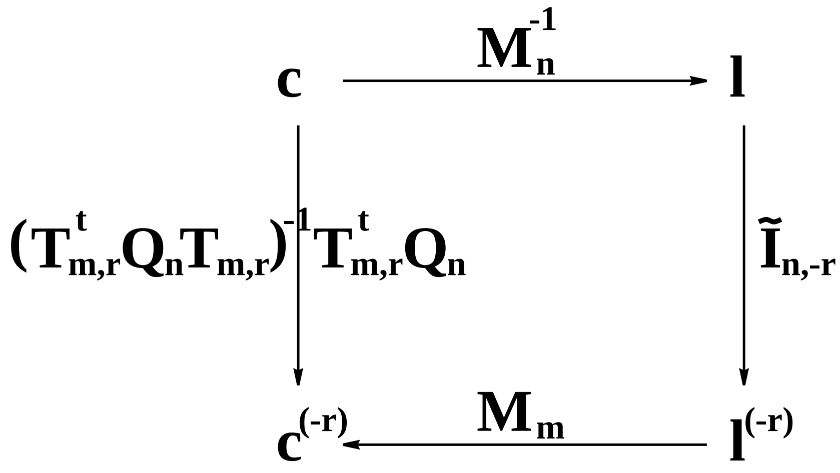
Degree Reduction with Legendre coefficients

By the orthogonality of Legendre basis, the degree reduction of a polynomial with Legendre basis is given by

$$\begin{aligned} l &= (l_0, l_1, \ldots, l_n)^t \\ l^{(-1)} &= (l_0, l_1, \ldots, l_{n-1})^t. \end{aligned}$$

After r degree reductions, we have a linear system

$$l^{(-r)} = \tilde{I}_{n,-r} l.$$



THEOREM 9 *The degree reduction matrix can be expressed in* M_m *and* M_n^{-1} *as*

$$\left(T^t_{m,r} Q_n T_{m,r}\right)^{-1} T^t_{m,r} Q_n = M_m \tilde{I}_{n,-r} M_n^{-1}.$$

Example [Parametric case, n=4]

$$c = [0, 1, 2, 1, 0]^t,$$

$$l = M_4^{-1} c = [\frac{4}{5}, 0, -\frac{6\sqrt{5}}{35}, 0, \frac{2}{105}]^t,$$

$$\|c^4\|_2{}^2 = \left(\frac{4}{5}\right)^2 + \left(\frac{6\sqrt{5}}{35}\right)^2 + \left(\frac{2}{105}\right)^2,$$

$$M_2 \tilde{I}_{4,-2} M_4^{-1} = \frac{1}{35} \begin{bmatrix} 31 & 9 & -3 & -5 & 3 \\ -13 & 17 & 27 & 17 & -13 \\ 3 & -5 & -3 & 9 & 31 \end{bmatrix},$$

$$l^{(-2)} = \tilde{I}_{4,-2} M_4^{-1} c = [\frac{4}{5}, 0, -\frac{6\sqrt{5}}{35}]^t,$$

$$c^{(-2)} = M_2 \tilde{I}_{4,-2} M_4^{-1} c = [\frac{-2}{35}, \frac{88}{35}, \frac{-2}{35}]^t,$$

$$\|c^{(-2)}\|_2{}^2 = \left(\frac{4}{5}\right)^2 + \left(\frac{6\sqrt{5}}{35}\right)^2.$$

1. For the degree reduction with Bernstein basis, we can use the explicit matrix form of $M_m \tilde{I}_{n,-r} M_n^{-1}$ instead of finding the inverse matrix of $T_{m,r}^t Q_n T_{m,r}$.

2. Our method using the relationship of transformations between Legendre and Bernstein basis is a simple and efficient method for optimal multiple degree reductions with respect to the L_2-norm.

3. This best approximation does not interpolate the given curve at its endpoints. Thus we have to consider the smoothness of our method for the practical use.

Thank you.