Orthogonal polynomials and its application to computer graphics

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- A moment functional is a linear map $\sigma : \mathcal{P} \to \mathbb{R}$. And the set of all moment functionals will be denoted by $\mathcal{M}$.
- Due to the representation theorems by Boas and Duran, any moment functional σ has an integral representation of the form

$$< \sigma, p >= \int_{\mathbb{R}} p \, d\mu \quad or \quad \sigma(p) = \int_{\mathbb{R}} p(x)w(x)dx \qquad (p \in \mathcal{P})$$

  for some finite, signed Borel measure μ on $\mathbb{R}$ and a function $w(x)$ of the Schwartz class.
- We denote $\sigma(p)$ by $< \sigma, p >= \sigma(p)$ and the moments of σ are defined to be $\sigma_n =< \sigma, x^n >$.
- A PS $\{p_n\}_{n=0}^{\infty}$ is an orthogonal polynomial system (OPS) with respect to $\sigma \in \mathcal{M}$ if for each $n \geq 0$, $deg(p_n) = n$ and there exists $K_n > 0$ such that

$$< \sigma, p_n p_m >= K_n \delta_{nm} \qquad (n, m \in \mathbb{N}_0).$$

Examples

1. $< \sigma_1, p >= \int_{-1}^{1} p(x)dx,$ the Legendre polynomials.
2. $< \sigma_2, p >= \sum_{n=0}^{\infty} \frac{a^n p(n)}{n!} (a > 0)$, the Charlier polynomials.
3. $< \sigma_3, p >= \int_{-\infty}^{\infty} p(x)e^{-x^2} dx$, the Hermite polynomials.
4. $< \sigma_5, x^n >= a^n, a > 0$, no OPS for σ_5 exists.
5. Let $p_n(x) = x^n, n \geq 0$. Then $\{p_n(x)\}_{n=0}^{\infty}$ is not an OPS.
6. (stieltjes, 1894) Define

$$g(x) = \begin{cases} 0 & for \quad x < 0 \\ e^{-x^{1/4}} \sin x^{1/4} & for \quad x \geq 0. \end{cases}$$

($g(x)$ is called the Stieltjes ghost function or polynomial killer) Then

$$\sigma(p) = \int_0^{\infty} p(x)g(x)dx = 0, \text{for all } p \in \mathcal{P}.$$

That is, $\sigma = 0$.

Question : On what condition does $\sigma \in \mathcal{M}$ have an OPS?

THEOREM 1 . *Let $\sigma \in \mathcal{M}$ and let $\{\sigma_n\}$ be the moments of σ. Then σ has an OPS if and only if*

$$\Delta_n := \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} > 0, \quad n \geq 0.$$

Moreover, MOPS $\{P_n(x)\}_{n=0}^{\infty}$ rel. to σ are given by

$$P_0(x) = 1, \quad P_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}, \quad n \geq 1$$

THEOREM 2 . *(Farvard's Theorem) A monic PS $\{P_n(x)\}_{n=0}^{\infty}$ is a monic OPS if and only if $\{P_n(x)\}_{n=0}^{\infty}$ satisfies a three-term recurrence relation(TTRR in short):*

$$P_{n+1}(x) = (x - b_n)P_n(x) - c_n P_{n-1}(x), \quad n \geq 0, \quad (P_{-1}(x) = 0)$$

where $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are real numbers with $c_n > 0, n \geq 1$.

The Classical Orthogonal Polynomials

DEFINITION 1 . *We call an OPS* $\{P_n(x)\}_{n=0}^{\infty}$ *rel. to* σ *a classical OPS (COPS) if* $\{P_n(x)\}_{n=0}^{\infty}$ *satisfy*

$$L[y](x) = \alpha(x)y''(x) + \beta(x)y'(x) = \lambda_n y(x), \tag{1}$$

where $\alpha(x) = ax^2 + bx + c, \quad \beta(x) = dx + e.$

THEOREM 3 *For an OPS* $\{P_n(x)\}_{n=0}^{\infty}$ *rel. to* σ*, TFAE*

1. $\{P_n(x)\}_{n=0}^{\infty}$ *is a classical OPS satisfying (1).*
2. $\{P_n'(x)\}_{n=1}^{\infty}$ *is also an OPS.*
3. σ *satisfies Pearson-type functional equation :*

$$(\alpha(x)\sigma)' = \beta(x)\sigma.$$

Classification of classical Orthogonal Polynomials

- S. Bochner characterized all sequences of polynomials that satisfy $\ell_2(x)y'' + \ell_1(x)y' = \lambda y$
  1. Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$:
     - $(1-x^2)y'' + (\beta - \alpha - (\alpha+\beta+2)x)y' = \lambda y$,
     - $d\mu(x) = (1-x)^\alpha(1+x)^\beta H(1-x^2)dx$.
  2. Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$:
     - $xy'' + (\alpha + 1 - x)y' = \lambda y$,
     - $d\mu(x) = x^\alpha e^{-x} H(x)dx$
  3. Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$:
     - $y'' - 2xy' = \lambda y$,
     - $d\mu(x) = e^{-x^2}dx$,
  4. Bessel polynomials $\{B_n^\alpha(x)\}_{n=0}^{\infty}$:
     - $x^2y'' + (2\alpha x + 2)y' = \lambda y$,
- Kwon and Littlejohn have classified classical orthogonal polynomials
  5. twisted Jacobi polynomials $\{\tilde{P}_n^{(d,e)}(x)\}_{n=0}^{\infty}$: $(1+x^2)y'' + (dx+e)y' = \lambda y$
  6. twisted Hermite polynomials $\{\tilde{H}_n(x)\}_{n=0}^{\infty}$: $y'' + 2xy' = \lambda y$,

Bochner-Krall orthogonal polynomials

Consider the N^{th}-order differential equation

$$L_N[y](x) := \sum_{i=1}^{N} \ell_i(x) y^{(i)}(x) = \sum_{i=1}^{N} \sum_{j=0}^{i} \ell_{ij} x^j y^{(i)}(x) = \lambda y(x),$$

where $N \in \mathbb{N}, \lambda \in \mathbb{R}$, and $\ell_i(x)$ are real polynomials.
I have been interested in the case that $L_N[\cdot]$ has a sequence of polynomial eigenfunctions orthogonal with respect to a real bilinear form of the type

$$(p, q) = \int_{\mathbb{R}} p(x) q(x) d\mu(x)$$

where $\mu(x)$ is a finite, signed Borel measure on $\mathbb{R}$.

THEOREM 4 **[G. J. Yoon et al.]** *If the differential equation*

$$L_N[y](x) = \sum_{i=1}^{N} \ell_i(x) y^{(i)}(x) = \lambda_n y(x)$$

has an OPS $\{P_n(x)\}_{n=0}^{\infty}$ *of solutions, then the differential operator* $L_N[\cdot]$ *is symmetrizable on any open interval not containing any root of* $\ell_N(x)$ *with*

$$s(x) = \frac{1}{\ell_N(x)} \exp\left[\frac{1}{r} \int \frac{\ell_{N-1}(x)}{\ell_N(x)}\, dx\right]$$

as a symmetry factor.

THEOREM 5 **[G. J. Yoon et al.]** *Let* $\{P_n(x)\}_{n=0}^{\infty}$ *be an OPS relative to* σ *satisfying the differential equation*

$$L_N[y](x) = \sum_{i=1}^{N} \ell_i(x) y^{(i)}(x) = \lambda_n y(x).$$

Then $\{P_n(x)\}_{n=0}^{\infty}$ *is Hermite polynomials if and only if the leading coefficient* $\ell_N(x)$ *is a constant.*

Table of all known BKOPS's

OPS	Order	Weight
Hermite	2	e^{-x^2}
Laguerre	2	$x^\alpha e^{-x} H(x)$
Jacobi	2	$(1-x)^\alpha (1+x)^\beta H(1-x^2)$
Bessel	2	
Laguerre type	4	$e^{-x} H(x) + A\delta(x)$
Jacobi type	4	$(1-x)^\alpha H(1-x^2) + A\delta(x+1)$
Legendre type	4	$\frac{\alpha}{2} H(1-x^2) + \frac{1}{2}\delta(x+1) + \frac{1}{2}\delta(x-1)$
Krall	6 , 8	$CH(1-x^2) + \frac{1}{A}\delta(x+1) + \frac{1}{B}\delta(x-1)$
Laguerre-Krall	10	$e^{-x} + A\delta(x) - B\delta'(x)$
Generalized Laguerre	$\infty \ (\alpha \notin \mathbb{N}_0)$ $2\alpha + 4 \ (\alpha \in \mathbb{N}_0)$	$x^\alpha e^{-x} + A\delta(x)$
Generalized Jacobi	$\infty \ (\alpha, \beta \notin \mathbb{N}_0)$ $2\alpha + 4 \ (A = 0, \alpha \in \mathbb{N}_0)$ $2\beta + 4 \ (B = 0, \beta \in \mathbb{N}_0)$ $2\alpha + 2\beta + 6 \ (\alpha, \beta \in \mathbb{N}_0)$	$(1-x)^\alpha (1+x)^\beta H(1-x^2)$ $+A\delta(x+1) + B\delta(x-1)$

where $H(x)$ is the heaviside step function. The weights are of in the form:

$$d\mu(x) = w(x)dx + d\mu_s(x)$$

where $w(x)$ is one of classical weights and μ_s is a discrete measure of which the supporting set consists of the end points of the supporting set of $w(x)$.

Theorem 7.1 [G. J. Yoon et al.] Let $\nu = \sum_{k=1}^{m} \sum_{j=0}^{m_k} c_{k,j} \delta^{(j)}(x - x_k)$ be a distribution with finite support in $\mathbb{C}$, and let $\tau := \sigma + \nu$ be a point-mass perturbation of σ, where $\nu \neq 0$. If τ is also quasi-definite and gives rise to a BKOPS $\{Q_n(x)\}_{n=0}^{\infty}$ of order $\leq 2r$ satisfying

$$L_{2r}[y](x) = \sum_{i=1}^{2r} \ell_i(x) y^{(i)}(x) = \lambda_n y(x)$$

then

(i) $supp(\nu) \subseteq \{x \in \mathbb{C} | A(x) = 0\}$ so that $supp(\nu)$ contains only one or two point;

(ii) σ must be $\sigma_J^{(\alpha,\beta)}$ or $\sigma_L^{(\alpha)}$ or $\sigma_{\check{J}}^{(d,e)}$ and

1. if $\sigma = \sigma_J^{(\alpha,\beta)}$ and $1 \in supp(\nu)$ $(resp.\ -1 \in supp(\nu))$, then α $(resp.\ \beta)$ must be a non-negative integer;
2. if $\sigma = \sigma_L^{(\alpha)}$, then α must be a non-negative integer;
3. if $\sigma = \sigma_{\check{J}}^{(d,e)}$, then $e = 0$ and $d = 2k$ for some integer $k \geq 1$.

Its application to computer graphics

Bézier Curves

The Bézier curves are basically and widely used in CAGD – short for Computer Aided Geometric Design. Bézier curves and surfaces are now established as the mathematical basis of many CAD systems, they have also become a major tool for the development of new methods for curve and surface descriptions.

The Bézier representation uses Bernstein polynomials as basis functions for the linear space of polynomials. In terms of the Bernstein polynomials of degree n,

$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i, \qquad 0 \le t \le 1,$$

a parametric polynomial curve of degree n in the plane, can be expressed as

$$b^n(t) = \sum_{i=0}^{n} b_i B_i^n(t).$$

The points b_i, $i = 0, \cdots, n$ are called the *control points* for the polynomial, and the polygon formed by joining successive control points is the *control polygon*.

Degree Reduction of Bézier Curves

PROBLEM 1 (L_2-NORM DEGREE REDUCTION) *Let* $\{c_i\}_{i=0}^n$ *be a given set of control points which define the Bézier curve*

$$c^n(t) = \sum_{i=0}^{n} c_i B_i^n(t)$$

of degree n*. Then find another point set* $\{b_i\}_{i=0}^m$ *defining the approximative Bézier curve*

$$b^m(t) = \sum_{i=0}^{m} b_i B_i^m(t)$$

of lower degree $m < n$ *so that an* L_2*-norm distance function* $d_2(b^m, c^n)$ *between* b^m *and* c^n *is minimized:*

$$d_2^2(b^m, c^n) = \int_0^1 |b^m(t) - c^n(t)|^2 dt = \int_0^1 |\sum_{i=0}^{m} b_i B_i^m(t) - \sum_{i=0}^{n} c_i B_i^n(t)|^2 dt.$$

Degree elevation with Bernstein coefficients

T_n is an $(n+2)\times(n+1)$ matrix such that for a polynomial $f(t)$ of degree n,

$$f(t) = \sum_{i=0}^{n} c_i B_i^n(t) = \sum_{i=0}^{n+1} c_i^{(1)} B_i^{n+1}(t), \quad c^{(1)} = T_n c$$

$$T_n = \frac{1}{n+1}\begin{pmatrix} n+1 & 0 & \dots & 0 & 0 & 0 \\ 1 & n & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & n-1 & 2 & 0 \\ 0 & 0 & \dots & 0 & n & 1 \\ 0 & 0 & \dots & 0 & 0 & n+1 \end{pmatrix}$$

$$c^{(r)} = T_{n,r}c, \quad T_{n,r} = T_{n+r-1}T_{n+r-2}\dots T_{n+1}T_n, \quad T_{n,r}(i,j) = \frac{\binom{n}{j}\binom{r}{i-j}}{\binom{n+r}{i}}$$

THEOREM 6 *The L_2-norm distance between the two Bézier curves b^m and c^n is*

$$d_2^2(b^m, c^n) = d_2^2(b^{(r)}, c^n) = A^t Q_n A$$

where $A = c - T_{m,r} b$, $b = (b_0, b_1, \ldots, b_m)^t$ and $c = (c_0, c_1, \ldots, c_n)^t$ and Q_n is the Gram matrix of Bernstein polynomials $\{B_i^n(x)\}_{i=0}^n$.

The vector $\hat{b}$ that minimizes $A^t Q_n A$ is shown to satisfy the equation

$$T_{m,r}^t Q_n T_{m,r} \hat{b} = T_{m,r}^t Q_n c.$$

THEOREM 7 **[B. G. Lee and Y. B. Park].** *The* $n \times n$ *matrix* $T_{n-1}^t Q_n T_{n-1}$ *has the following property:*

$$T_{n-1}^t Q_n T_{n-1} = Q_{n-1}.$$

Since Q_{n-1} is invertible, we have the unique solution for $\hat{b}$,

$$\hat{b} = (T_{m,r}^t Q_n T_{m,r})^{-1} T_{m,r}^t Q_n c.$$

This approximate curve is the best approximation with respect to the L_2-norm.

Basis Transformations

Consider a polynomial $f(t)$ of degree n, expressed in the degree n Bernstein and Legendre bases on $t \in [0,1]$:

$$f(t) = \sum_{j=0}^{n} c_j B_j^n(t) = \sum_{k=0}^{n} l_k L_k(t).$$

We are interested in the linear transformation M_n

$$c = M_n l$$

that maps the Legendre coefficients $l = (l_0, l_1, \ldots, l_n)^t$ into the Bernstein coefficients $c = (c_0, c_1, \ldots, c_n)^t$, and its inverse.

THEOREM 8

$$M_n(j,k) = \frac{\sqrt{2k+1}}{\binom{n}{j}} \sum_{i=\max(0,j+k-n)}^{\min(j,k)} (-1)^{k+i} \binom{k}{i} \binom{k}{i} \binom{n-k}{j-i}$$

$$M_n^{-1}(j,k) = \frac{\sqrt{2j+1}}{n+j+1} \frac{1}{\binom{n+j}{n}} \sum_{i=0}^{j} (-1)^{j+i} \binom{j}{i} \binom{k+i}{k} \binom{n-k+j-i}{n-k}$$

EXAMPLE 1

$$M_3 = \begin{bmatrix} 1 & -\sqrt{3} & \sqrt{5} & -\sqrt{7} \\ 1 & -\frac{\sqrt{3}}{3} & -\sqrt{5} & 3\sqrt{7} \\ 1 & \frac{\sqrt{3}}{3} & -\sqrt{5} & -3\sqrt{7} \\ 1 & \sqrt{3} & \sqrt{5} & \sqrt{7} \end{bmatrix}, M_3^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{3\sqrt{3}}{20} & -\frac{\sqrt{3}}{20} & \frac{\sqrt{3}}{20} & \frac{3\sqrt{3}}{20} \\ \frac{\sqrt{5}}{20} & -\frac{\sqrt{5}}{20} & -\frac{\sqrt{5}}{20} & \frac{\sqrt{5}}{20} \\ -\frac{\sqrt{7}}{140} & \frac{3\sqrt{7}}{140} & -\frac{3\sqrt{7}}{140} & \frac{\sqrt{7}}{140} \end{bmatrix}$$

Degree elevation and reduction with Legendre coefficients

The Legendre polynomials $L_n(t)$ on $t \in [0,1]$ constitute an orthonomal basis with L_2-norm distance function.

$$L_n(t) = \sqrt{2n+1} \sum_{i=0}^{n} \binom{n}{i}^2 (t-1)^i t^{n-i}.$$

By the orthogonality of Legendre basis, the degree elevation of a polynomial $f(t)$

$$f(t) = \sum_{i=0}^{n} l_i L_i(t) = \sum_{i=0}^{n+r} l_i L_i(t), \quad l_i = 0, \quad n+1 \le i \le n+r,$$

with Legendre basis is given by $I_{n,r} l = l^{(r)}$, where the $(n+r+1) \times (n+1)$ $I_{n,r}$ matrix has elements $I_{n,r}(i,j) = \delta_{ij}$ and

$$l = (l_0, l_1, \dots, l_n)^t, \quad l^{(r)} = (l_0, l_1, \dots, l_n, \underbrace{0, \dots, 0}_{r})^t.$$

And the $(n-r) \times n$ degree elevation matrix $I_{n,-r}$ corresponds a polynomial $f(t)$ to the polynomial

$$\tilde{f}(t) = \sum_{i=0}^{n-r} l_i L_i(t), \quad I_{n,-r} l = l^{(-r)}, \quad I_{n,-r}(i,j) = \delta_{ij}$$

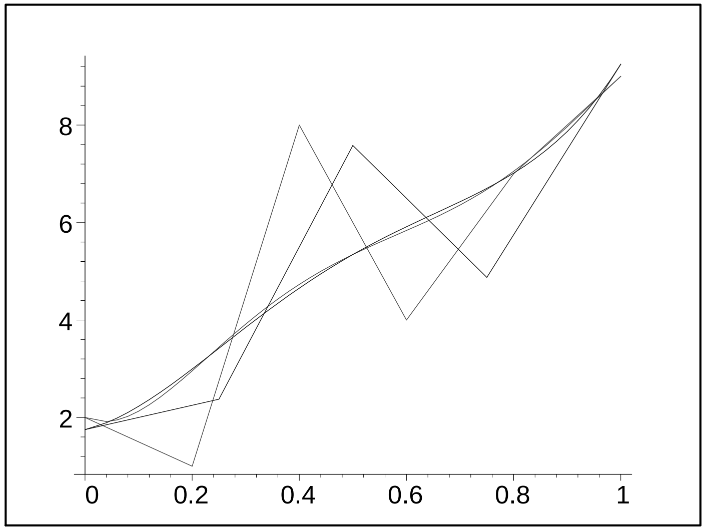
$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{c} \arrow[r, shift left, "\mathbf{M_n^{-1}}"] \arrow[d, shift right, "(\mathbf{T_{m,r}^t T_{m,r}})^{-1}\mathbf{T_{m,r}^t}"'] & \mathbf{l} \arrow[l, shift left, "\mathbf{M_n}"] \arrow[d, shift right, "\mathbf{I_{n,-r}}"'] \\
\mathbf{c^{(-r)}} \arrow[r, shift left, "\mathbf{M_m^{-1}}"] \arrow[u, shift right, "\mathbf{T_{m,r}}"'] & \mathbf{l^{(-r)}} \arrow[l, shift left, "\mathbf{M_m}"] \arrow[u, shift right, "\mathbf{I_{m,r}}"']
\end{tikzcd}$$

THEOREM 9 . *The degree elevation matrix $T_{n,r}$ can be expressed in M_n^{-1}, $I_{n,r}$ and M_{n+r} as*

$$T_{n,r} = M_{n+r} I_{n,r} M_n^{-1}.$$

THEOREM 10 .[**B. G. Lee, Y. B Park and J. C. Yoo**] *The degree reduction matrix can be expressed in M_m and M_n^{-1} as*

$$\left(T_{m,r}^t T_{m,r}\right)^{-1} T_{m,r}^t = M_m I_{n,-r} M_n^{-1}, \quad (r = n - m).$$


8
6
4
2
0
0.2
0.4
0.6
0.8
1

Constrained Degree Reduction with Bernstein coefficients

PROBLEM 2 (L_2-NORM DEGREE REDUCTION WITH CONSTRAINTS) *Let* $\{c_i\}_{i=0}^n$ *be a given set of control points which define the Bézier curve*

$$c^n(t) = \sum_{i=0}^{n} c_i B_i^n(t)$$

of degree n*. Then find another point set* $\{b_i\}_{i=0}^m$ *defining the approximative Bézier curve*

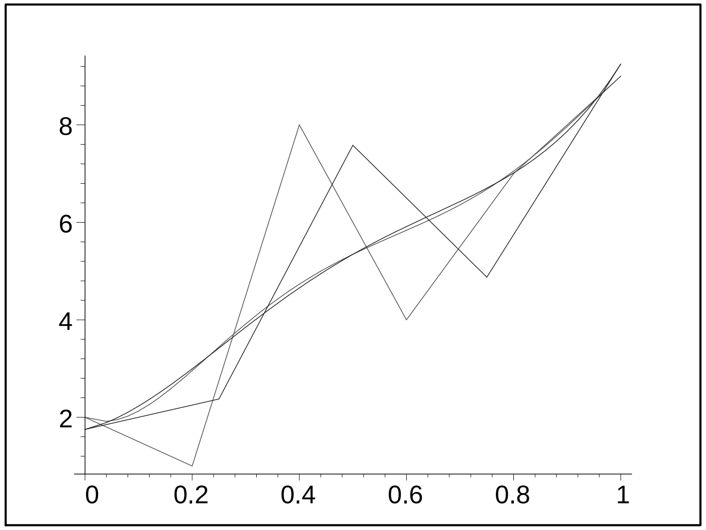
$$b^m(t) = \sum_{i=0}^{m} b_i B_i^m(t)$$

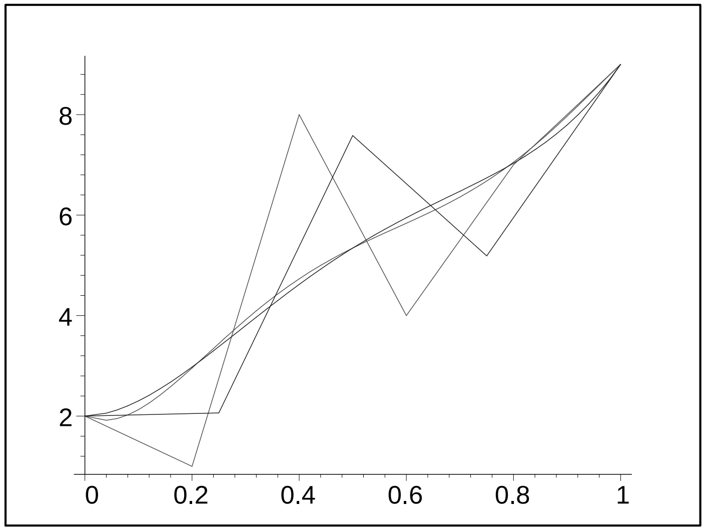
of lower degree $m < n(r = n - m)$ *so that an* L_2*-norm distance function* $d_2(c^n, b^m)$ *between* c^n *and* b^m *is minimized subjected to* $C_{\alpha-1}$*-continuity at* $t = 0$ *and* $t = 1$*.*

$$\begin{tikzcd}
\mathbf{c} \arrow[r, shift left, "\mathbf{M_n^{-1}}"] \arrow[d, shift right, "(\mathbf{T_{m,r}^t W_n^\alpha T_{m,r}})^{-1}\mathbf{T_{m,r}^t W_n^\alpha}"'] & \mathbf{l} \arrow[l, shift left, "\mathbf{M_n}"] \arrow[d, shift right, "\mathbf{I_{n,-r}^\alpha}"'] \\
\mathbf{c^{(-r)}} \arrow[u, shift right, "\mathbf{T_{m,r}}"'] \arrow[r, shift left, "\mathbf{M_m^{-1}}"] & \mathbf{l^{(-r)}} \arrow[l, shift left, "\mathbf{M_m}"] \arrow[u, shift right, "\mathbf{I_{m,r}}"']
\end{tikzcd}$$

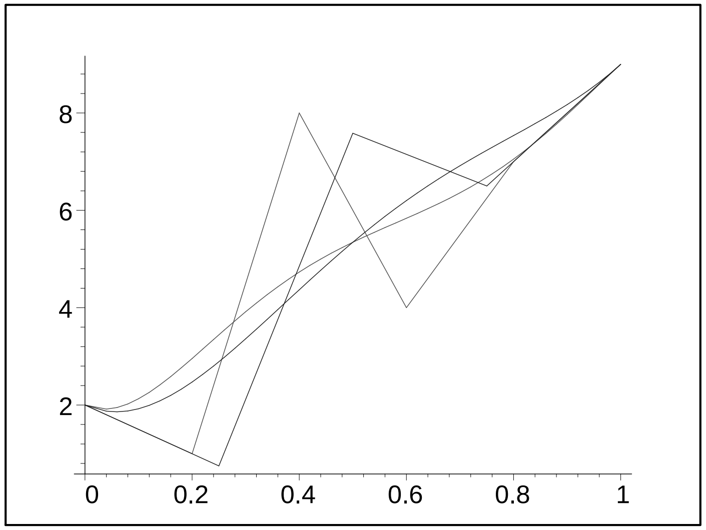
THEOREM 11

$$I_{n,-r}^{\alpha} = M_m^{-1}\left(T_{m,r}^t W_n^{\alpha} T_{m,r}\right)^{-1} T_{m,r}^t W_n^{\alpha} M_n$$





$\alpha = 1$



$\alpha = 2$

Further Works

- Let $\{P_n(x)\}_{n=0}^{\infty}$ be a BKOPS relative to τ of order $2r \geq 4$ satisfying the differential equation
$$L_{2r}[y](x) = \sum_{i=1}^{2r} \ell_i(x) y^{(i)}(x) = \lambda y(x).$$
Then
  - τ is a point masses perturbation $\tau = \sigma + \nu$ of a classical moment functional σ.
  - $\ell_{2r}(x) = A(x)^r$ for some polynomial $A(x)$ of degree ≤ 2.
- Find a representation $\omega(x)$ of the derivative $\delta'(x)$ of the Dirac delta distribution, that is, find a weight function $\omega(x)$ such that for every polynomial $P(x)$
$$\int_{-\infty}^{\infty} P(x)\omega(x)dx = -P'(x).$$
- Find a constrained degree reduction with Berstein coefficients without using the limit process.
- Find an algorithm on the constrain degree reduction using an orthogonal polynomials other than Legendre polynomials and check how efficient it is.
- Improve the results related to 2D to the 3D case.