# Multiple degree reduction and elevation of Bézier curves using Jacobi-Bernstein basis transformations

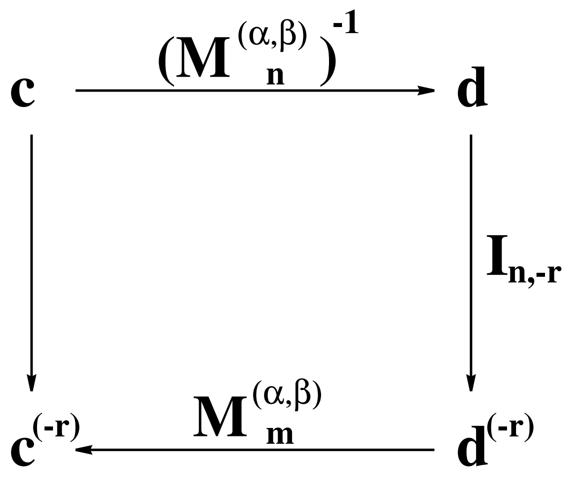
**Byung-Gook Lee Dongseo Univ.**

**Abedallah Rababah Jordan Univ.**

**Jaechil Yoo Dongeui Univ.**

**Nov 10∼13, 2003**
**SIAM Conference on Geometric Design and Computing**
**Seattle, Washington**

# Agenda




1. Bernstein and Jacobi polynomials
2. Basis Transformations
3. Relationships
4. Multiple degree elevation and reduction

## Bernstein polynomials

$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i$$

## Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = h_n^{(\alpha,\beta)} \sum_{\nu=0}^{n} \begin{pmatrix} n+\alpha \\ n-\nu \end{pmatrix} \begin{pmatrix} n+\beta \\ \nu \end{pmatrix} (x-1)^{\nu} x^{n-\nu},$$

where

$$h_n^{(\alpha,\beta)} = \sqrt{\frac{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}}.$$

## Polynomials of degree $n$ and Basis Transformations

Consider a polynomial $f(x)$ of degree $n$, expressed in the degree $n$ Bernstein and Jacobi bases on $x \in [0,1]$:

$$f(x) = \sum_{\nu=0}^{n} d_\nu P_\nu^{(\alpha,\beta)}(x) = \sum_{\mu=0}^{n} c_\mu B_\mu^n(x).$$

We are interested in the linear transformation

$$c_\mu = \sum_{\nu=0}^{n} M_n^{(\alpha,\beta)}(\mu,\nu) d_\nu, \qquad \mu = 0,\ldots,n,$$

that maps the Jacobi coefficients $d_0, d_1, \ldots, d_n$ into the Bernstein coefficients $c_0, c_1, \ldots, c_n$, and its inverse.

LEMMA 1 *The orthonormal Jacobi polynomial* $P_n^{(\alpha,\beta)}(x)$ *of degree* $n$ *has the following representation in the Bernstein basis form* $B_\nu^n(x), \nu = 0, 1, \ldots, n$ *of degree* $n$*:*

$$P_n^{(\alpha,\beta)}(x) = h_n^{(\alpha,\beta)} \sum_{\nu=0}^{n} (-1)^{n-\nu} \frac{\binom{n+\alpha}{\nu}\binom{n+\beta}{n-\nu}}{\binom{n}{\nu}} B_\nu^n(x).$$

LEMMA 2 *Let* $P_\mu^{(\alpha,\beta)}(x)$ *be the orthonormal Jacobi polynomial of degree* $\mu$*, and* $B_\nu^n(x)$ *be the Bernstein polynomial of degree* $n$*. Then for all* $\mu, \nu = 0, 1, \ldots, n$ *the following integral has the value*

$$\begin{aligned} & \int_0^1 w(x) B_\nu^n(x) P_\mu^{(\alpha,\beta)}(x) dx \\ = \quad & 2^{\alpha+\beta} h_n^{(\alpha,\beta)} \binom{n}{\nu} \sum_{i=0}^{\mu} (-1)^{\mu-i} \binom{\mu+\alpha}{i} \binom{\mu+\beta}{\mu-i} b_n^{(\alpha,\beta)}(\mu-\nu-i, \nu+i), \end{aligned}$$

*where* $b_n^{(\alpha,\beta)}(x, y) = B(n + \alpha + x + 1, \beta + y + 1)$.

THEOREM 3 *The entries* $M_n^{(\alpha,\beta)}(\mu,\nu), \mu,\nu = 0,1,\ldots,n$ *of the matrix of transformation of Bernstein polynomial basis into Jacobi polynomial basis of degree* $n$ *are given by*

$$M_n^{(\alpha,\beta)}(\mu,\nu) = \frac{h_\nu^{(\alpha,\beta)}}{\binom{n}{\mu}} \sum_{i=max(0,\mu+\nu-n)}^{min(\mu,\nu)} (-1)^{\nu-i} \binom{n-\nu}{\mu-i} \binom{\nu+\alpha}{i} \binom{\nu+\beta}{\nu-i}.$$

THEOREM 4 *The elements* $(M_n^{(\alpha,\beta)})^{-1}(\mu,\nu), \mu,\nu = 0,1,\ldots,n$ *of the matrix of transformation of the orthonormal Jacobi polynomial basis into the Bernstein polynomial basis of degree* $n$ *are given by*

$$\begin{aligned} &(M_n^{(\alpha,\beta)})^{-1}(\mu,\nu) \\ = \quad & 2^{\alpha+\beta} h_\mu^{(\alpha,\beta)} \binom{n}{\nu} \sum_{i=0}^{\mu} (-1)^{\mu-i} \binom{\mu+\alpha}{i} \binom{\mu+\beta}{\mu-i} b_n^{(\alpha,\beta)}(\mu-\nu-i,\nu+i). \end{aligned}$$

# Weighted $L_2$-norm of the $f(x)$ with Bernstein bases

$$
\begin{aligned}
\|f\|_w^2 &= \int_0^1 w(x) \left| \sum_{i=0}^{n} c_i B_i^n(x) \right|^2 dx \\
&= \sum_{i,j} c_i c_j \int_0^1 2^{\alpha+\beta} (1-x)^\alpha x^\beta B_i^n(x) B_j^n(x) dx \\
&= \sum_{i,j} c_i c_j \int_0^1 2^{\alpha+\beta} (1-x)^\alpha x^\beta \binom{n}{i} (1-x)^{n-i} x^i \binom{n}{j} (1-x)^{n-j} x^j dx \\
&= \sum_{i,j} c_i c_j 2^{\alpha+\beta} \binom{n}{i} \binom{n}{j} b_n^{(\alpha,\beta)}(n-i-j, i+j).
\end{aligned}
$$

where $b_n^{(\alpha,\beta)}(x,y) = B(n+\alpha+x+1, \beta+y+1)$.

Weighted $L_2$-norm of the polynomial $f(x)$ is

$$\|f\|_w^2 = c^t Q_n^{(\alpha,\beta)} c.$$

where the elements of the Gram matrix $Q_n^{(\alpha,\beta)}$ of Bernstein basis are given by

$$Q_n^{(\alpha,\beta)}(i,j) = 2^{\alpha+\beta}\binom{n}{i}\binom{n}{j} b_n^{(\alpha,\beta)}(n-i-j, i+j), \qquad i,j = 0,1,\ldots,n.$$

$Q_n^{(\alpha,\beta)}$ is real symmetric matrix, so there is a similar matrix $D_n^{(\alpha,\beta)}$ to the Gram matrix $Q_n^{(\alpha,\beta)}$ with $M_n^{(\alpha,\beta)}$ and can be written in the form,

$$D_n^{(\alpha,\beta)} = (M_n^{(\alpha,\beta)})^{-1} Q_n^{(\alpha,\beta)} M_n^{(\alpha,\beta)}.$$

# $L_2$-norm of the $f(x)$ with Jacobi bases

$$\begin{aligned} \|f\|_w^2 &= \int_0^1 w(x) \left| \sum_{i=0}^{n} d_i P_i^{(\alpha,\beta)}(x) \right|^2 dx \\ &= \sum_{i,j} d_i d_j \int_0^1 w(x) P_i^{(\alpha,\beta)}(x) P_j^{(\alpha,\beta)}(x) dx = d^t d. \end{aligned}$$

THEOREM 5 *The matrix of transformation from the Bernstein basis into the orthonormal Jacobi basis can be computed by*

$$(M_n^{(\alpha,\beta)})^{-1} = D_n^{(\alpha,\beta)} (M_n^{(\alpha,\beta)})^t.$$

# Degree elevation with Bernstein coefficients

$$c^{(1)} = T_n c$$

$$T_n = \frac{1}{n+1}\begin{pmatrix} n+1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & n & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & n-1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n-1 & 2 & 0 \\ 0 & 0 & 0 & \dots & 0 & n & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & n+1 \end{pmatrix}$$

$$c^{(r)} = T_{n,r} c$$

$$T_{n,r} = T_{n+r-1} T_{n+r-2} \dots T_{n+1} T_n$$

$$T_{n,r}(i,j) = \frac{\binom{n}{j}\binom{r}{i-j}}{\binom{n+r}{i}}$$

## Degree elevation with Jacobi coefficients

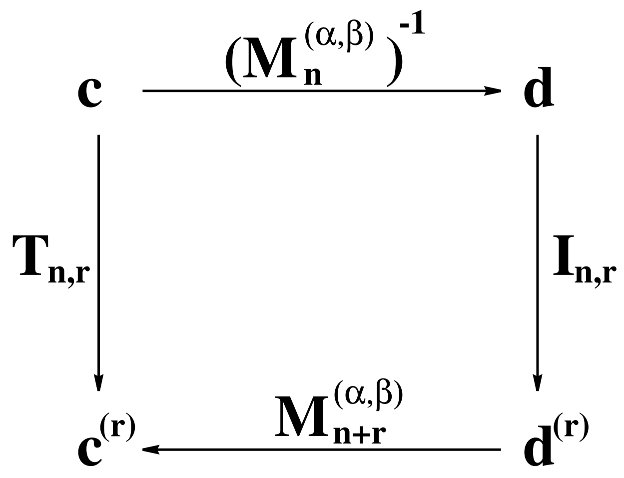
By the orthogonality of Jacobi basis, the degree elevation of a polynomial with Jacobi basis is given by

$$d = (d_0, d_1, \ldots, d_n)^t,$$

$$d^{(r)} = (d_0, d_1, \ldots, d_n, \underbrace{0, \ldots, 0}_{r})^t.$$

After $r$ degree elevations, we have a linear system $I_{n,r} l = l^{(r)}$, where the $(n+r+1) \times (n+1)$ $I_{n,r}$ matrix has elements

$$I_{n,r}(i,j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$




THEOREM 6 *The r times degree elevation matrix* $T_{n,r}$ *is obtained by multiplying the matrices* $(M_n^{(\alpha,\beta)})^{-1}$, $I_{n,r}$ *and* $M_{n+r}^{(\alpha,\beta)}$ *as follows*

$$T_{n,r} = M_{n+r}^{(\alpha,\beta)} I_{n,r} (M_n^{(\alpha,\beta)})^{-1}.$$

# Degree Reduction with Bernstein coefficients

PROBLEM 1 (WEIGHTED $L_2$ DEGREE REDUCTION) *Let* $\{c_i\}_{i=0}^n$ *be a given set of control points which define the Bézier curve*

$$c^n(t) = \sum_{i=0}^{n} c_i B_i^n(t)$$

*of degree* $n$. *Then find another point set* $\{b_i\}_{i=0}^m$ *defining the approximative Bézier curve*

$$b^m(t) = \sum_{i=0}^{m} b_i B_i^m(t)$$

*of lower degree* $m < n$ *so that the weighted* $L_2$*-norm distance between* $b^m$ *and* $c^n$ *is minimum.*

THEOREM 7 *The weighted $L_2$-distance between the Bézier curves $c^n$ and $b^m$ is given by*

$$\|b^m - c^n\|_w^2 = \|b^{(r)} - c^n\|_w^2 = A^t Q_n^{(\alpha,\beta)} A,$$

*where* $A := c - T_{m,r}b$, $b = (b_0, b_1, \ldots, b_m)^t$ *and* $c = (c_0, c_1, \ldots, c_n)^t$.

To simplify the last result of $\|b^m - c^n\|_w^2$, we substitute the value $A = c - T_{m,r}b$ to get

$$\|b^m - c^n\|_w^2 = c^t Q_n^{(\alpha,\beta)} c - 2b^t T_{m,r}^t Q_n^{(\alpha,\beta)} c + b^t T_{m,r}^t Q_n^{(\alpha,\beta)} T_{m,r} b. \qquad (1)$$

The error in the last equation is a function of the elements of the vector $b$. The minimum $\hat{b}$ in the sense of the least squares method occurs when the first partial derivatives $\partial(A^t Q_n^{(\alpha,\beta)} A)/\partial b$ are equal to zero. Thus, we get the following normal equations

$$T_{m,r}^t Q_n^{(\alpha,\beta)} T_{m,r} \hat{b} = T_{m,r}^t Q_n^{(\alpha,\beta)} c.$$

The matrix $T_{m,r}^t Q_n^{(\alpha,\beta)} T_{m,r}$ of dimension $(m+r+1) \times (m+1)$ is invertible, because $Q_{n-r}^{(\alpha,\beta)}$ is invertible, and $T_{m,r}^t Q_n^{(\alpha,\beta)} T_{m,r} = Q_{n-r}^{(\alpha,\beta)}$. And thus it follows that the normal equations are uniquely solvable and the solution is given by

$$\hat{b} = (T_{m,r}^t Q_n^{(\alpha,\beta)} T_{m,r})^{-1} T_{m,r}^t Q_n^{(\alpha,\beta)} c.$$

The vector $\hat{b}$ of Bézier points describes the Bézier curve of best approximation in the least-squares sense with respect to the weighted $L_2$-norm.

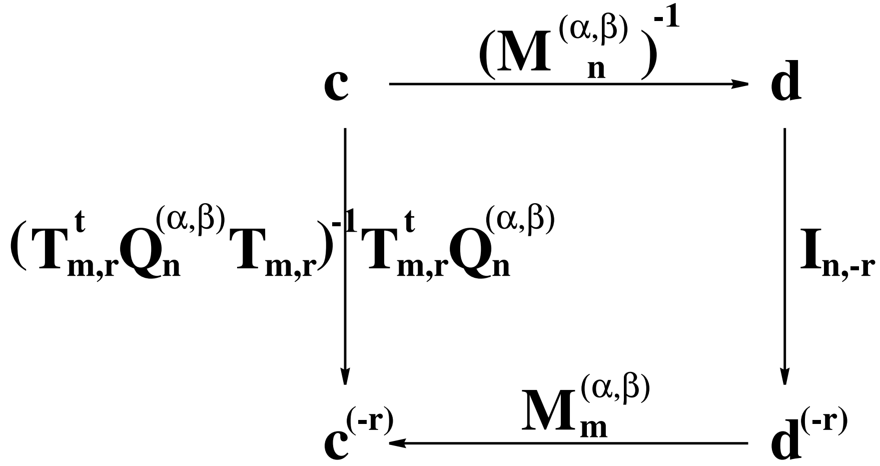
## Degree Reduction with Jacobi coefficients

By the orthogonality of Jacobi basis, the degree reduction of a polynomial with Jacobi basis is given by

$$\begin{aligned} d &= (d_0, d_1, \ldots, d_n)^t \\ d^{(-1)} &= (d_0, d_1, \ldots, d_{n-1})^t. \end{aligned}$$

After $r$ degree reductions, we have a linear system

$$d^{(-r)} = I_{n,-r} d.$$




THEOREM 8 *The degree reduction matrix can be expressed in* $(M_n^{(\alpha,\beta)})^{-1}$ , $I_{n,-r}$ *and* $M_m^{(\alpha,\beta)}$ *as*

$$R_{m,r}^{(\alpha,\beta)} = \left(T_{m,r}^t Q_n^{(\alpha,\beta)} T_{m,r}\right)^{-1} T_{m,r}^t Q_n^{(\alpha,\beta)} = M_m^{(\alpha,\beta)} I_{n,-r} (M_n^{(\alpha,\beta)})^{-1}.$$

## Approximation Error

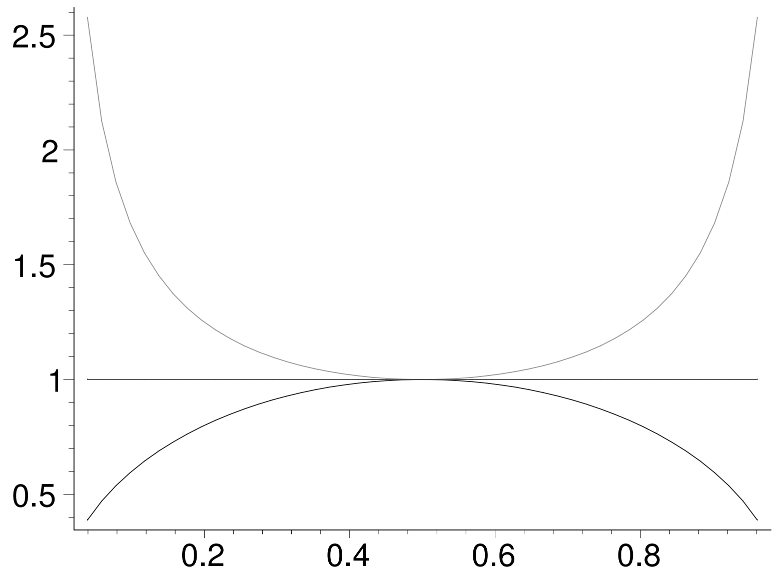
THEOREM 9 *The error of the solution $\hat{b}$ of the $r$-times degree reduction with respect to the weighted $L_2$-norm is*

$$\epsilon_w^2 = c^t E_{m,r}^{(\alpha,\beta)} c$$

*where*

$$E_{m,r}^{(\alpha,\beta)} = Q_n^{(\alpha,\beta)} \left[ I - T_{m,r} \left( T_{m,r}^t Q_n^{(\alpha,\beta)} T_{m,r} \right)^{-1} T_{m,r}^t Q_n^{(\alpha,\beta)} \right]$$

# weight function $w(x) = (2-2x)^\alpha(2x)^\beta$



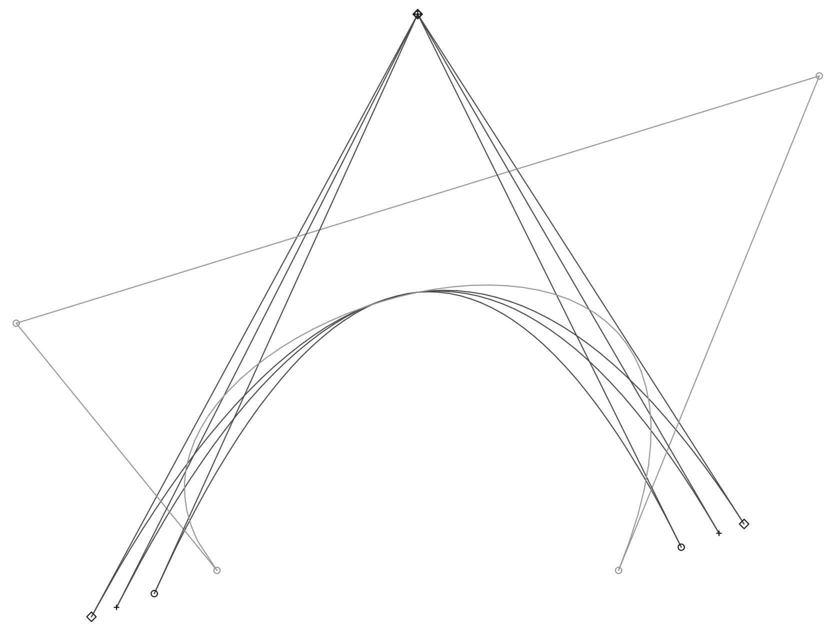

$$\alpha = \beta = -\frac{1}{2},$$
$$\alpha = \beta = 0,$$
$$\alpha = \beta = \frac{1}{2}.$$

EXAMPLE 1 (N=3, R=1, M=2) *The matrices of degree reduction* $R_{m,r}^{(\alpha,\beta)}$ *and error* $E_{m,r}^{(\alpha,\beta)}$ *are computed for the reduction of a polynomial of degree* 3 *into polynomials of degree* 2 *for the values* $\alpha = \beta = -1/2$, $\alpha = \beta = 0$, *and* $\alpha = \beta = 1/2$.

$$R_{2,1}^{(-\frac{1}{2},-\frac{1}{2})} = \begin{bmatrix} \frac{31}{32} & \frac{3}{32} & -\frac{3}{32} & \frac{1}{32} \\ -\frac{1}{4} & \frac{3}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{32} & -\frac{3}{32} & \frac{3}{32} & \frac{31}{32} \end{bmatrix}, E_{2,1}^{(-\frac{1}{2},-\frac{1}{2})} = \frac{5\pi}{1024} \begin{bmatrix} \frac{1}{20} & -\frac{3}{20} & \frac{3}{20} & -\frac{1}{20} \\ -\frac{3}{20} & \frac{9}{20} & -\frac{9}{20} & \frac{3}{20} \\ \frac{3}{20} & -\frac{9}{20} & \frac{9}{20} & -\frac{3}{20} \\ -\frac{1}{20} & \frac{3}{20} & -\frac{3}{20} & \frac{1}{20} \end{bmatrix},$$

$$R_{2,1}^{(0,0)} = \begin{bmatrix} \frac{19}{20} & \frac{3}{20} & -\frac{3}{20} & \frac{1}{20} \\ -\frac{1}{4} & \frac{3}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{20} & -\frac{3}{20} & \frac{3}{20} & \frac{19}{20} \end{bmatrix}, E_{2,1}^{(0,0)} = \frac{1}{140} \begin{bmatrix} \frac{1}{20} & -\frac{3}{20} & \frac{3}{20} & -\frac{1}{20} \\ -\frac{3}{20} & \frac{9}{20} & -\frac{9}{20} & \frac{3}{20} \\ \frac{3}{20} & -\frac{9}{20} & \frac{9}{20} & -\frac{3}{20} \\ -\frac{1}{20} & \frac{3}{20} & -\frac{3}{20} & \frac{1}{20} \end{bmatrix},$$

$$R_{2,1}^{(\frac{1}{2},\frac{1}{2})} = \begin{bmatrix} \frac{15}{16} & \frac{3}{16} & -\frac{3}{16} & \frac{1}{16} \\ -\frac{1}{4} & \frac{3}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{16} & -\frac{3}{16} & \frac{3}{16} & \frac{15}{16} \end{bmatrix}, E_{2,1}^{(\frac{1}{2},\frac{1}{2})} = \frac{5\pi}{4096} \begin{bmatrix} \frac{1}{20} & -\frac{3}{20} & \frac{3}{20} & -\frac{1}{20} \\ -\frac{3}{20} & \frac{9}{20} & -\frac{9}{20} & \frac{3}{20} \\ \frac{3}{20} & -\frac{9}{20} & \frac{9}{20} & -\frac{3}{20} \\ -\frac{1}{20} & \frac{3}{20} & -\frac{3}{20} & \frac{1}{20} \end{bmatrix}.$$



$$\circ \quad : \quad \alpha = \beta = -\frac{1}{2},$$
$$+ \quad : \quad \alpha = \beta = 0,$$
$$\diamond \quad : \quad \alpha = \beta = \frac{1}{2}.$$

EXAMPLE 2 (N=4, R=1, M=3) *We compute the matrices of degree reduction* $R_{m,r}^{(\alpha,\beta)}$ *and error* $E_{m,r}^{(\alpha,\beta)}$ *for the reduction of a polynomial of degree* 4 *into a polynomial of degree* 3 *using Theorem 8 for the values* $\alpha = \beta = -1$, $\alpha = \beta = -1/2$, $\alpha = \beta = 0$, $\alpha = \beta = 1/2$.

$$R_{3,1}^{(-\frac{1}{2},-\frac{1}{2})} = \begin{bmatrix} \frac{127}{128} & \frac{1}{32} & -\frac{3}{64} & \frac{1}{32} & -\frac{1}{128} \\ -\frac{33}{128} & \frac{33}{32} & \frac{29}{64} & -\frac{29}{96} & \frac{29}{384} \\ \frac{29}{384} & -\frac{29}{96} & \frac{29}{64} & \frac{33}{32} & -\frac{33}{128} \\ -\frac{1}{128} & \frac{1}{32} & -\frac{3}{64} & \frac{1}{32} & \frac{127}{128} \end{bmatrix},$$

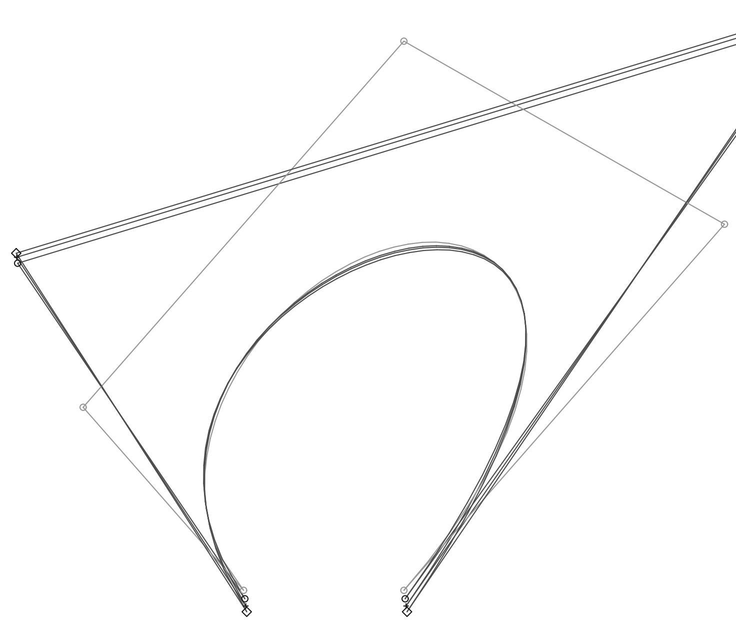
$$E_{3,1}^{(-\frac{1}{2},-\frac{1}{2})} = \frac{35\pi}{32768} \begin{bmatrix} \frac{1}{70} & -\frac{2}{35} & \frac{3}{35} & -\frac{2}{35} & \frac{1}{70} \\ -\frac{2}{35} & \frac{8}{35} & -\frac{12}{35} & \frac{8}{35} & -\frac{2}{35} \\ \frac{3}{35} & -\frac{12}{35} & \frac{18}{35} & -\frac{12}{35} & \frac{3}{35} \\ -\frac{2}{35} & \frac{8}{35} & -\frac{12}{35} & \frac{8}{35} & -\frac{2}{35} \\ \frac{1}{70} & -\frac{2}{35} & \frac{3}{35} & -\frac{2}{35} & \frac{1}{70} \end{bmatrix},$$

$$R_{3,1}^{(0,0)} = \begin{bmatrix} \frac{69}{70} & \frac{2}{35} & -\frac{3}{35} & \frac{2}{35} & -\frac{1}{70} \\ -\frac{53}{210} & \frac{106}{105} & \frac{17}{35} & -\frac{34}{105} & \frac{17}{210} \\ \frac{17}{210} & -\frac{34}{105} & \frac{17}{35} & \frac{106}{105} & -\frac{53}{210} \\ -\frac{1}{70} & \frac{2}{35} & -\frac{3}{35} & \frac{2}{35} & \frac{69}{70} \end{bmatrix},$$

$$E_{3,1}^{(0,0)} = \frac{1}{630} \begin{bmatrix} \frac{1}{70} & -\frac{2}{35} & \frac{3}{35} & -\frac{2}{35} & \frac{1}{70} \\ -\frac{2}{35} & \frac{8}{35} & -\frac{12}{35} & \frac{8}{35} & -\frac{2}{35} \\ \frac{3}{35} & -\frac{12}{35} & \frac{18}{35} & -\frac{12}{35} & \frac{3}{35} \\ -\frac{2}{35} & \frac{8}{35} & -\frac{12}{35} & \frac{8}{35} & -\frac{2}{35} \\ \frac{1}{70} & -\frac{2}{35} & \frac{3}{35} & -\frac{2}{35} & \frac{1}{70} \end{bmatrix},$$

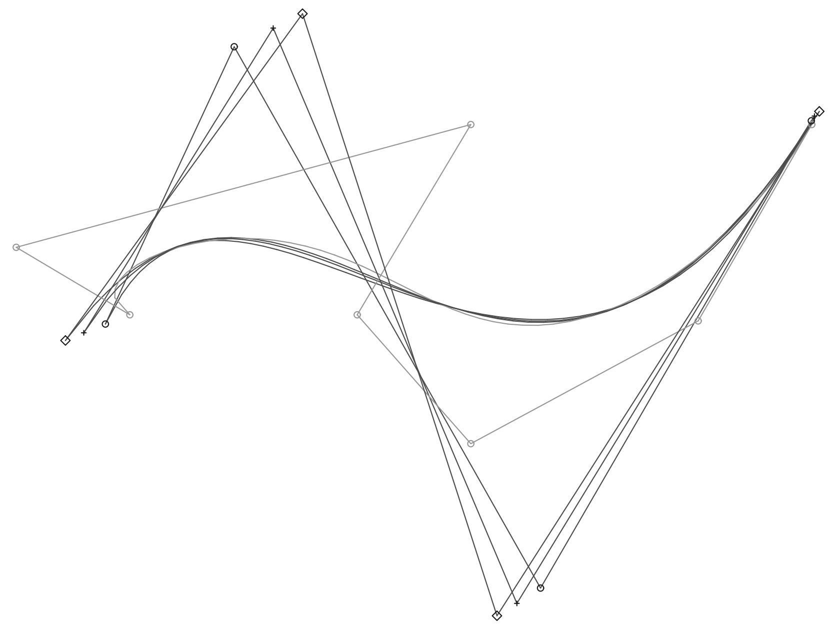
$$R_{3,1}^{(\frac{1}{2},\frac{1}{2})} = \begin{bmatrix} \frac{251}{256} & \frac{5}{64} & -\frac{15}{128} & \frac{5}{64} & -\frac{5}{256} \\ -\frac{191}{768} & \frac{191}{192} & \frac{65}{128} & -\frac{65}{192} & \frac{65}{768} \\ \frac{65}{768} & -\frac{65}{192} & \frac{65}{128} & \frac{191}{192} & -\frac{191}{768} \\ -\frac{5}{256} & \frac{5}{64} & -\frac{15}{128} & \frac{5}{64} & \frac{251}{256} \end{bmatrix},$$

$$E_{3,1}^{(\frac{1}{2},\frac{1}{2})} = \frac{35\pi}{131072} \begin{bmatrix} \frac{1}{70} & -\frac{2}{35} & \frac{3}{35} & -\frac{2}{35} & \frac{1}{70} \\ -\frac{2}{35} & \frac{8}{35} & -\frac{12}{35} & \frac{8}{35} & -\frac{2}{35} \\ \frac{3}{35} & -\frac{12}{35} & \frac{18}{35} & -\frac{12}{35} & \frac{3}{35} \\ -\frac{2}{35} & \frac{8}{35} & -\frac{12}{35} & \frac{8}{35} & -\frac{2}{35} \\ \frac{1}{70} & -\frac{2}{35} & \frac{3}{35} & -\frac{2}{35} & \frac{1}{70} \end{bmatrix}.$$



$$\circ \; : \; \alpha = \beta = -\frac{1}{2},$$
$$+ \; : \; \alpha = \beta = 0,$$
$$\diamond \; : \; \alpha = \beta = \frac{1}{2}.$$

EXAMPLE 3 (N=6, R=3, M=3) *The* 3*-times degree reduction of the Bézier curve of degree* 6 *to the approximating curves of degree* 3*.*



$$\circ : \alpha = \beta = -\frac{1}{2},$$
$$+ : \alpha = \beta = 0,$$
$$\diamond : \alpha = \beta = \frac{1}{2}.$$

1. For the degree reduction with Bernstein basis, we can use the explicit matrix form of $M_m^{(\alpha,\beta)} I_{n,-r} (M_n^{(\alpha,\beta)})^{-1}$ instead of finding the inverse matrix of $T_{m,r}^t Q_n^{(\alpha,\beta)} T_{m,r}$.

2. Our method using the relationship of transformations between Jacobi and Bernstein basis is a simple and efficient method for optimal multiple degree reductions with respect to the weighted $L_2$-norm.

3. This best approximation does not interpolate the given curve at its endpoints. Thus we have to consider the smoothness of our method for the practical use.

*Thank you.*