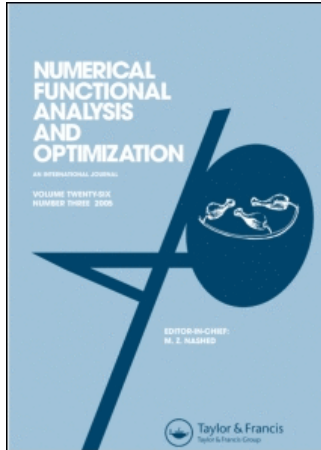


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MULTIPLE DEGREE REDUCTION AND ELEVATION OF BÉZIER CURVES USING JACOBI–BERNSTEIN BASIS TRANSFORMATIONS

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□ *In this article, we find the optimal r times degree reduction of Bézier curves with respect to the Jacobi-weighted L_2 -norm on the interval $[0, 1]$. This method describes a simple and efficient algorithm based on matrix computations. Also, our method includes many previous results for the best approximation with L_1 , L_2 , and L_∞ -norms. We give some examples and figures to demonstrate these methods.*

Keywords Basis transformation; Bernstein polynomials; Bézier curves; Degree reduction; Jacobi polynomials; Least-squares approximation; Orthogonal polynomials.

AMS Subject Classification 33C45; 41A10; 41A50; 65D17.

1. INTRODUCTION

The use of the orthonormal polynomial basis simplifies the least-squares approximation of a continuous function $f(x)$. The matrix of the normal equations becomes the unity matrix, and thus it gives a compact form for the coefficients of the least-squares approximation. Also, by computing one more coefficient, the approximating polynomial of degree n can be easily computed from the approximating polynomial of

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degree $n - 1$. So, we are encouraged to use the orthonormal polynomial basis from all possible kinds of basis.

On the other hand, the Bernstein basis has interesting geometric properties, symmetric relations, and the property of partition of unity. These properties make the Bernstein polynomials the basis for the Bézier curves and surfaces on the computer aided design (CAD) systems; see a book on computer aided geometric design (CAGD), for example [1] and [2]. However, the use of the Bernstein polynomials in the least-squares approximation is limited, because the Bernstein polynomials are not orthogonal, see [3].

In addition, the CAD systems use the basis of Bernstein, monomials, and B-spline functions. We need to convert data from one basis to another. This transformation is numerically exact for a polynomial of degree n only to polynomials of the same or higher degrees. However, basis transformation between different kinds of basis, and other kinds of operations on polynomials, like degree elevation and degree reduction, are challenging problems in CAGD. The use of orthogonal basis had been found to be optimal in the reduction, conversion, and exchange of data and degree, see for example [4] and [5].

Basis transformations between different kinds of basis have been discussed in [4, 6–11]. It is important to choose the appropriate basis if we wish to preserve specific properties of the curve or surface. Degree reduction application schemes can be found in [12–18].

In this article, we find the Gram matrix $Q_n^{(\alpha,\beta)}$ and a similar matrix $D_n^{(\alpha,\beta)}$ with the Jacobi–Bernstein basis transformation matrix $M_n^{(\alpha,\beta)}$. The relation between $M_n^{(\alpha,\beta)}$ and the Bernstein–Jacobi basis transformation matrix $(M_n^{(\alpha,\beta)})^{-1}$ is given. The relationships between the basis transformation matrices $M_n^{(\alpha,\beta)}$, $(M_n^{(\alpha,\beta)})^{-1}$ and the degree elevation matrix T_n , the basis transformation matrices and the degree reduction matrix are derived. The Legendre–Bernstein basis transformation [15] and the Chebyshev–Bernstein basis transformation [17] become special cases.

First, some properties of the Bernstein and Jacobi polynomials are given in the second section. We derive the matrices of transformation between the orthonormal Jacobi polynomials and Bernstein polynomial basis in the third section. In the fourth section, we discuss the relationship among transformations, $M_n^{(\alpha,\beta)}$, $(M_n^{(\alpha,\beta)})^{-1}$ and a similar matrix $D_n^{(\alpha,\beta)}$ of Gram matrix $Q_n^{(\alpha,\beta)}$. An application to degree elevation of Bézier curves is given in the fifth section. The degree reduction of Bézier curves is discussed in the sixth section. Some examples are plotted in the seventh section, and conclusions are presented in the eighth section.

2. BERNSTEIN AND JACOBI POLYNOMIALS

The Bernstein–Bézier form of a parametric polynomial curve $f(x)$ of degree n is given by

$$f(x) = \sum_{i=0}^n c_i B_i^n(x), \quad 0 \leq x \leq 1, \quad (2.1)$$

where c_0, c_1, \dots, c_n are the $(n + 1)$ Bézier points, and $B_0^n(x), B_1^n(x), \dots, B_n^n(x)$ are the Bernstein polynomials of degree n defined by

$$B_i^n(x) = \binom{n}{i} (1-x)^{n-i} x^i, \quad i = 0, 1, \dots, n. \quad (2.2)$$

In [19], it is shown that each Bernstein polynomial of degree v where $v \leq n$ can be written in the Bernstein polynomial of degree n using the following formula of degree raising

$$B_i^v(x) = \sum_{\mu=i}^{n-v+i} \frac{\binom{v}{i} \binom{n-v}{\mu-i}}{\binom{n}{\mu}} B_\mu^n(x), \quad i = 0, 1, \dots, v.$$

The area under a Bernstein polynomial $B_i^n(x)$ for all $i = 0, \dots, n$ of degree n has the value

$$\int_0^1 B_i^n(x) dx = \frac{1}{n+1}. \quad (2.3)$$

The product of the Bernstein polynomials $B_i^n(x)$ and $B_j^m(x)$ of degree n and m respectively is the Bernstein polynomial of degree $n + m$ and has the form

$$B_i^n(x) B_j^m(x) = \frac{\binom{n}{i} \binom{m}{j}}{\binom{n+m}{i+j}} B_{i+j}^{n+m}(x). \quad (2.4)$$

The Bernstein polynomial basis is known to be optimally stable, see [20]. For more properties about the Bernstein polynomials, see [1] and [2].

The Jacobi polynomials are traditionally defined on the interval $[-1, +1]$, see [21]. The Bernstein polynomials are traditionally defined on the interval $[0, 1]$. Throughout this article, we use the interval $[0, 1]$ for both polynomials. The Jacobi polynomials constitute an orthonormal basis on the interval $[0, 1]$ with respect to the weight

function $w(x) = (2 - 2x)^\alpha (2x)^\beta$ where $\alpha, \beta > -1$. The orthonormal Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ on $[0, 1]$ have the following formula

$$P_n^{(\alpha, \beta)}(x) = h_n^{(\alpha, \beta)} \sum_{v=0}^n \binom{n+\alpha}{n-v} \binom{n+\beta}{v} (x-1)^v x^{n-v}, \quad (2.5)$$

where

$$h_n^{(\alpha, \beta)} = \sqrt{\frac{(2n + \alpha + \beta + 1)\Gamma(n+1)\Gamma(n + \alpha + \beta + 1)}{2^{\alpha+\beta}\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}}.$$

If $n = 0$, then

$$h_0^{(\alpha, \beta)} = \sqrt{\frac{(\alpha + \beta + 1)\Gamma(\alpha + \beta + 1)}{2^{\alpha+\beta}\Gamma(\alpha + 1)\Gamma(\beta + 1)}} = \frac{1}{\sqrt{2^{\alpha+\beta}B(\alpha + 1, \beta + 1)}},$$

where $B(p, q)$ is the beta function $B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx$.

There are many interesting properties for the Jacobi polynomials, and many applications in the quadrature, quantum mechanics, and the least-squares. For more properties about the Jacobi polynomials, see [21].

It is known that the Bernstein polynomials are not orthogonal and thus can not be applied in the least-squares approximation. We aim to combine the superior performance of the Jacobi polynomials in the least-squares approximation with the geometric properties of the Bernstein polynomial basis.

3. JACOBI-BERNSTEIN TRANSFORMATION

In this section, we derive the matrices of transformation between the orthonormal Jacobi polynomials $P_v^{(\alpha, \beta)}(x)$, $v = 0, 1, \dots, n$ of degree v and the Bernstein polynomial basis $B_v^n(x)$, $v = 0, 1, \dots, n$ of degree n . We proceed as in [10] to get the following analogous results.

Lemma 3.1. *The orthonormal Jacobi polynomial $P_n^{(\alpha, \beta)}(x)$ of degree n has the following representation in the Bernstein basis form $B_v^n(x)$, $v = 0, 1, \dots, n$ of degree n :*

$$P_n^{(\alpha, \beta)}(x) = h_n^{(\alpha, \beta)} \sum_{v=0}^n (-1)^{n-v} \frac{\binom{n+\alpha}{v} \binom{n+\beta}{n-v}}{\binom{n}{v}} B_v^n(x).$$

Lemma 3.2. Let $P_\mu^{(\alpha,\beta)}(x)$ be the orthonormal Jacobi polynomial of degree μ , and $B_\nu^n(x)$ be the Bernstein polynomial of degree n . Then for all $\mu, \nu = 0, 1, \dots, n$ the following integral has the value

$$\begin{aligned} & \int_0^1 w(x) B_\nu^n(x) P_\mu^{(\alpha,\beta)}(x) dx \\ &= 2^{\alpha+\beta} h_n^{(\alpha,\beta)} \binom{n}{\nu} \sum_{i=0}^{\mu} (-1)^{\mu-i} \binom{\mu+\alpha}{i} \binom{\mu+\beta}{\mu-i} b_n^{(\alpha,\beta)}(\mu-\nu-i, \nu+i), \end{aligned}$$

where $b_n^{(\alpha,\beta)}(x, y) = B(n + \alpha + x + 1, \beta + y + 1)$.

We write a polynomial $f(x)$, $x \in [0, 1]$ in the form of the orthonormal Jacobi polynomial basis and the form of the Bernstein polynomial basis as:

$$f(x) = \sum_{\nu=0}^n d_\nu P_\nu^{(\alpha,\beta)}(x) = \sum_{\mu=0}^n c_\mu B_\mu^n(x).$$

The following theorem gives the entries of the transformation Matrix $M_n^{(\alpha,\beta)}$, which transforms the orthonormal Jacobi coefficients $\{d_\nu\}_{\nu=0}^n$ into the Bernstein coefficients $\{c_\mu\}_{\mu=0}^n$.

Theorem 3.3. The entries $M_n^{(\alpha,\beta)}(\mu, \nu)$, $\mu, \nu = 0, 1, \dots, n$ of the matrix of transformation of Bernstein polynomial basis into Jacobi polynomial basis of degree n are given by

$$M_n^{(\alpha,\beta)}(\mu, \nu) = \frac{h_\nu^{(\alpha,\beta)}}{\binom{n}{\mu}} \sum_{i=\max(0, \mu+\nu-n)}^{\min(\mu, \nu)} (-1)^{\nu-i} \binom{n-\nu}{\mu-i} \binom{\nu+\alpha}{i} \binom{\nu+\beta}{\nu-i}.$$

We have the following theorem for writing the orthonormal Jacobi polynomial basis into Bernstein polynomial basis of degree n .

Theorem 3.4. The elements $(M_n^{(\alpha,\beta)})^{-1}(\mu, \nu)$, $\mu, \nu = 0, 1, \dots, n$ of the matrix of transformation of the orthonormal Jacobi polynomial basis into the Bernstein polynomial basis of degree n are given by

$$\begin{aligned} & (M_n^{(\alpha,\beta)})^{-1}(\mu, \nu) \\ &= 2^{\alpha+\beta} h_\mu^{(\alpha,\beta)} \binom{n}{\nu} \sum_{i=0}^{\mu} (-1)^{\mu-i} \binom{\mu+\alpha}{i} \binom{\mu+\beta}{\mu-i} b_n^{(\alpha,\beta)}(\mu-\nu-i, \nu+i). \end{aligned}$$

4. JACOBI-WEIGHTED L_2 -NORM

In this section, we compute the Jacobi-weighted L_2 -norm of the Bézier curve $f(x)$ of degree n in (2.1) and end this section by writing the transformation matrix $(M_n^{(\alpha,\beta)})^{-1}$ from Bernstein to Jacobi basis as the product of the similar matrix $D_n^{(\alpha,\beta)}$ to the Gram matrix $Q_n^{(\alpha,\beta)}$ with $M_n^{(\alpha,\beta)}$.

In the following computations, the equations (2.3) and (2.4) are used to simplify the L_2 -norm as follows

$$\begin{aligned} \|f\|_w^2 &= \int_0^1 w(x) \left| \sum_{i=0}^n c_i B_i^n(x) \right|^2 dx \\ &= \sum_{i,j} c_i c_j \int_0^1 2^{\alpha+\beta} (1-x)^\alpha x^\beta B_i^n(x) B_j^n(x) dx \\ &= \sum_{i,j} c_i c_j \int_0^1 2^{\alpha+\beta} (1-x)^\alpha x^\beta \binom{n}{i} (1-x)^{n-i} x^i \binom{n}{j} (1-x)^{n-j} x^j dx \\ &= \sum_{i,j} c_i c_j 2^{\alpha+\beta} \binom{n}{i} \binom{n}{j} b_n^{(\alpha,\beta)}(n-i-j, i+j). \end{aligned}$$

The elements of the Gram matrix $Q_n^{(\alpha,\beta)}$ of the Bernstein basis are defined by

$$Q_n^{(\alpha,\beta)}(i,j) = 2^{\alpha+\beta} \binom{n}{i} \binom{n}{j} b_n^{(\alpha,\beta)}(n-i-j, i+j), \quad i, j = 0, 1, \dots, n. \quad (4.1)$$

Because the combinatorial coefficients are symmetric, the matrix $Q_n^{(\alpha,\beta)}$ is a real symmetric matrix. And because the left-hand side in the definition is positive, the matrix $Q_n^{(\alpha,\beta)}$ is positive definite. Thus the Gram matrix $Q_n^{(\alpha,\beta)}$ is a real symmetric positive definite matrix. And thus, the weighted L_2 -norm of the polynomial $f(x)$ can be written in the form

$$\|f\|_w^2 = c^t Q_n^{(\alpha,\beta)} c. \quad (4.2)$$

Thus there is a similar matrix $D_n^{(\alpha,\beta)}$ to the Gram matrix $Q_n^{(\alpha,\beta)}$ with $M_n^{(\alpha,\beta)}$. It can be written in the form (see [22]),

$$D_n^{(\alpha,\beta)} = (M_n^{(\alpha,\beta)})^{-1} Q_n^{(\alpha,\beta)} M_n^{(\alpha,\beta)}. \quad (4.3)$$

Solving for $Q_n^{(\alpha,\beta)}$ gives

$$Q_n^{(\alpha,\beta)} = M_n^{(\alpha,\beta)} D_n^{(\alpha,\beta)} (M_n^{(\alpha,\beta)})^{-1}. \quad (4.4)$$

The weighted L_2 -norm of the polynomial $f(x)$ with Jacobi basis is computed using the orthonormality property of the orthonormal Jacobi basis to get

$$\|f\|_w^2 = \int_0^1 w(x) \left| \sum_{i=0}^n d_i P_i^{(\alpha,\beta)}(x) \right|^2 dx \tag{4.5}$$

$$= \sum_{i,j} d_i d_j \int_0^1 w(x) P_i^{(\alpha,\beta)}(x) P_j^{(\alpha,\beta)}(x) dx = d^t d. \tag{4.6}$$

Combining the previous results of the similar matrix $D_n^{(\alpha,\beta)}$ in (4.3), we get the relationship between $(M_n^{(\alpha,\beta)})^{-1}$, $D_n^{(\alpha,\beta)}$, and $M_n^{(\alpha,\beta)}$ in the following theorem.

Theorem 4.1. *The matrix of transformation from the Bernstein basis into the orthonormal Jacobi basis can be computed by*

$$(M_n^{(\alpha,\beta)})^{-1} = D_n^{(\alpha,\beta)} (M_n^{(\alpha,\beta)})^t.$$

Proof. Using (4.2) and (4.5), we get

$$c^t Q_n^{(\alpha,\beta)} c = d^t d.$$

Because $d = (M_n^{(\alpha,\beta)})^{-1} c$, the weighted L_2 -norm of the polynomial $f(x)$ can be written in the form

$$c^t Q_n^{(\alpha,\beta)} c = c^t ((M_n^{(\alpha,\beta)})^{-1})^t (M_n^{(\alpha,\beta)})^{-1} c. \tag{4.7}$$

And thus

$$Q_n^{(\alpha,\beta)} = ((M_n^{(\alpha,\beta)})^{-1})^t (M_n^{(\alpha,\beta)})^{-1}.$$

Combining the result in (4.4) and the last result gives

$$((M_n^{(\alpha,\beta)})^{-1})^t (M_n^{(\alpha,\beta)})^{-1} = M_n^{(\alpha,\beta)} D_n^{(\alpha,\beta)} (M_n^{(\alpha,\beta)})^{-1}.$$

We multiply both sides of the last equation by $M_n^{(\alpha,\beta)}$ from right, and then take the transpose of both sides to complete the proof. \square

5. DEGREE ELEVATION

The technique of degree elevation is a tool to increase the flexibility of a CAD system. Using the degree elevation, we can write a given Bézier curve of degree n into Bézier curve of degree $>n$. Elevating the degree

by 1 produces the new vertices $c_i^{(1)}$ of the new polygon from the old polygon by using the following formula (see [1]):

$$c_i^{(1)} = \frac{i}{n+1}c_{i-1} + \left(1 - \frac{i}{n+1}\right)c_i, \quad i = 0, 1, \dots, n+1. \quad (5.1)$$

Using the matrix T_n of dimension $(n+2) \times (n+1)$, the formula (5.1) can be written in the following matrix form

$$T_n c = c^{(1)},$$

where

$$T_n = \frac{1}{n+1} \begin{pmatrix} n+1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & n & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & n-1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n-1 & 2 & 0 \\ 0 & 0 & 0 & \dots & 0 & n & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & n+1 \end{pmatrix}$$

and the vector c has dimension $(n+1)$ and $c^{(1)}$ has dimension $(n+2)$ with the form

$$c = (c_0, c_1, \dots, c_n)^t, \\ c^{(1)} = (c_0^{(1)}, c_1^{(1)}, \dots, c_{n+1}^{(1)})^t.$$

Repeating the degree elevation r times gives the new sequence of control points $c^{(r)}$, where

$$c^{(r)} = T_{n,r} c.$$

The matrix $T_{n,r}$ of dimension $(n+r+1) \times (n+1)$ is given by

$$T_{n,r} = T_{n+r-1} T_{n+r-2} \dots T_{n+1} T_n$$

and its elements are given by

$$T_{n,r}(i, j) = \frac{\binom{n}{j} \binom{r}{i-j}}{\binom{n+r}{i}}, \quad i = 0, 1, \dots, n+r \text{ and } j = 0, 1, \dots, n.$$

Because the orthonormal Jacobi polynomials are orthogonal, the degree elevation of a polynomial in the orthonormal Jacobi basis is reduced to

$$d = (d_0, d_1, \dots, d_n)^t,$$

$$d^{(1)} = (d_0, d_1, \dots, d_n, 0)^t.$$

We repeat the degree elevation r times to get $d^{(r)} = I_{n,r}d$, where the elements of the matrix $I_{n,r}$ of dimension $(n + r + 1) \times (n + 1)$ have the form

$$I_{n,r}(i, j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

We get the r times degree elevation matrix $T_{n,r}$ by first transforming the Bernstein coefficients into the Jacobi coefficients using $(M_n^{(\alpha,\beta)})^{-1}$, and then elevating the degree r times by using $I_{n,r}$, and finally finding the Bernstein coefficients using $M_{n+r}^{(\alpha,\beta)}$. This is summarized in the following theorem.

Theorem 5.1. *The r times degree elevation matrix $T_{n,r}$ is obtained by multiplying the matrices $(M_n^{(\alpha,\beta)})^{-1}$, $I_{n,r}$ and $M_{n+r}^{(\alpha,\beta)}$ as follows:*

$$T_{n,r} = M_{n+r}^{(\alpha,\beta)} I_{n,r} (M_n^{(\alpha,\beta)})^{-1}.$$

6. DEGREE REDUCTION

The inverse process of degree elevation is called degree reduction. The degree reduction process aims to represent a given curve of degree n by one of degree $m < n$. In general, exact degree reduction is not possible. In the degree reduction process, we use the weighted L_2 -norm to find the best approximation.

For a given Bézier curve of degree n in (2.1) with the set of Bézier points $\{c_i\}_{i=0}^n$, find an approximative Bézier curve

$$b^m(x) = \sum_{i=0}^m b_i B_i^m(x)$$

of lower degree $m < n$ with the set of Bézier points $\{b_i\}_{i=0}^m$, so that the weighted L_2 -norm distance between b^m and f is minimum.

To solve this problem, we first elevate the degree of b^m to n using the r times degree elevation matrix $T_{m,r}$, $r = n - m$, to get

$$b^{(r)} = T_{m,r}b.$$

Thus, the curve b^m of degree m is written as a curve of degree n as follows:

$$b^m(x) = b^{(r)}(x) = \sum_{i=0}^n b_i^{(r)} B_i^n(x).$$

We are now able to write the weighted L_2 -distance of the two Bézier curves f and b^m in the form

$$\|b^m - f\|_w^2 = \|b^{(r)} - f\|_w^2 = \int_0^1 w(x) \left| \sum_{i=0}^n (b_i^{(r)} - c_i) B_i^n(x) \right|^2 dx.$$

Substituting the previous computations for the weighted L_2 -norm into the last equation gives the matrix form for the weighted L_2 -norm distance between the Bézier curve f of degree n and the Bézier curve b^m of degree m in the following theorem.

Theorem 6.1. *The weighted L_2 -distance between the Bézier curves f and b^m is given by*

$$\|b^m - f\|_w^2 = \|b^{(r)} - f\|_w^2 = A^t Q_n^{(\alpha, \beta)} A,$$

where $A := c - T_{m,r} b$, $b = (b_0, b_1, \dots, b_m)^t$ and $c = (c_0, c_1, \dots, c_n)^t$.

To simplify the last result of $\|b^m - f\|_w^2$, we substitute the value $A = c - T_{m,r} b$ to get

$$\|b^m - f\|_w^2 = c^t Q_n^{(\alpha, \beta)} c - 2b^t T_{m,r}^t Q_n^{(\alpha, \beta)} c + b^t T_{m,r}^t Q_n^{(\alpha, \beta)} T_{m,r} b. \quad (6.1)$$

The error in the last equation is a function of the elements of the vector b . The minimum \hat{b} in the sense of the least-squares method occurs when the first partial derivatives $\partial(A^t Q_n^{(\alpha, \beta)} A) / \partial b$ are equal to zero, see [3]. Thus, we get the following normal equations

$$T_{m,r}^t Q_n^{(\alpha, \beta)} T_{m,r} \hat{b} = T_{m,r}^t Q_n^{(\alpha, \beta)} c.$$

Similar approaches have been done in [14] and [17]. The matrix $T_{m,r}^t Q_n^{(\alpha, \beta)} T_{m,r}$ of dimension $(m+1) \times (m+1)$ is invertible, because $Q_{n-r}^{(\alpha, \beta)}$ is invertible, and $T_{m,r}^t Q_n^{(\alpha, \beta)} T_{m,r} = Q_{n-r}^{(\alpha, \beta)}$. And thus it follows that the normal equations are uniquely solvable and the solution is given by

$$\hat{b} = (T_{m,r}^t Q_n^{(\alpha, \beta)} T_{m,r})^{-1} T_{m,r}^t Q_n^{(\alpha, \beta)} c.$$

The vector \hat{b} of Bézier points describes the Bézier curve of best approximation in the least-squares sense with respect to the Jacobi-weighted L_2 -norm.

Because the orthonormal Jacobi polynomials are orthogonal, the degree reduction of a polynomial in the form of the orthonormal Jacobi polynomials is reduced to

$$d = (d_0, d_1, \dots, d_n)^t, \\ d^{(-1)} = (d_0, d_1, \dots, d_{n-1})^t.$$

Reducing the degree of the curve r times gives the Bézier points in the matrix form

$$d^{(-r)} = I_{n,-r}d,$$

where the matrix $I_{n,-r}$ of dimension $(n - r + 1) \times (n + 1)$ is the augmented matrix of the identity matrix of dimension $(n - r + 1)$ and the zero matrix, i.e.,

$$I_{n,-r} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix}. \tag{6.2}$$

First, we transform the Bernstein coefficients into the orthonormal Jacobi coefficients by $(M_n^{(\alpha,\beta)})^{-1}$. Next, we reduce the degree using $I_{n,-r}$. And finally, we find the Bernstein coefficients using the matrix $M_m^{(\alpha,\beta)}$. This process produces the matrix $R_{m,r}^{(\alpha,\beta)}$ of r times degree reduction. And thus we get the following theorem.

Theorem 6.2. *The r times degree reduction matrix $R_{m,r}^{(\alpha,\beta)}$ is computed using the Jacobi–Bernstein transformation matrices $(M_n^{(\alpha,\beta)})^{-1}$, $M_m^{(\alpha,\beta)}$, the Gram matrix $Q_n^{(\alpha,\beta)}$, and the matrix $I_{n,-r}$ as follows:*

$$R_{m,r}^{(\alpha,\beta)} = (T_{m,r}^t Q_n^{(\alpha,\beta)} T_{m,r})^{-1} T_{m,r}^t Q_n^{(\alpha,\beta)} = M_m^{(\alpha,\beta)} I_{n,-r} (M_n^{(\alpha,\beta)})^{-1}.$$

The error of approximation ϵ_w is given after some simplifications in the following theorem.

Theorem 6.3. *The error of the solution \hat{b} of the r -times degree reduction with respect to the weighted L_2 -norm is*

$$\epsilon_w^2 = c^t E_{m,r}^{(\alpha,\beta)} c,$$

where

$$E_{m,r}^{(\alpha,\beta)} = Q_n^{(\alpha,\beta)} \left[I - T_{m,r} (T_{m,r}^t Q_n^{(\alpha,\beta)} T_{m,r})^{-1} T_{m,r}^t Q_n^{(\alpha,\beta)} \right].$$

Theorem 6.2 combined with Theorems 3.3 and 3.4 and equation (6.2) describe a matrix form for optimal r times degree reduction with respect to the weighted L_2 -norm. The examples and figures in the next section show the efficiency and simplicity of applying this method.

7. EXAMPLES

We give some examples and draw some figures to show the efficiency of the algorithm described in Theorem 6.2. In Fig. 1, the weight function $w(x) = (2 - 2x)^2(2x)^\beta$ is plotted for the values of $\alpha = \beta = -1$, $\alpha = \beta = -1/2$, $\alpha = \beta = 0$, $\alpha = \beta = 1/2$, and $\alpha = \beta = 1$.

Example 7.1. ($n = 3, r = 1, m = 2$.) The matrices of degree reduction $R_{m,r}^{(\alpha,\beta)}$ and error $E_{m,r}^{(\alpha,\beta)}$ are computed for the reduction of a polynomial of degree 3 into polynomials of degree 2 for the values $\alpha = \beta = -1$, $\alpha = \beta = -1/2$, $\alpha = \beta = 0$, $\alpha = \beta = 1/2$.

$$R_{2,1}^{(-1,-1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{4} & \frac{3}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

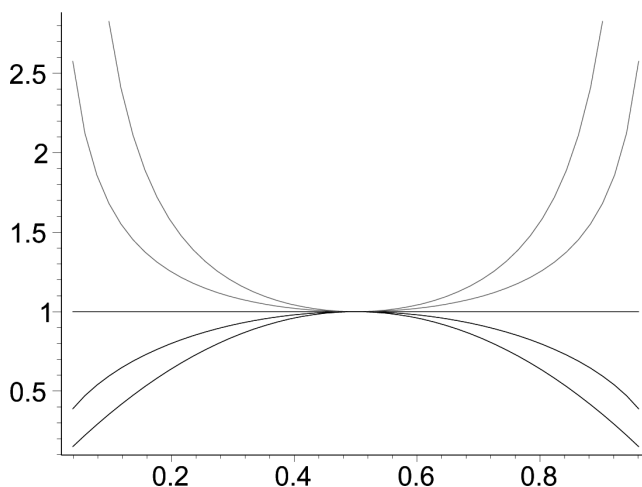


FIGURE 1 Weight function $w(x) = (2 - 2x)^2(2x)^\beta$, $\alpha = \beta = -1$, $\alpha = \beta = -1/2$, $\alpha = \beta = 0$, $\alpha = \beta = 1/2$, $\alpha = \beta = 1$.

$$E_{2,1}^{(-1,-1)} = \frac{1}{24} \begin{bmatrix} \frac{1}{20} & -\frac{3}{20} & \frac{3}{20} & -\frac{1}{20} \\ -\frac{3}{20} & \frac{9}{20} & -\frac{9}{20} & \frac{3}{20} \\ \frac{3}{20} & -\frac{9}{20} & \frac{9}{20} & -\frac{3}{20} \\ -\frac{1}{20} & \frac{3}{20} & -\frac{3}{20} & \frac{1}{20} \end{bmatrix},$$

$$R_{2,1}^{(-\frac{1}{2},-\frac{1}{2})} = \begin{bmatrix} \frac{31}{32} & \frac{3}{32} & -\frac{3}{32} & \frac{1}{32} \\ -\frac{1}{4} & \frac{3}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{32} & -\frac{3}{32} & \frac{3}{32} & \frac{31}{32} \end{bmatrix},$$

$$E_{2,1}^{(-\frac{1}{2},-\frac{1}{2})} = \frac{5\pi}{1024} \begin{bmatrix} \frac{1}{20} & -\frac{3}{20} & \frac{3}{20} & -\frac{1}{20} \\ -\frac{3}{20} & \frac{9}{20} & -\frac{9}{20} & \frac{3}{20} \\ \frac{3}{20} & -\frac{9}{20} & \frac{9}{20} & -\frac{3}{20} \\ -\frac{1}{20} & \frac{3}{20} & -\frac{3}{20} & \frac{1}{20} \end{bmatrix},$$

$$R_{2,1}^{(0,0)} = \begin{bmatrix} \frac{19}{20} & \frac{3}{20} & -\frac{3}{20} & \frac{1}{20} \\ -\frac{1}{4} & \frac{3}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{20} & -\frac{3}{20} & \frac{3}{20} & \frac{19}{20} \end{bmatrix},$$

$$E_{2,1}^{(0,0)} = \frac{1}{140} \begin{bmatrix} \frac{1}{20} & -\frac{3}{20} & \frac{3}{20} & -\frac{1}{20} \\ -\frac{3}{20} & \frac{9}{20} & -\frac{9}{20} & \frac{3}{20} \\ \frac{3}{20} & -\frac{9}{20} & \frac{9}{20} & -\frac{3}{20} \\ -\frac{1}{20} & \frac{3}{20} & -\frac{3}{20} & \frac{1}{20} \end{bmatrix},$$

$$R_{2,1}^{(\frac{1}{2},\frac{1}{2})} = \begin{bmatrix} \frac{15}{16} & \frac{3}{16} & -\frac{3}{16} & \frac{1}{16} \\ -\frac{1}{4} & \frac{3}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{16} & -\frac{3}{16} & \frac{3}{16} & \frac{15}{16} \end{bmatrix},$$

$$E_{2,1}^{(\frac{1}{2},\frac{1}{2})} = \frac{5\pi}{4096} \begin{bmatrix} \frac{1}{20} & -\frac{3}{20} & \frac{3}{20} & -\frac{1}{20} \\ -\frac{3}{20} & \frac{9}{20} & -\frac{9}{20} & \frac{3}{20} \\ \frac{3}{20} & -\frac{9}{20} & \frac{9}{20} & -\frac{3}{20} \\ -\frac{1}{20} & \frac{3}{20} & -\frac{3}{20} & \frac{1}{20} \end{bmatrix}.$$

Figure 2 shows the Bézier curve of degree 3 and the approximating curves of degree 2 with Bézier net using Theorem 6.2 for the values of $\alpha = \beta = -1$, $\alpha = \beta = -1/2$, $\alpha = \beta = 0$, $\alpha = \beta = 1$, and $\alpha = \beta = \infty$.

Example 7.2 ($n = 4, r = 1, m = 3$). We compute the matrices of degree reduction $R_{m,r}^{(\alpha,\beta)}$ and error $E_{m,r}^{(\alpha,\beta)}$ for the reduction of a polynomial of

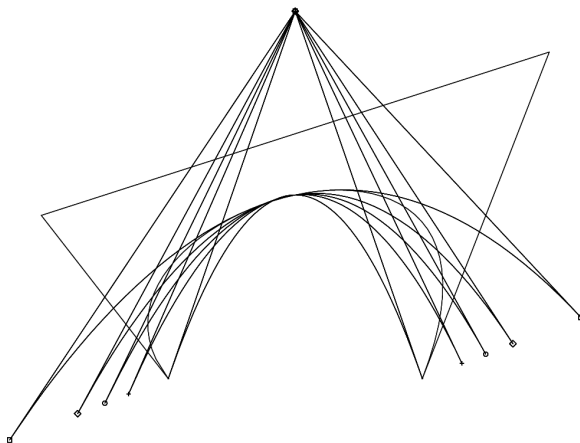


FIGURE 2 $n = 3, r = 1, m = 2$, \cdot : $\alpha = \beta = -1$, $+$: $\alpha = \beta = -1/2$, \circ : $\alpha = \beta = 0$, \diamond : $\alpha = \beta = 1$, \square : $\alpha = \beta = \infty$.

degree 4 into polynomials of degree 3 using Theorem 6.2 for the values $\alpha = \beta = -1$, $\alpha = \beta = -1/2$, $\alpha = \beta = 0$, $\alpha = \beta = 1/2$.

$$R_{3,1}^{(-1,-1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{4}{15} & \frac{16}{15} & \frac{2}{5} & -\frac{4}{15} & \frac{1}{15} \\ \frac{1}{15} & -\frac{4}{15} & \frac{2}{5} & \frac{16}{15} & -\frac{4}{15} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$E_{3,1}^{(-1,-1)} = \frac{1}{120} \begin{bmatrix} \frac{1}{70} & -\frac{2}{35} & \frac{3}{35} & -\frac{2}{35} & \frac{1}{70} \\ -\frac{2}{35} & \frac{8}{35} & -\frac{12}{35} & \frac{8}{35} & -\frac{2}{35} \\ \frac{3}{35} & -\frac{12}{35} & \frac{18}{35} & -\frac{12}{35} & \frac{3}{35} \\ -\frac{2}{35} & \frac{8}{35} & -\frac{12}{35} & \frac{8}{35} & -\frac{2}{35} \\ \frac{1}{70} & -\frac{2}{35} & \frac{3}{35} & -\frac{2}{35} & \frac{1}{70} \end{bmatrix},$$

$$R_{3,1}^{\left(-\frac{1}{2}, -\frac{1}{2}\right)} = \begin{bmatrix} \frac{127}{128} & \frac{1}{32} & -\frac{3}{64} & \frac{1}{32} & -\frac{1}{128} \\ -\frac{33}{128} & \frac{33}{32} & \frac{29}{64} & -\frac{29}{96} & \frac{29}{384} \\ \frac{29}{384} & -\frac{29}{96} & \frac{29}{64} & \frac{33}{32} & -\frac{33}{128} \\ -\frac{1}{128} & \frac{1}{32} & -\frac{3}{64} & \frac{1}{32} & \frac{127}{128} \end{bmatrix},$$

$$E_{3,1}^{\left(-\frac{1}{2}, -\frac{1}{2}\right)} = \frac{35\pi}{32768} \begin{bmatrix} \frac{1}{70} & -\frac{2}{35} & \frac{3}{35} & -\frac{2}{35} & \frac{1}{70} \\ -\frac{2}{35} & \frac{8}{35} & -\frac{12}{35} & \frac{8}{35} & -\frac{2}{35} \\ \frac{3}{35} & -\frac{12}{35} & \frac{18}{35} & -\frac{12}{35} & \frac{3}{35} \\ -\frac{2}{35} & \frac{8}{35} & -\frac{12}{35} & \frac{8}{35} & -\frac{2}{35} \\ \frac{1}{70} & -\frac{2}{35} & \frac{3}{35} & -\frac{2}{35} & \frac{1}{70} \end{bmatrix},$$

$$R_{3,1}^{(0,0)} = \begin{bmatrix} \frac{69}{70} & \frac{2}{35} & -\frac{3}{35} & \frac{2}{35} & -\frac{1}{70} \\ -\frac{53}{210} & \frac{106}{105} & \frac{17}{35} & -\frac{34}{105} & \frac{17}{210} \\ \frac{17}{210} & -\frac{34}{105} & \frac{17}{35} & \frac{106}{105} & -\frac{53}{210} \\ -\frac{1}{70} & \frac{2}{35} & -\frac{3}{35} & \frac{2}{35} & \frac{69}{70} \end{bmatrix},$$

$$E_{3,1}^{(0,0)} = \frac{1}{630} \begin{bmatrix} \frac{1}{70} & -\frac{2}{35} & \frac{3}{35} & -\frac{2}{35} & \frac{1}{70} \\ -\frac{2}{35} & \frac{8}{35} & -\frac{12}{35} & \frac{8}{35} & -\frac{2}{35} \\ \frac{3}{35} & -\frac{12}{35} & \frac{18}{35} & -\frac{12}{35} & \frac{3}{35} \\ -\frac{2}{35} & \frac{8}{35} & -\frac{12}{35} & \frac{8}{35} & -\frac{2}{35} \\ \frac{1}{70} & -\frac{2}{35} & \frac{3}{35} & -\frac{2}{35} & \frac{1}{70} \end{bmatrix},$$

$$R_{3,1}^{\left(\frac{1}{2}, \frac{1}{2}\right)} = \begin{bmatrix} \frac{251}{256} & \frac{5}{64} & -\frac{15}{128} & \frac{5}{64} & -\frac{5}{256} \\ -\frac{191}{768} & \frac{191}{192} & \frac{65}{128} & -\frac{65}{192} & \frac{65}{768} \\ \frac{65}{768} & -\frac{65}{192} & \frac{65}{128} & \frac{191}{192} & -\frac{191}{768} \\ -\frac{5}{256} & \frac{5}{64} & -\frac{15}{128} & \frac{5}{64} & \frac{251}{256} \end{bmatrix},$$

$$E_{3,1}^{\left(\frac{1}{2}, \frac{1}{2}\right)} = \frac{35\pi}{131072} \begin{bmatrix} \frac{1}{70} & -\frac{2}{35} & \frac{3}{35} & -\frac{2}{35} & \frac{1}{70} \\ -\frac{2}{35} & \frac{8}{35} & -\frac{12}{35} & \frac{8}{35} & -\frac{2}{35} \\ \frac{3}{35} & -\frac{12}{35} & \frac{18}{35} & -\frac{12}{35} & \frac{3}{35} \\ -\frac{2}{35} & \frac{8}{35} & -\frac{12}{35} & \frac{8}{35} & -\frac{2}{35} \\ \frac{1}{70} & -\frac{2}{35} & \frac{3}{35} & -\frac{2}{35} & \frac{1}{70} \end{bmatrix}.$$

The Bézier nets and Bézier curve of degree 4 and the approximating curves of degree 3 using Theorem 6.2 are plotted in Fig. 3 for the values of $\alpha = \beta = -1$, $\alpha = \beta = -1/2$, $\alpha = \beta = 0$, $\alpha = \beta = 1$, and $\alpha = \beta = \infty$.

Example 7.3 ($n = 6, r = 3, m = 3$). The 3-times degree reduction of the Bézier curve of degree 6 to the approximating curves of degree 3 using Theorem 6.2 are plotted in Fig. 4 for the values of $\alpha = \beta = -1$, $\alpha = \beta = -1/2$, $\alpha = \beta = 0$, $\alpha = \beta = 1$, and $\alpha = \beta = \infty$.

8. CONCLUSIONS

In this article, we derived a simple and efficient method based on matrix computations given in Theorems 3.3, 3.4, and 6.2, and Eq. (6.2) to write r times degree reduction with respect to the weighted L_2 -norm. The error term is given in Theorem 6.3. We use the explicit matrix forms

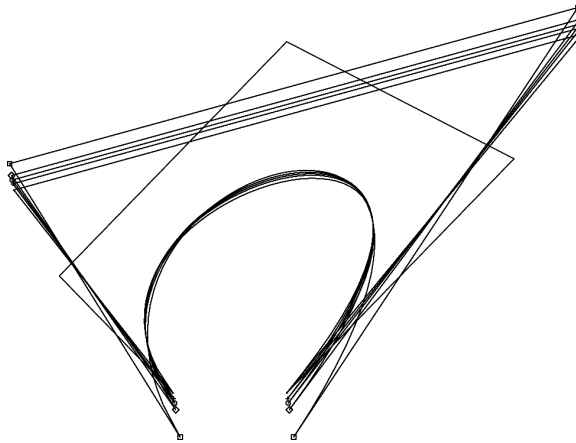


FIGURE 3 $n = 4, r = 1, m = 3, \cdot : \alpha = \beta = -1, + : \alpha = \beta = -1/2, \circ : \alpha = \beta = 0, \diamond : \alpha = \beta = 1, \square : \alpha = \beta = \infty.$

of $M_m^{(\alpha, \beta)}, (M_n^{(\alpha, \beta)})^{-1}$ and $I_{n, -r}$, given by Theorems 3.3, 3.4 and Eq. (6.2), to compute $M_m^{(\alpha, \beta)} I_{n, -r} (M_n^{(\alpha, \beta)})^{-1}$ in Theorem 6.2.

With the help of [23], we can explain the geometric situation of the Bézier points of the degree reduction. The Bézier points of the optimal component-wise degree reduction lie on parallel lines for any norm. Note that our result includes many previous results.

- (1) For the C_0 constrained best approximation with respect to L_2 -norm, we use $\alpha = \beta = -1$ (constrained Jacobi polynomials).

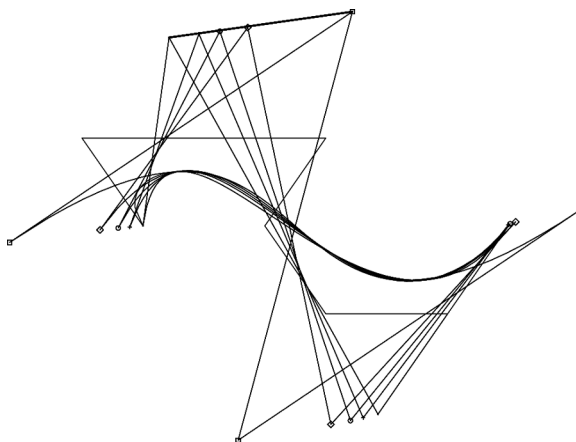


FIGURE 4 $n = 6, r = 3, m = 3, \cdot : \alpha = \beta = -1, + : \alpha = \beta = -1/2, \circ : \alpha = \beta = 0, \diamond : \alpha = \beta = 1, \square : \alpha = \beta = \infty.$

- (2) For the best approximation with respect to L_∞ -norm, we use $\alpha = \beta = -1/2$ (Chebyshev polynomials of first kind).
- (3) For the best approximation with respect to L_2 -norm, we use $\alpha = \beta = 0$ (Legendre polynomials).
- (4) For the best approximation with respect to L_1 -norm, we use $\alpha = \beta = 1/2$ (Chebyshev polynomials of second kind).

The examples and figures show the simplicity and efficiency of the method.

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