APPROXIMATE CONVERSION OF RATIONAL BÉZIER CURVES *

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Abstract. It is frequently important to approximate a rational Bézier curve by an integral, i.e., polynomial one. This need will arise when a rational Bézier curve is produced in one CAD system and is to be imported into another system, which can only handle polynomial curves. The objective of this paper is to present an algorithm to approximate rational Bézier curves with polynomial curves of higher degree.

Key words. CAD, CAGD, rational Bézier curve, approximation, degree elevation

AMS subject classifications. 65D17, 68U05

1. Introduction. A polynomial Bézier curve of degree n can be represented by

$$b^n(t) = \sum_{k=0}^{n} b_k B_k^n(t), \tag{1}$$

where $B_k^n(t)$ are Bernstein polynomials of degree n, and $b_k (k = 0, \cdots, n)$ are control points of $b^n(t)$. A rational Bézier curve is defined by

$$b^n(t) = \frac{\sum_{k=0}^{n} w_k b_k B_k^n(t)}{\sum_{k=0}^{n} w_k B_k^n(t)}, \tag{2}$$

where w_k are weights of the control points b_k. Clearly (1) is a genuine generation of (2): we obtain the polynomial case by setting all w_k's are same values.

A great majority of CAD systems for free-form curve and surface modelling use polynomial representations; nowadays rational representations are also introduced because of the larger degree of freedom that they offer and the exact representation of conics. The fundamental idea of the rational Bézier algorithms is to evaluate and manipulate the curves and surfaces by a small number of control points and weights. The new industrial interface STEP also deals with rational curves and surfaces. With the availability of a fast growing variety of modelling systems, the demand has risen to exchange curve and surface descriptions between CAD systems. It is frequently important to approximate a rational Bézier curve by an integral, i.e., polynomial one. This need will arise when a rational Bézier curve is produced in one CAD system and is to be imported into another system, which can only handle polynomial curves. This paper describes two algorithms to approximate rational Bézier curves with polynomial curves of higher degree.

2. Least Squares Approximation by Polynomial Bézier Curve. The most appropriate metric for the curves in geometrical terms would be the *Hausdorff distance* ([2]). Suppose (M, d) is a metric space with subsets A and B. We define the Hausdorff metric d_H by

$$d_H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}, \text{where } d(x,B) = \inf_{y\in B} d(x,y).$$

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If we regard a planar curve as simply a locus of points without any underlying parameterization, the Hausdorff metric for two such curves is essentially the radius of the largest circle with its center on one curve and touching the other curve. For general parametric curves, this measure is truly independent of the relative parameterizations of two curves. Emery([3]) presents a method for explicit computation of Hausdorff metric for piecewise linear curves, but the computation of Hausdorff distance d_H of two nonlinear curves is not so easy. So we define and use the L_2 distance for the Bézier curves. In general least squares approximation of rational Bézier curves address the following problem.

PROBLEM 1 (Least Squares Approximation). Find another points set $\{c_i\}_{i=0}^m$ defining the approximative polynomial Bézier curve of higher degree $m > n$ so that the least squares distance function

$$d_{LS}(b^n, c^m) = \sqrt{\int_0^1 \|b^n(t) - c^m(t)\|^2 dt}$$

between b^n and c^m is minimized on the interval $[0,1]$, where $\|\cdot\|$ denotes the Euclidean distance.

We define the Hadamard product and Hadamard division of the matrices. Let A and B be the $n \times m$ matrices with $A = (a_{i,j})_{n\times m}$ and $B = (b_{i,j})_{n\times m}$. Then the Hadamard product is $A \odot B = (a_{i,j} \times b_{i,j})_{n\times m}$ and the Hadamard division is $A \oslash B = (a_{i,j}/b_{i,j})_{n\times m}$.

Let the $W = (w_0, w_1, \ldots, w_n)^t$ and $B = (b_0, b_1, \ldots, b_n)^t$ be the $(n+1)-$vectors. Then, we can write the degree elevation formula ([9], [10]) of rational Bézier curve as a linear system $W^{(r)} = T_{n,r}W$ and $W^{(r)} \odot B^{(r)} = T_{n,r}(W \odot B)$, the $(n+r+1)\times(n+1)$ matrix $T_{n,r} = T_{n+r-1}T_{n+r-2}\ldots T_{n+1}T_n$ has elements

$$t_{i+j,i} = \frac{\binom{n}{i}\binom{r}{j}}{\binom{n+r}{i+j}}, \quad \begin{cases} i = 0,1,\ldots,n \\ j = 0,1,\ldots,r. \end{cases}$$

Let us consider the least squares distance function.

$$\begin{aligned} d_{LS}(b^n, c^m)^2 &= \int_0^1 \|b^n(t) - c^m(t)\|^2 dt \\ &= \lim_{N\to\infty} \frac{1}{N}\sum_{i=0}^{N} \|b^n(\frac{i}{N}) - c^m(\frac{i}{N})\|^2 \\ &= \lim_{r\to\infty} \frac{1}{n+r}\sum_{i=0}^{n+r} \|b_i^{(r)} - c_i^{(r-l)}\|^2 \quad \text{,where } l = m-n \\ &= \lim_{r\to\infty} \frac{1}{n+r}\sum_{i=0}^{n+r} [b_i^{(r)} - c_i^{(r-l)}][b_i^{(r)} - c_i^{(r-l)}] \\ &= \frac{1}{n+r}[B^{(r)} - C^{(r-l)}]^t[B^{(r)} - C^{(r-l)}] \quad \text{for some large r} \\ &= \frac{1}{n+r}D^t D, \end{aligned}$$

where $D = B^{(r)} - C^{(r-l)} = [T_{n,r}(W \odot B)] \oslash [T_{n,r}W] - T_{m,r-l}C$.

For some large r, it is sufficient from now on to investigate the Euclidian distance with respect to the control points. One method of obtaining the vector C is so-called method of least squares ([4]). Choosing the vector $\hat{C}$ that value of C which minimize D^tD involves differentiating D^tD with respect to the elements of C. Equating $\partial(D^tD)/\partial C$ to zero and writing the resulting equations in terms of $\hat{C}$, we find that these equations are

$$T^t_{m,r-l}T_{m,r-l}\hat{C} = T^t_{m,r-l}\{[T_{n,r}(W \odot B)] \oslash [T_{n,r}W]\}.$$

They are known as the normal equations. Provided $(T^t_{m,r}T_{m,r})^{-1}$ exists they have the unique solution for $\hat{C}$,

$$\hat{C} = (T^t_{m,r-l}T_{m,r-l})^{-1}T^t_{m,r-l}\{[T_{n,r}(W \odot B)] \oslash [T_{n,r}W]\}.$$

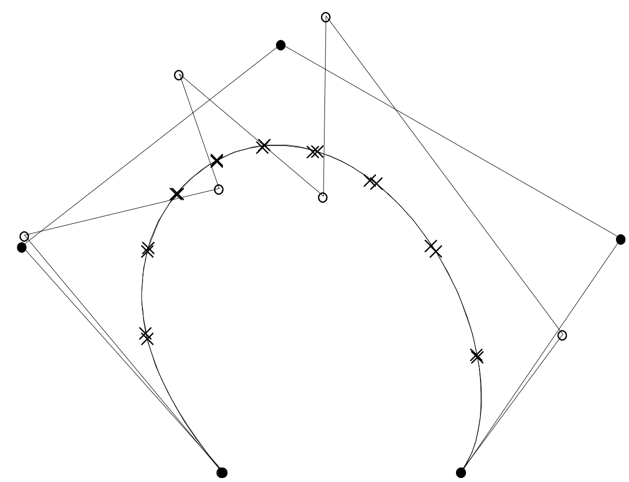

FIG. 2.1. *Least squares approximation by polynomial Bézier curve(open). The rational Bézier curve(solid) is over the weight vector* $(1,2,3,1,1), n = 4, m = 7, r = 30$. *Times marks indicate the images of curves evaluated at 10 equal parameter intervals.*

3. Approximate Conversion of Rational Bézier Curve. In general, we will not be able to converse rational Bézier curve to polynomial one. But, we can be approximately represented as a polynomial Bézier curve of sufficiently large degree m,

$$\frac{\sum_{i=0}^n w_i b_i B_i^n(t)}{\sum_{i=0}^n w_i B_i^n(t)} \approx \sum_{j=0}^m c_j B_j^m(t) \text{ for some large } m.$$

$$\begin{aligned}\sum_{i=0}^n w_i b_i B_i^n(t) &\approx \sum_{i=0}^n w_i B_i^n(t) \sum_{j=0}^m c_j B_j^m(t)\\ &= \sum_{i=0}^n \sum_{j=0}^m w_i c_j B_i^n(t) B_j^m(t)\\ &= \sum_{i=0}^n \sum_{j=0}^m \frac{\binom{n}{i}\binom{m}{j}}{\binom{n+m}{i+j}} w_i c_j B_{i+j}^{n+m}(t).\end{aligned}$$

The degree of left-hand side curve can be elevated from n to $n+m$. Then we make use of the generalized degree elevation formula:

$$\sum_{k=0}^{n+m} \sum_{i+j=k} \frac{\binom{n}{i}\binom{m}{j}}{\binom{n+m}{k}} w_i b_i B_k^{n+m}(t) \approx \sum_{k=0}^{n+m} \sum_{i+j=k} \frac{\binom{n}{i}\binom{m}{j}}{\binom{n+m}{k}} w_i c_j B_k^{n+m}(t).$$

This simplifies enough so that we can compare coefficients:

$$\sum_{i+j=k} \frac{\binom{n}{i}\binom{m}{j}}{\binom{n+m}{k}} w_i b_i = \sum_{i+j=k} \frac{\binom{n}{i}\binom{m}{j}}{\binom{n+m}{k}} w_i c_j.$$

Then, we can rewrite the formula as a linear system

$$T_{m,n}^{w} C = T_{n,m}(W \odot B),$$

where the $(m+n+1) \times (m+1)$ matrix $T_{m,n}^{w}$ has elements $(w_j t_{i+j,i})$. The linear equation $T_{m,n}^{w} C = T_{n,m}(W \odot B)$ are consistent for some large m , i.e.

$$T_{m,n}^{w} (T_{m,n}^{w^t} T_{m,n}^{w})^{-1} T_{m,n}^{w^t} T_{n,m} = T_{n,m}$$

and have

$$C = (T_{m,n}^{w^t} T_{m,n}^{w})^{-1} T_{m,n}^{w^t} T_{n,m}(W \odot B)$$

as a solution.

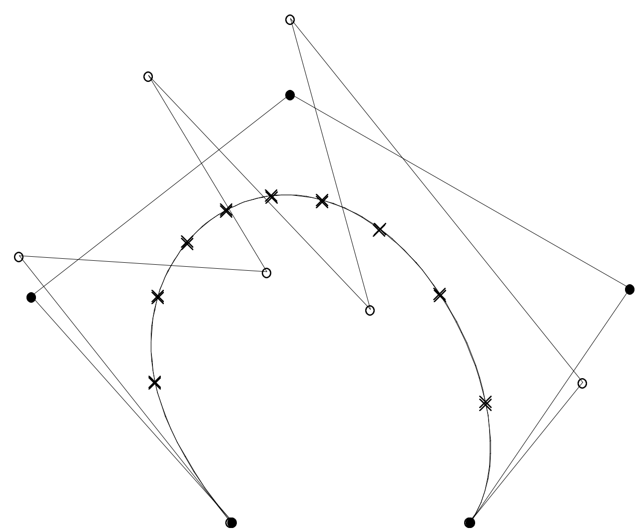

FIG. 3.1. *Approximate Conversion of rational Bézier curve. The rational Bézier curve(solid) is over the weight vector* $(1,2,3,1,1), n=4, m=30$. *The Bézier curve(open) after degree reduction from* m *to 7. Times marks indicate the images of curves evaluated at 10 equal parameter intervals.*

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