

Constrained polynomial degree reduction in the L_2 -norm
equals best weighted Euclidean approximation of Bézier coefficients

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- Eck, M., 1995. Least squares degree reduction of Bézier curves, Computer-Aided Design 27, 845-851.
- Lutterkort, D., Peters, J., Reif, U., 1999. Polynomial degree reduction in the L_2 -norm equals best Euclidean approximation of Bézier coefficients, Computer Aided Geometric Design 16, 607-612.
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Constrained Legendre Polynomials

It is well known that the best approximation of a given Bézier curve $f(t)$ of degree n by the Bézier curve of degree $n - 1$ in L_2 -norm is

$$\bar{f}(t) = f(t) - a_n l_n(2t - 1), \quad t \in [0, 1]$$

where a_n is the leading coefficient of $f(t)$ and

$$l_n(t) = \sum_{i=0}^n (-1)^{n+i} \binom{n}{i} B_i^n \left(\frac{t+1}{2} \right)$$

is the Legendre polynomial of degree n .

However, the best approximation $\bar{f}(t)$ does not agree with $f(t)$ for $t = 0$ and 1 .

Constrained Legendre Polynomials

Eck(1995) extends the concept in such a way that the first $\alpha - 1$ derivatives of the two curves $f(t)$ and $\bar{f}(t)$ at $t = 0$ and $t = 1$ agree for the integer $\alpha \geq 0$. Then the best

approximation of given Bézier curve $f(t)$ of degree n by the Bézier curve of degree $n - 1$

in L_2 -norm satisfying

$$\frac{d^i f}{dt^i}(t) = \frac{d^i \bar{f}}{dt^i}(t) \quad i = 0, \dots, \alpha - 1$$

at $t = 0$ and $t = 1$ is

$$\bar{f}(t) = f(t) - a_\alpha l_\alpha^n(2t - 1), \quad t \in [0, 1]$$

where $l_\alpha^n(t)$ is called by *constrained Legendre polynomial* and has the Bernstein-Bézier

representation

$$l_\alpha^n(t) = \sum_{i=\alpha}^{n-\alpha} (-1)^{n+i} \binom{n}{i} w_i B_n^i\left(\frac{t+1}{2}\right)$$

with the weights

$$w_i = \frac{\binom{n}{i} \binom{i-\alpha}{n} \binom{i+\alpha}{n}}{\binom{n}{i}^2}$$

Here a_α must be chosen so that $\bar{f}(t)$ is a polynomial of degree $n - 1$.

Equivalence of Orthogonal Complements

Let \mathbf{P}_n be the linear space of polynomials of degree less than or equal to n . Also for $m > n$ and $\alpha = 0, \dots, [m/2] + 1$, let

$$\mathbf{P}_\alpha^m = \{f(t) \in \mathbf{P}_m : \frac{d^\alpha f}{dt^\alpha}(t) = 0 \text{ at } t = 0, 1 \text{ for } i = 0, \dots, \alpha - 1\},$$

$$\mathbf{Q}_\alpha^m = \{f(t) \in \mathbf{P}_m : f(i) = 0 \text{ for } i = 0, \dots, \alpha - 1 \text{ and } i = n - \alpha + 1, \dots, n\}.$$

Note that $\mathbf{P}_0^m = \mathbf{Q}_0^m = \mathbf{P}_m$. Let B_n and \hat{Q}_n be the row vectors of Bernstein

polynomials and Lagrange polynomials

$$B_n := [B_n^0, \dots, B_n^n], \text{ where } B_n^i(t) := \binom{n}{i} t^i (1-t)^{n-i},$$

$$\hat{Q}_n := [Q_n^0, \dots, Q_n^n], \text{ where } Q_n^i(t) := \prod_{\substack{j=0, j \neq i \\ j=n}} \frac{t-j}{i-j}.$$

With $b \in \mathbf{R}^{n+1}$, a column vector of coefficients, we write polynomials in BB form and Lagrange form as $B_n b$ and $\hat{Q}_n b$, respectively.

The followings are the extensions of Lemma 2.1 and Theorem 3.1 in Lutterkort et al. (CAGD 16, 607-612, 1999).

LEMMA 1 A polynomial $B^n b$ is of degree $\leq m$ with $b(i) = 0$ for $i = 0, \dots, \alpha - 1$ and $i = n - \alpha + 1, \dots, n$ if and only if the vector of coefficients is a polynomial of degree $\leq m$ with zeros at $i = 0, \dots, \alpha - 1$ and $i = n - \alpha + 1, \dots, n$ in its index, i.e.,

$$B^n b \in \mathbf{P}_\alpha^m \Leftrightarrow Q^n b \in \mathbf{Q}_m^m.$$

THEOREM 2 The orthogonal complements of \mathbf{P}_α^m in \mathbf{P}_α^n ($m < n$) with respect to the L_2 -inner product

$$(1) \quad \langle f, g \rangle_L := \int_1^0 f(t)g(t)dt$$

and the weighted Euclidean inner product of the BB coefficients

$$(2) \quad \langle B^n b, B^n c \rangle_w := \sum_{i=\alpha}^{n-\alpha} b_i c_i w_i$$

are equal, where $w_i = \frac{\binom{n}{i} \binom{n-\alpha}{i+\alpha}}{\binom{n}{2}}$.

Constrained Degree Reduction

COROLLARY 3 Given a polynomial $B^n b$ of degree n , the approximation problem

$$\min_{p \in \mathbf{P}^m} \{ \|B^n b - p\| : \frac{d^i}{dt^i} B^n(t)b = \frac{d^i}{dt^i} p(t) \text{ at } t = 0, 1 \text{ for } i = 0, \dots, \alpha - 1 \}$$

has the same minimizer for the norm induced either by the L_2 -inner product (1) or the weighted Euclidean inner product (2).

COROLLARY 4 Denote by $P_{m,n}^\alpha$ the linear operator mapping polynomials $B^n b \in \mathbf{P}^n$ to their best constrained L_2 -norm or weighted Euclidean approximant $p \in \mathbf{P}^m$. Then

$$P_{m,n}^\alpha = P_{m,\ell}^\alpha P_{\ell,n}^\alpha \quad m \leq \ell \leq n.$$

Asymmetric Constrained Degree Reduction

We extend the results in above sections to the case of asymmetric constraint.
 For the nonnegative integers α and β satisfying $\alpha + \beta \leq m + 1$, let

$$\mathbf{P}_{\alpha, \beta}^m = \{f(t) \in \mathbf{P}^m : \frac{d^i f}{dt^i}(0) = 0 \text{ for } i = 0, \dots, \alpha - 1, \text{ and } \frac{d^i f}{dt^i}(1) = 0 \text{ for } i = 0, \dots, \beta - 1\}$$

$$\mathbf{Q}_{\alpha, \beta}^m = \{f(t) \in \mathbf{P}^m : f(i) = 0 \text{ for } i = 0, \dots, \alpha - 1 \text{ and } i = n - \beta + 1, \dots, n\}.$$

The following lemma and corollaries are obtained from Lemma 1 and Theorem 2.

Asymmetric Constrained Degree Reduction

LEMMA 5

$$B_n b \in \mathbf{P}_{\alpha, \beta}^m \Leftrightarrow Q_n b \in \mathbf{Q}_{\alpha, \beta}^m.$$

COROLLARY 6 The orthogonal complements of $\mathbf{P}_{\alpha, \beta}^m$ in $\mathbf{P}_{\alpha, \beta}^n$ with respect to the L_2 -inner product and the weighted Euclidean inner product of the BB coefficients

$$(3) \quad \langle B_n b, B_n c \rangle_w := \sum_{i=\beta}^{n-\alpha} b_i c_i w_i$$

are equal, where

$$(4) \quad w_i = \frac{\binom{n}{i-\alpha} \binom{n}{i+\beta}}{\binom{n}{i}^2}.$$

Asymmetric Constrained Degree Reduction

COROLLARY 7 Given a polynomial $B^n b$ of degree n , the approximation problem

$$\min_{p \in \mathbf{P}^m} \{ \|B^n b - p\| : \frac{d^i}{dt^i} B^n(t)b = \frac{d^i}{dt^i} p(t) \text{ at } t = 0 \text{ for } i = 0, \dots, \alpha - 1, \}$$

and at $t = 1$ for $i = 0, \dots, \beta - 1$ }

has the same minimizer for the norm induced either by the L_2 -inner product (1) or the weighted Euclidean inner product (3).

COROLLARY 8 Denote by $P_{\alpha, \beta}^{m, n}$ the linear operator mapping polynomials $B^n b \in \mathbf{P}^n$

to their best constrained L_2 -norm or the weighted Euclidean approximant $p \in \mathbf{P}^m$.

Then

$$P_{\alpha, \beta}^{m, n} = P_{\alpha, \beta}^{m, \ell} P_{\alpha, \beta}^{\ell, n}, \quad m \leq \ell \leq n.$$

Examples

In practice, one is often interested in the BB form $p = B^m c$ of the constrained best degree reduction from the polynomial $B^n b$. In order to compare coefficients, p has to be represented in terms of B^n , i.e., $p = B^n c^{(r)}$. The degree raising $(n+1) \times (m+1)$ matrix $T_{m,r}$ for mapping the BB coefficients c to $c^{(r)}$ has elements

$$T_{m,r}(i, j) = \frac{\binom{m}{r} \binom{j}{i-j}}{\binom{m+r}{i}} , \quad i = 0, 1, \dots, m+r \text{ and } j = 0, 1, \dots, m.$$

Then the degree reduction amounts to solving the least squares problem

$$\min_{c \in \mathbf{R}^{m+1}} \|b - T_{m,r} c\|_w$$

with $C^{\alpha-1}$ -continuity at $t = 0$ and $C^{\beta-1}$ -continuity at $t = 1$.

To solve the least squares problem explicitly, let

$$d = b - T_{m,r} c,$$

where b is the given vector, and c is the unknown vector. Considering $C^{\alpha-1}$ -continuity at $t = 0$ and $C^{\beta-1}$ -continuity at $t = 1$, we can impose

$$\begin{aligned} \frac{n!}{m!} \Delta^i b_0 &= \frac{\binom{n-i}{m-i} i!}{m!} \Delta^i c_0 \text{ for } i = 0, 1, \dots, \alpha - 1 \\ \frac{n!}{m!} \Delta^i b_{n-i} &= \frac{\binom{n-i}{m-i} i!}{m!} \Delta^i c_{m-i} \text{ for } i = 0, 1, \dots, \beta - 1. \end{aligned}$$

With these conditions, we solve $b - T^{m,r}c$ and split it in two parts, namely the known part \tilde{b} and the unknown part \tilde{c} i.e.,

$$\tilde{d} = \tilde{b} - T^{m,r}\tilde{c}.$$

Let $A^{\alpha,\beta}$ be the submatrix of the $(n+1) \times (n+1)$ matrix A obtained by extracting rows α through $(n+1-\beta)$ and columns α through $(n+1-\beta)$ and let $v^{\alpha,\beta}$ be the subvector of the vector v obtained by extracting rows α through $(n+1-\beta)$.

Let W^n be the diagonal matrix whose diagonal elements are given by (4).

Then the solution $\tilde{c}^{\alpha,\beta}$ is given by the pseudo inverse $P^{\alpha,\beta}_{m,r}$ of the degree raising

matrix

$$\tilde{c}^{\alpha,\beta} P^{\alpha,\beta}_{m,r} = (T^{\alpha,\beta}_{m,r} W^{\alpha,\beta}_T M^{\alpha,\beta}_T (T^{\alpha,\beta}_{m,r})^{-1} (T^{\alpha,\beta}_{m,r} W^{\alpha,\beta}_T M^{\alpha,\beta}_T (T^{\alpha,\beta}_{m,r})^{-1}))^{-1} \tilde{v}^{\alpha,\beta}.$$

Example 1. $n = 5$, $r = 1$, C^0 -continuity, $\alpha = \beta = 1$

$$d = b - T_{4,1}c, \quad d = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} - \frac{1}{5} \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix},$$

where $c_0 = b_0$, and $c_4 = b_5$.

$$\tilde{d} = b - T_{4,1}\tilde{c}, \quad \tilde{d} = \begin{pmatrix} 0 \\ b_1 - \frac{5}{1}b_0 \\ b_2 \\ b_3 \\ b_4 - \frac{5}{1}b_5 \\ 0 \end{pmatrix} - \frac{1}{5} \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 0 \\ c_1 \\ c_2 \\ c_3 \\ 0 \end{pmatrix}.$$

$$\tilde{d}_{1,1}^p = b_{1,1} - T_{1,1}^{4,1} \tilde{c}_{1,1}^{4,1},$$

$$\tilde{d}_{1,1}^q = \begin{pmatrix} b_1 - \frac{5}{1}b_0 \\ b_2 \\ b_3 \\ b_4 - \frac{5}{1}b_5 \end{pmatrix} - \frac{5}{1} \begin{pmatrix} 4 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

$$P_{1,1}^{4,1} = \left((T_{1,1}^{4,1})^T W_{1,1}^5 (T_{1,1}^{4,1})^{-1} \right) \begin{pmatrix} \frac{48}{55} & -\frac{48}{5} & \frac{24}{5} \\ -\frac{48}{5} & \frac{6}{5} & \frac{24}{5} \\ \frac{48}{5} & -\frac{12}{5} & -\frac{24}{5} \end{pmatrix} = \begin{pmatrix} \frac{48}{5} & \frac{24}{5} & -\frac{24}{5} \\ -\frac{48}{5} & \frac{6}{5} & \frac{24}{5} \\ \frac{48}{5} & -\frac{12}{5} & -\frac{24}{5} \end{pmatrix},$$

where

$$W_{1,1}^5 = \begin{pmatrix} \frac{2}{5} & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{5} \end{pmatrix}.$$

$$\tilde{c}_{1,1}^2 = P_{1,1}^{4,1} \tilde{b}_{1,1}^{4,1}.$$

Example 2. $n = 5$, $r = 1$, $\alpha = 1$, $\beta = 2$

$$d = b - T_{4,1}c, \quad d = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} - \frac{1}{5} \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix},$$

where $c_0 = b_0$, $c_3 = b_5 - \frac{4}{5}(b_5 - b_4)$, and $c_4 = b_5$.

$$\tilde{d} = \tilde{b} - T_{4,1}\tilde{c},$$

$$\tilde{d} = \begin{pmatrix} 0 \\ b_1 - \frac{5}{1}b_0 \\ b_2 \\ b_3 - \frac{2}{1}b_4 + \frac{10}{1}b_5 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{5} \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 0 \\ c_1 \\ c_2 \\ c_2 \\ 0 \end{pmatrix}.$$

$$\tilde{d}_{1,2}^d = b_{1,2} - T_{1,2}^{4,1} \tilde{c}_{1,2},$$

$$\tilde{d}_{1,2}^d = \begin{pmatrix} b_1 - \frac{5}{1}b_0 & & & & \\ b_2 & & & & \\ b_3 - \frac{2}{1}b_4 + \frac{10}{1}b_5 & & & & \\ \frac{5}{1} & & & & \\ 0 & 4 & 0 & 3 & 0 \end{pmatrix} - \begin{pmatrix} \frac{5}{1} & & & & \\ 2 & & & & \\ 3 & & & & \\ 3 & & & & \\ 0 & 3 & 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

$$P_{1,2}^{4,1} = \left((T_{1,2}^{4,1})^T W_{1,2}^5 (T_{1,2}^{4,1})^{-1} \right) W_{1,2}^5 = \begin{pmatrix} \frac{35}{5} & & & & \\ \frac{36}{5} & & & & \\ \frac{9}{5} & & & & \\ \frac{10}{5} & & & & \\ \frac{27}{5} & & & & \end{pmatrix}.$$

where

$$W_{1,2}^5 = \begin{pmatrix} \frac{2}{5} & & & & \\ 0 & 4 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\tilde{c}_{1,2} = P_{1,2}^{4,1} b_{1,2}.$$

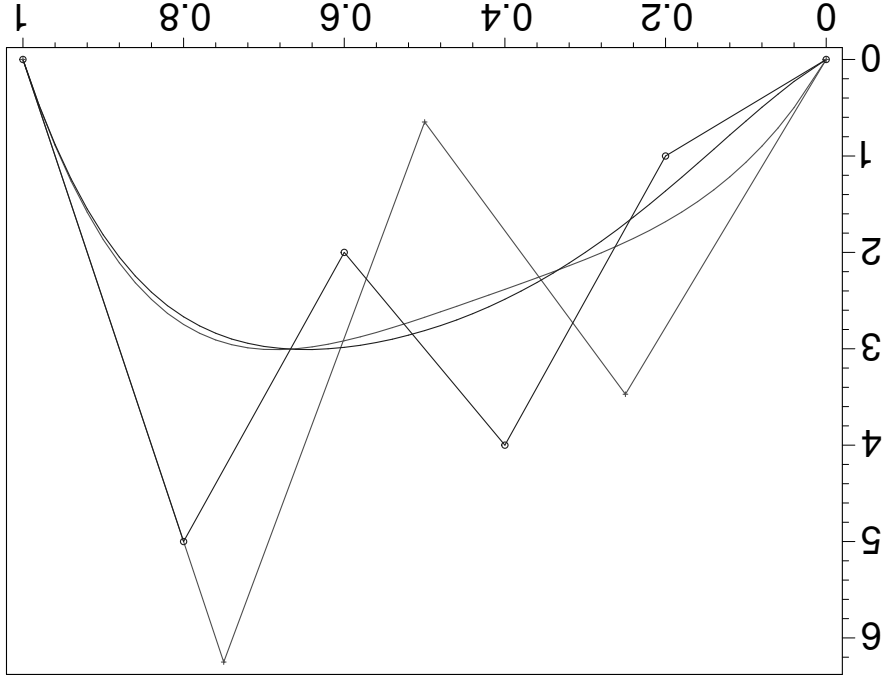
Degree reduction of Bézier curve from degree five to degree four with constraint $\alpha = 1$ and $\beta = 2$

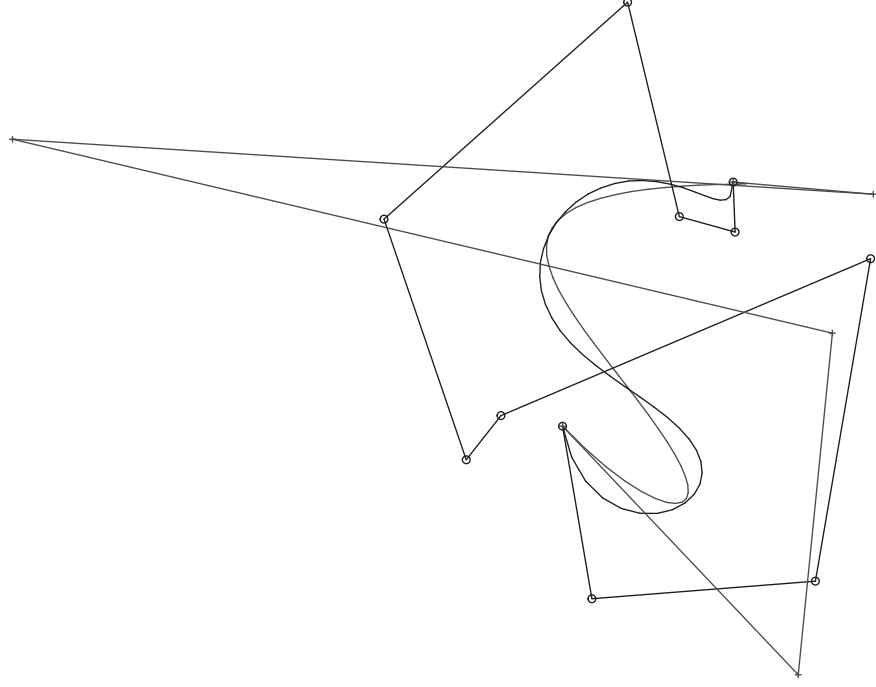
As an example, consider the polynomial $B^5 b$ with BB coefficients

$$b = [0, 1, 4, 2, 5, 0]^t.$$

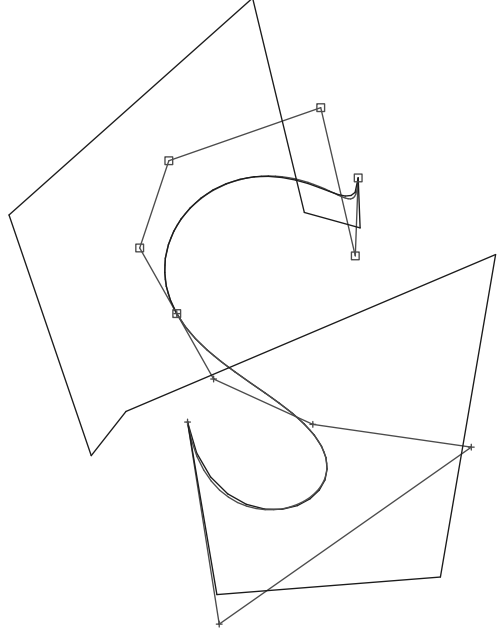
The best approximation $B^4 c$ with C^0 -continuity at $t = 0$ and C^1 -continuity at $t = 1$ has coefficients

$$c = [0, 125/36, 35/54, 25/4, 0].$$





Degree reduction of planar Bézier curve from degree ten to degree five with constraint $\alpha = \beta = 1$: circles are the control points of given curve and crosses are the control points of degree-reduced curve



Degree reductions after subdivision of the given curve: crosses and boxes are the control points of the degree-reduced curves with constraint $(\alpha, \beta) = (1, 2)$ and $(\alpha, \beta) = (2, 1)$, respectively

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