Mathematical and Computational Issues on Medial Axis Transform

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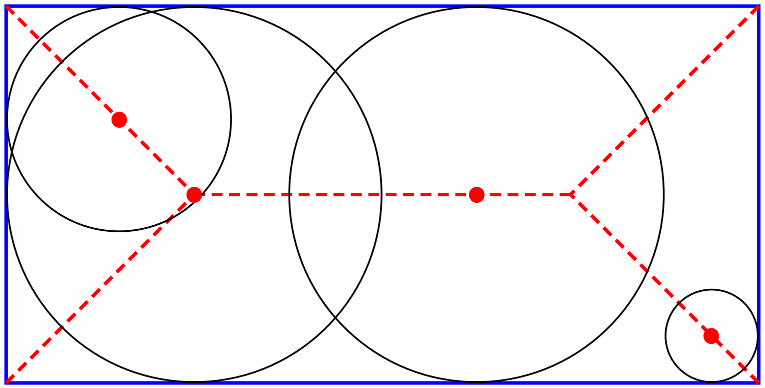
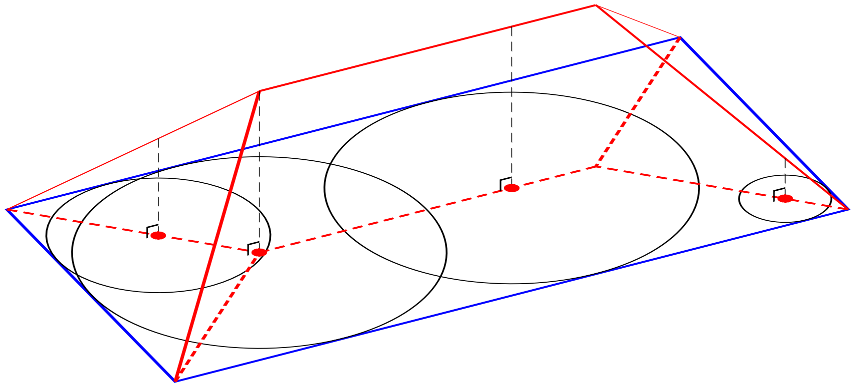
What is Medial Axis Transform?

$$\mathbf{MAT}(\Omega) = \{(p,r) \in \mathbb{R}^n \times \mathbb{R} \,|\, B_r(p) \text{ is a maximal ball in } \Omega\}$$

$$\mathbf{MA}(\Omega) = \{p \in \mathbb{R}^n \,|\, \exists r \geq 0 \; \textit{s.t.} \; (p,r) \in \mathbf{MAT}(\Omega)\}$$

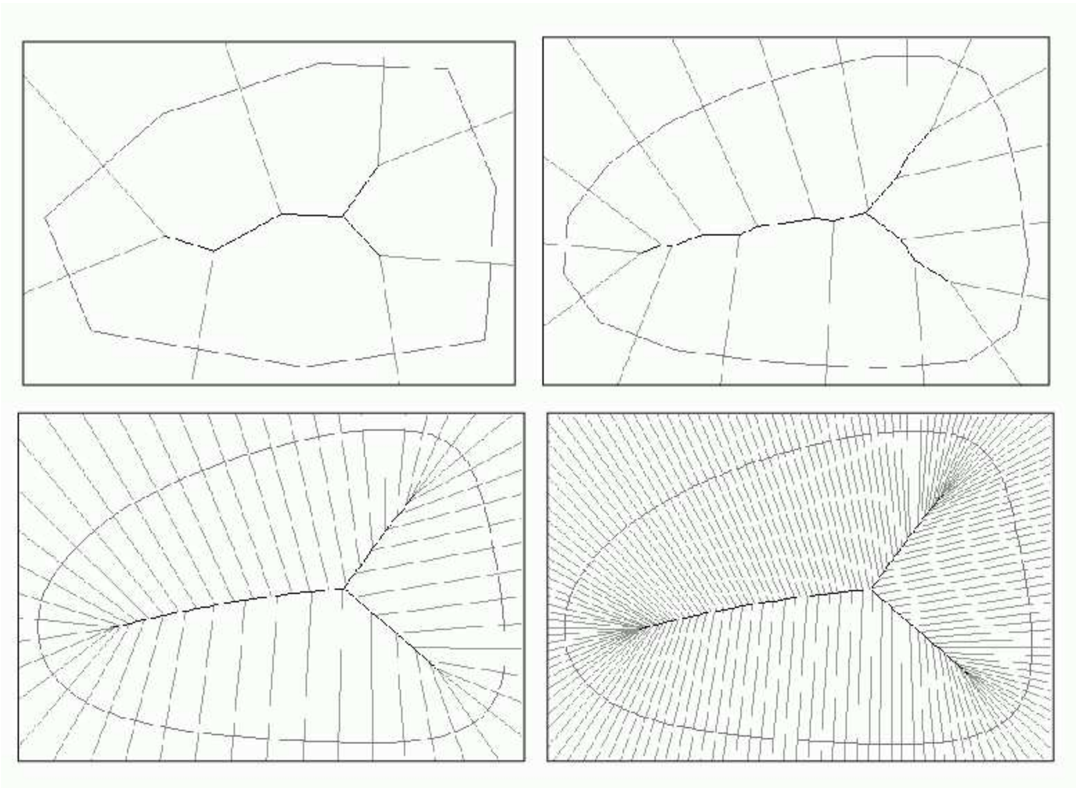




Other Definitions

- Grass Fire Model
  - points where wave starting from the boundary meet
- Cut Locus
  - points where the distance to the boundary stop minimizing
- Set of Singularities
  - points where the distance function to the boundary is not differentiable

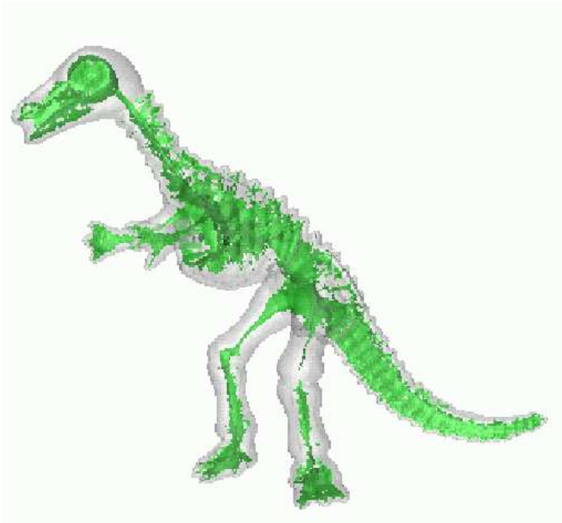
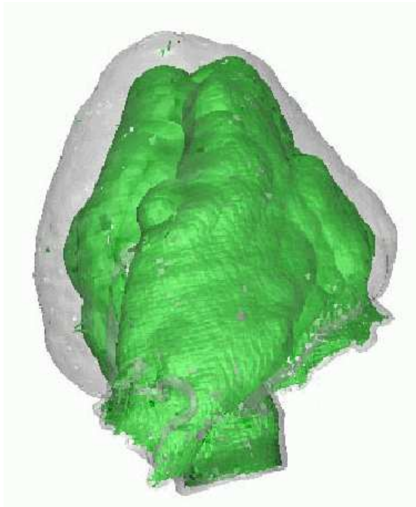
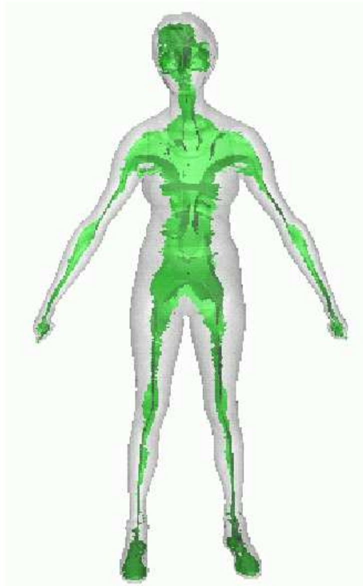
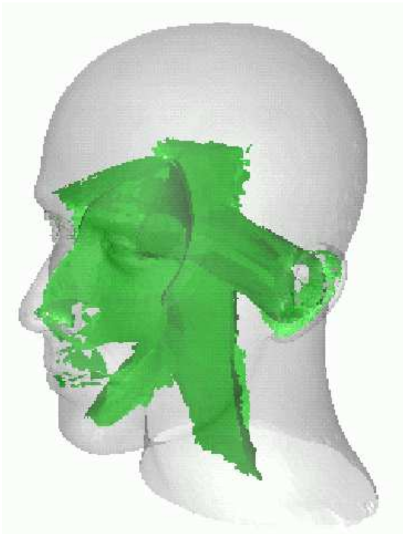
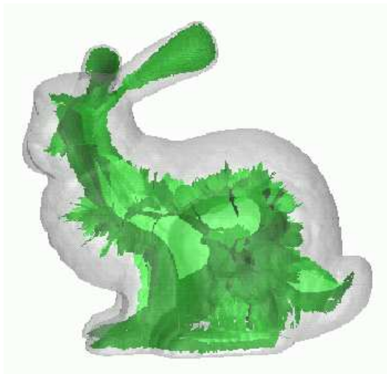
Relation with Voronoi Diagram



N. Amenta, *et al.*

Examples

More Examples


Good & Bad Aspects

- Homotopy equivalence: Preserves topological structure of domains.
- Compact representation (graph structure): Easy to store and process in computer.
- Wide range of applications (Shape Analysis)
  pattern recognition, computer vision, computer graphics, CAGD, mechanical engineering, biological/medical applications

Problems:

- Difficult to compute
- Sensitive to noise
- Poor mathematical analyses so far

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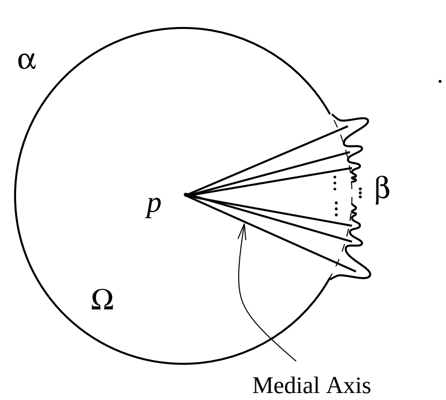
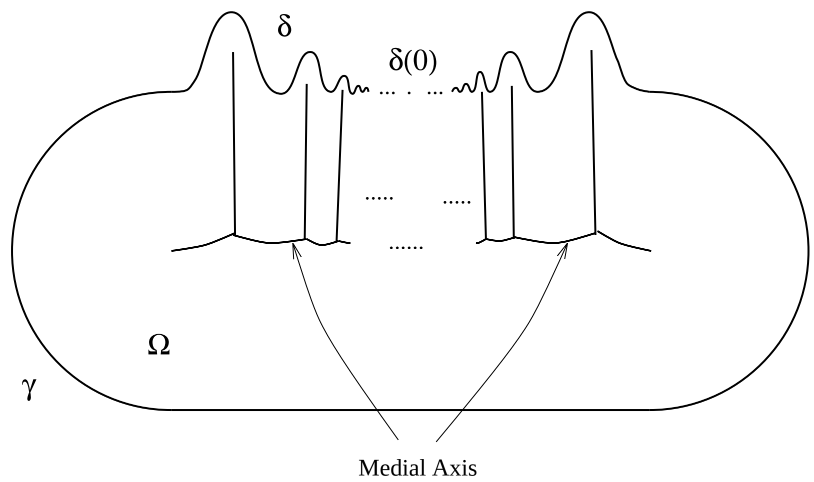
Pathological Shapes

- $\mathbf{MA}$ ($\mathbf{MAT}$) can exhibit pathological behaviours even for domains with C^∞ boundary.





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Normal Domain

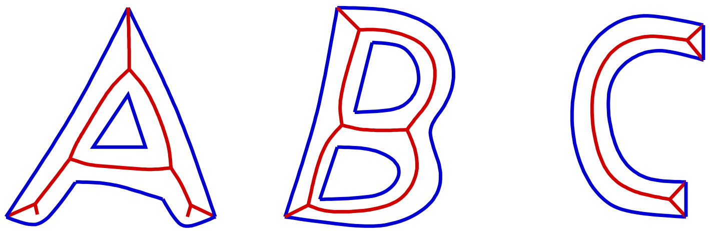
Ω: compact & connected in $\mathbb{R}^2$ is a *normal domain*, if

- $\partial\Omega$: *finite* number of *simple closed* curves
- each boundary component: *finite* number of *real analytic* curve pieces.

Theorem (*Pacific. J. Math.* 1997)

Ω: a normal domain. $\Longrightarrow$

- $\mathbf{MA}(\Omega)$ and $\mathbf{MAT}(\Omega)$ have *finite graph structures*.
- $\mathbf{MA}(\Omega)$ is a *strong deformation retract* of Ω.

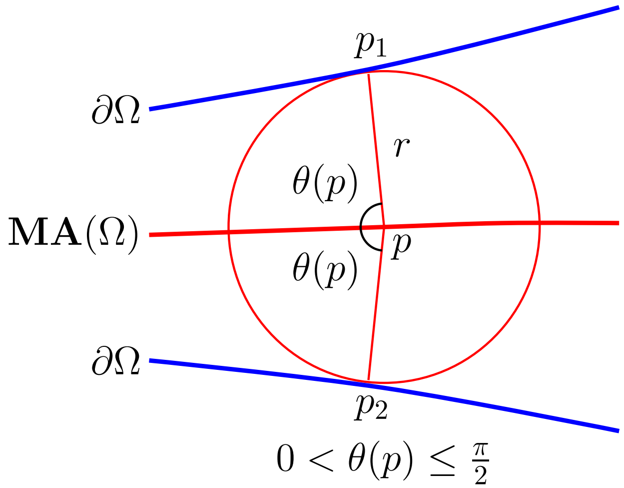


♣ Almost every domain in applications is normal.

Generic Points – 2D

$G(\Omega)$: the set of $\mathbf{MA}$ points with two contact points

$\mathbf{MA}(\Omega)$ except for finitely many points

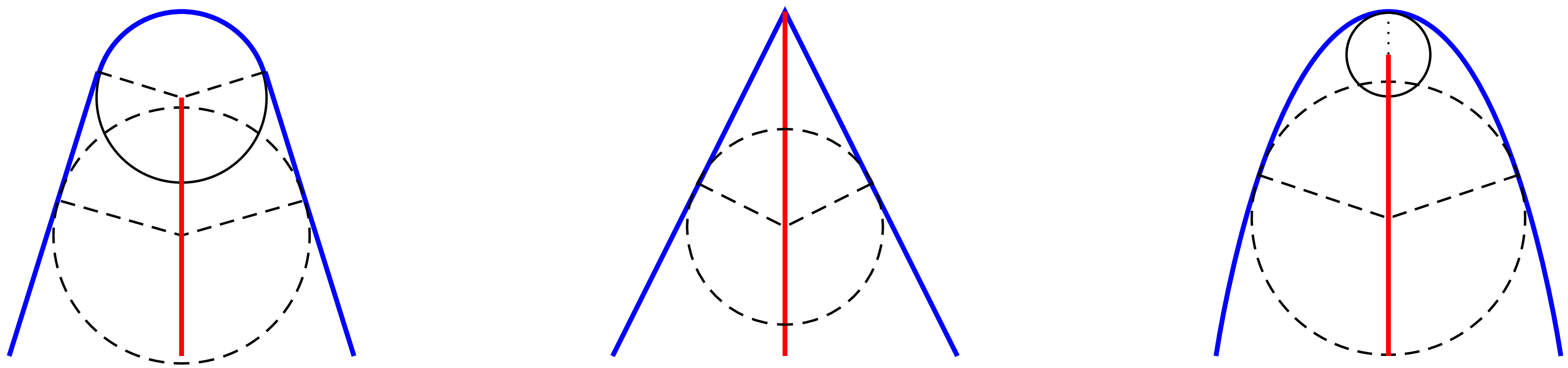


$$\theta_\Omega = \inf\{\theta(p) : p \in G(\Omega)\}$$

$$(0 \leq \theta_\Omega \leq \frac{\pi}{2})$$

Definition: Ω is *weakly injective*, if $\theta_\Omega > 0$.

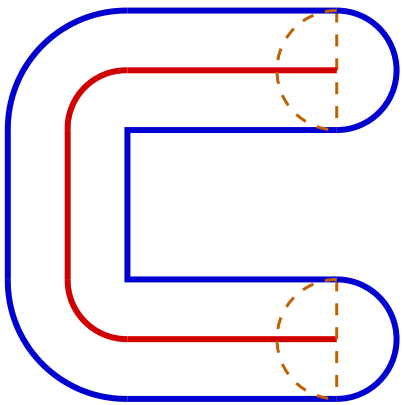
Three Types of 1-Prong Points – 2D

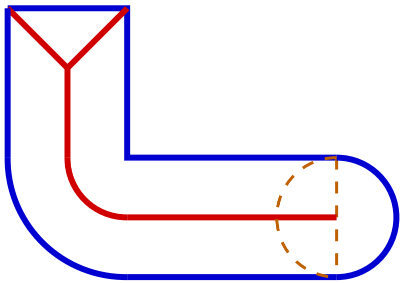


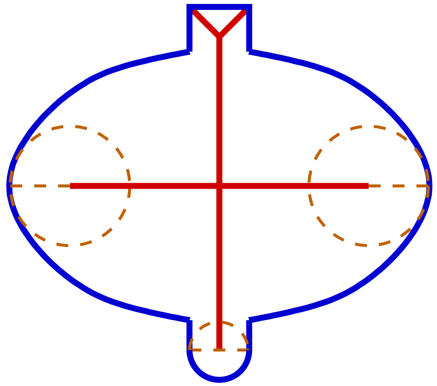
(a) (b) (c)

- $\theta_\Omega = 0 \iff \mathbf{MA}(\Omega)$ has a 1-prong point of type (c).
- Ω is weakly injective $\iff$ only (a) and (b)

Examples



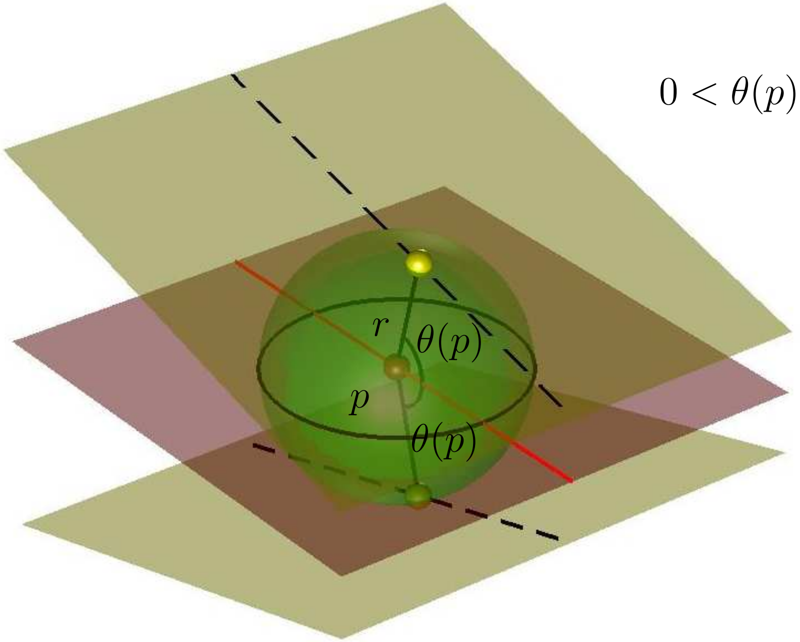




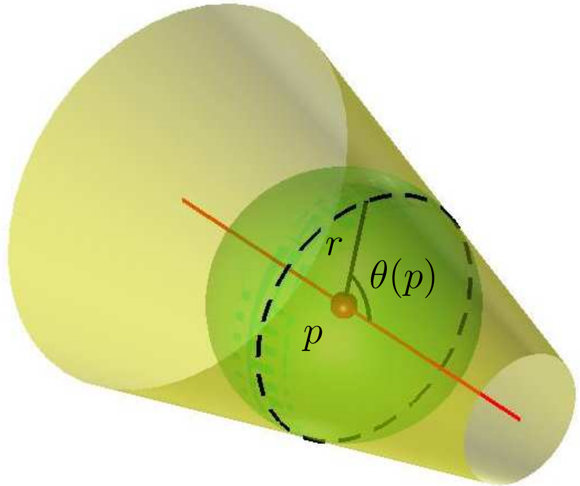
weakly injective

not weakly injective
(normal)

Generic Points – 3D



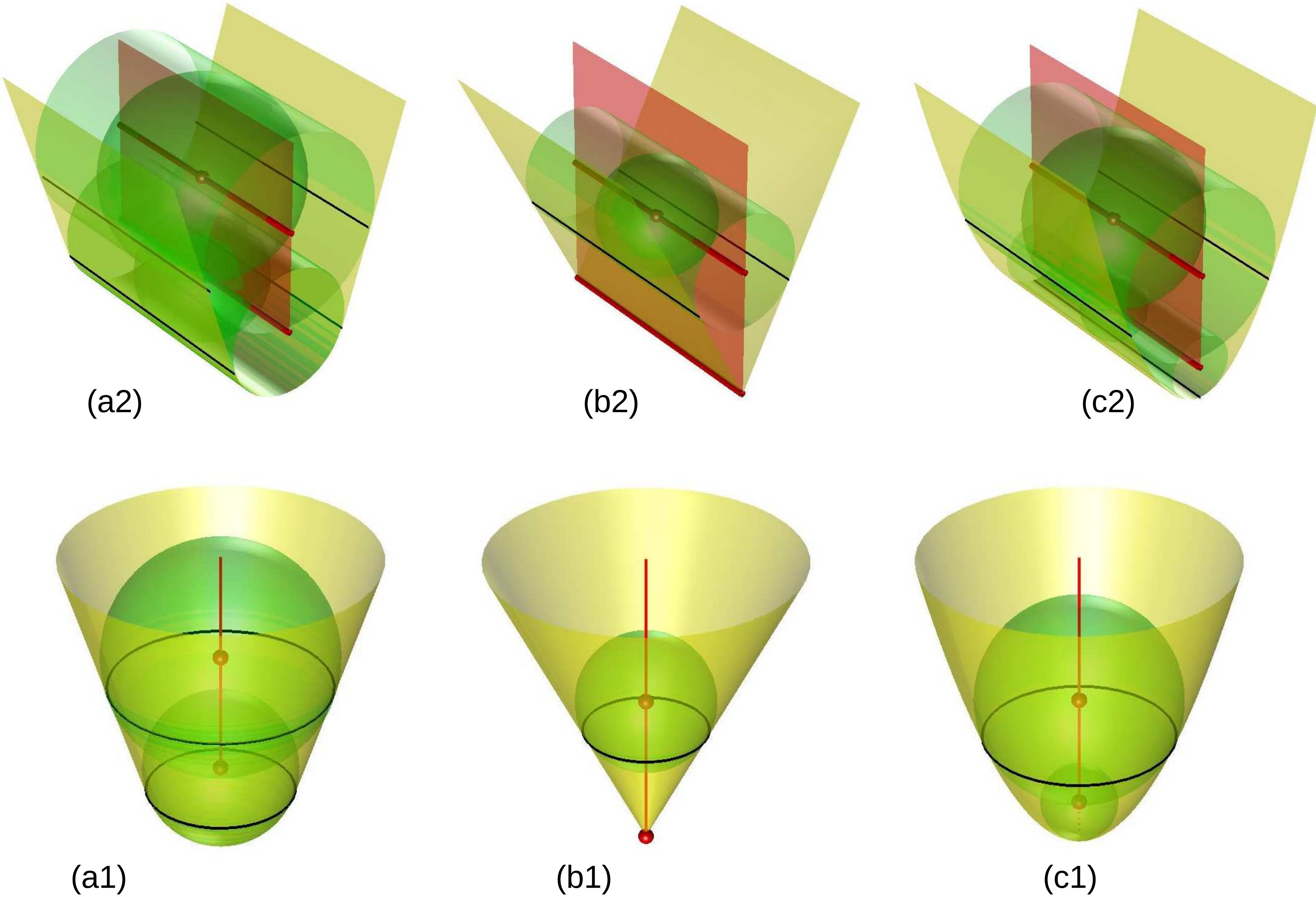
$0 < \theta(p) \leq \frac{\pi}{2}$



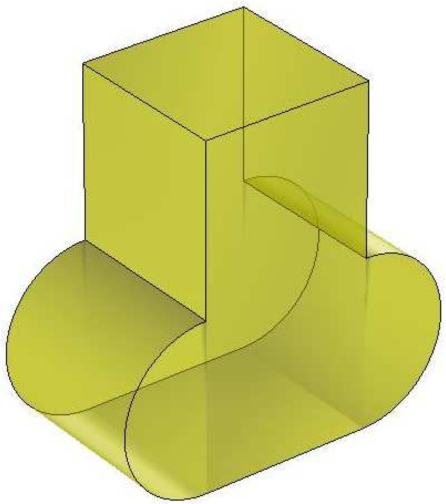
$$\theta_\Omega = \inf\{\theta(p) \,:\, p \in G(\Omega)\}$$

Definition: Ω is *weakly injective*, if $\theta_\Omega > 0$.

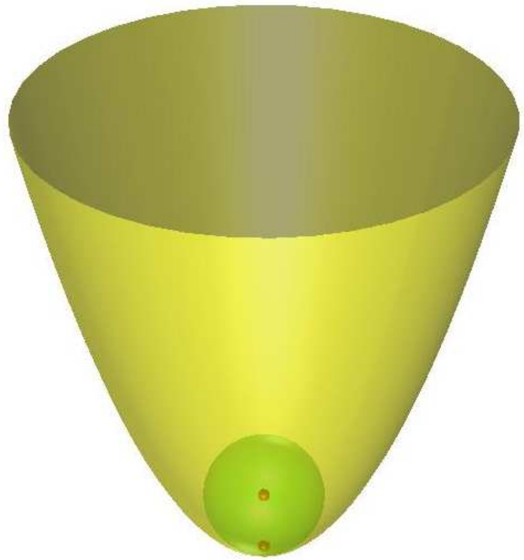
Types of 1-Prong Points – 3D



Examples



weakly injective



not weakly injective
(pseudonormal)

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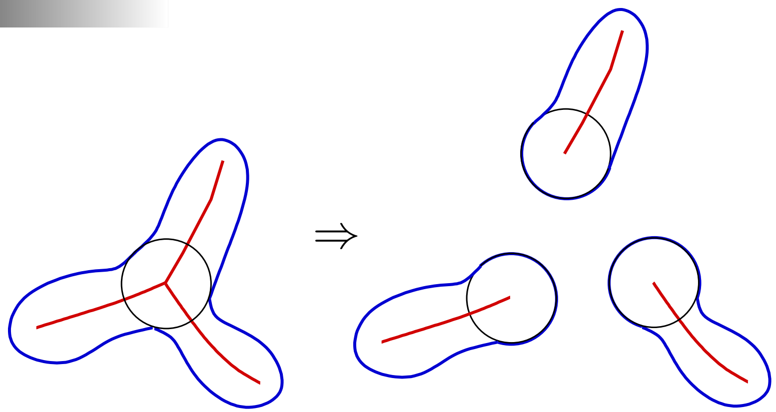
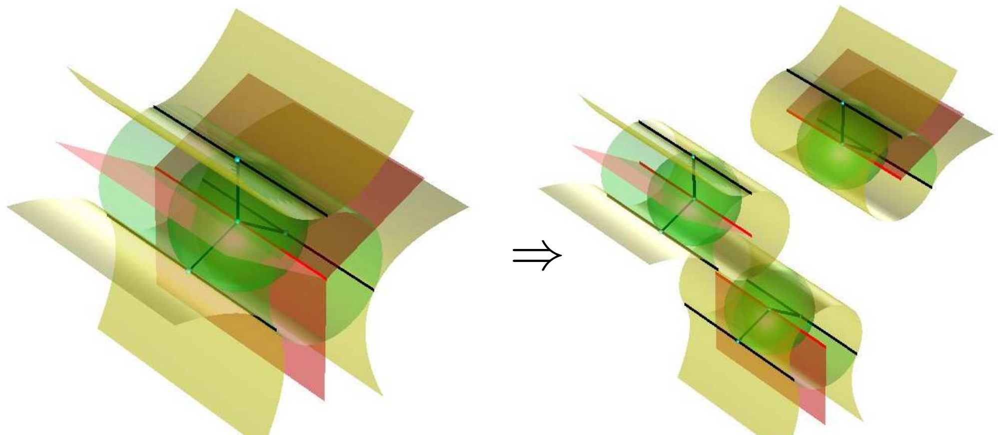
Algorithm



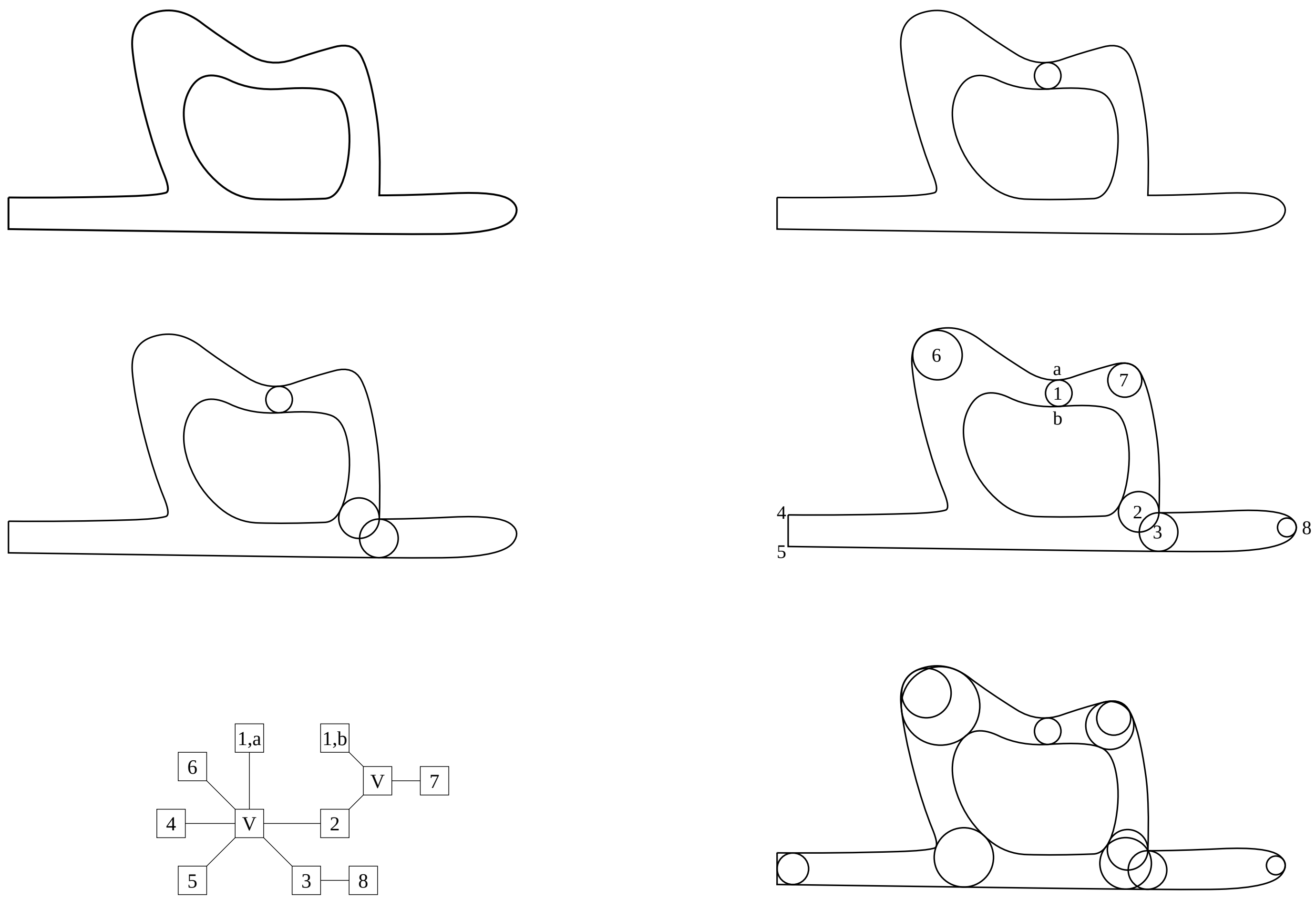


Essentially the first for domains *with free-form boundary* and *non-trivial homology* (*GMIP* 1997)

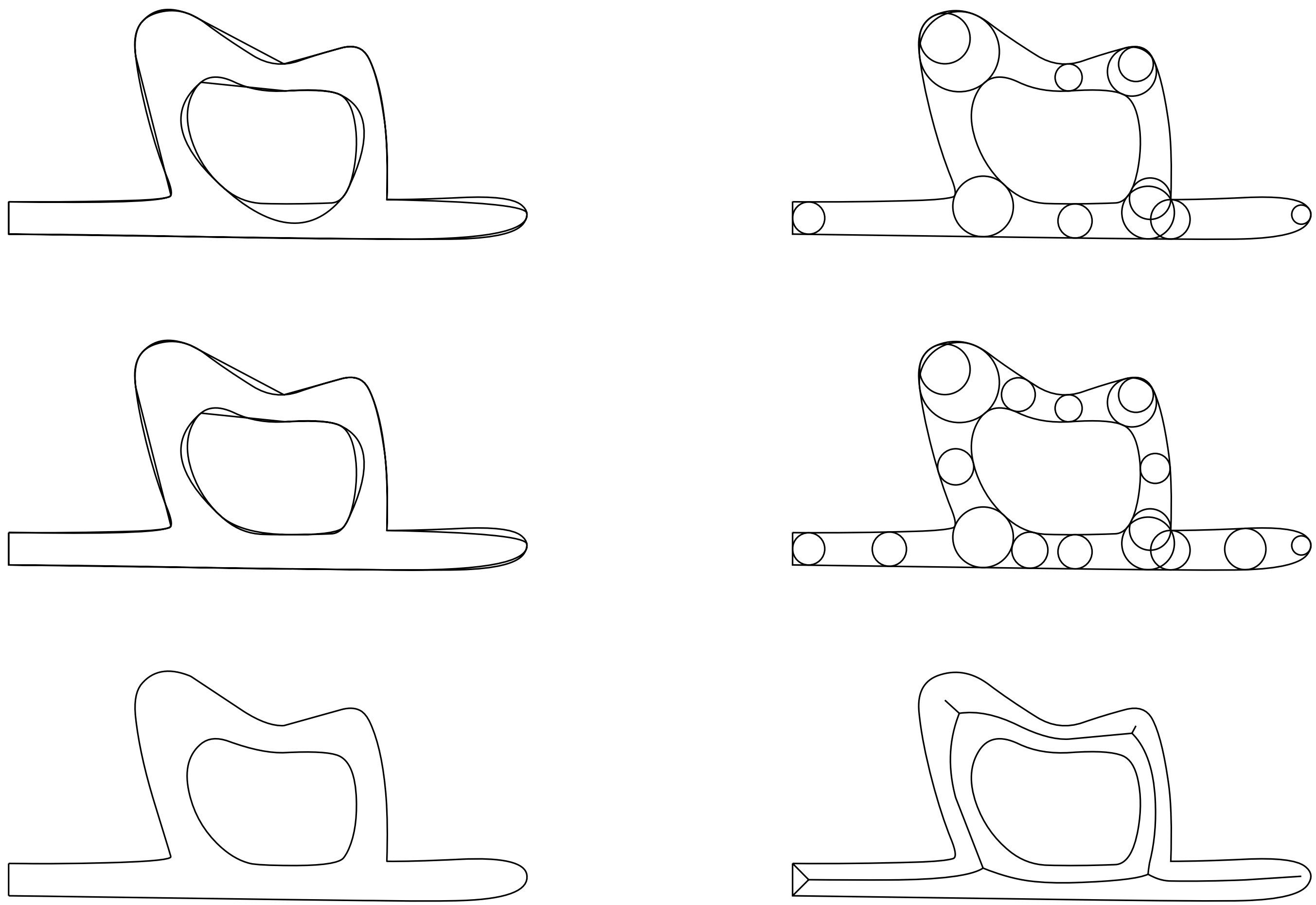
- approximation/interpolation (transcendental curves)
- domain decomposition (fast)
- updating tree data structure (topologically correct)

♣ 3D extension – need extensive theoretical analysis

Example

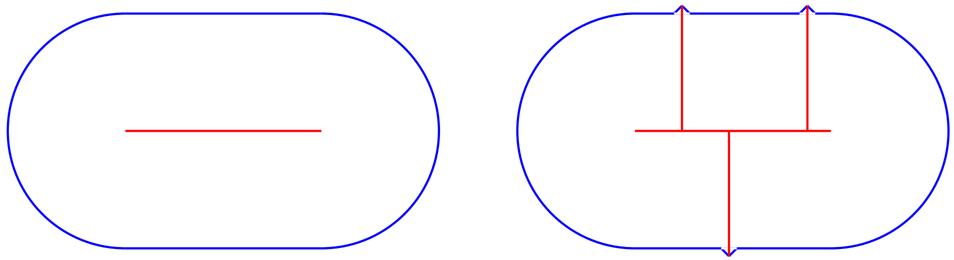


Example

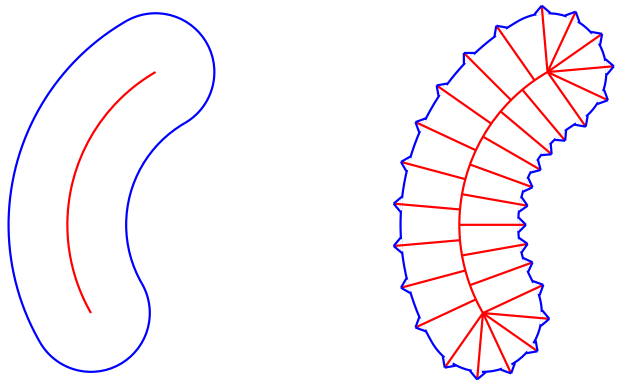


Instability of MAT

- Very sensitive to boundary perturbation.



- Introduces many *hairy* prongs.

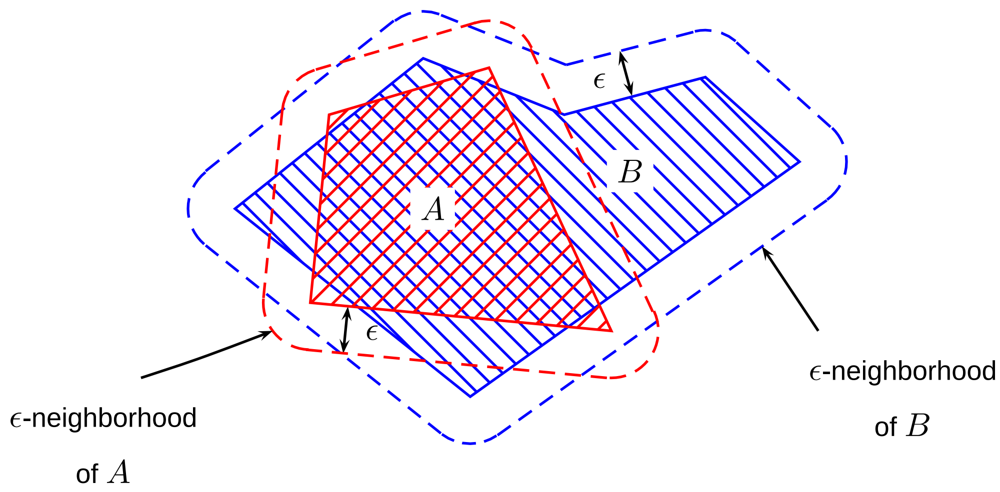


- No explicit and quantitative error analyses before.

One-sided Hausdorff Distance

A, B: compact sets in $\mathbb{R}^n$

$$\mathcal{H}(A|B) = \max_{p \in A} d(p, B)$$

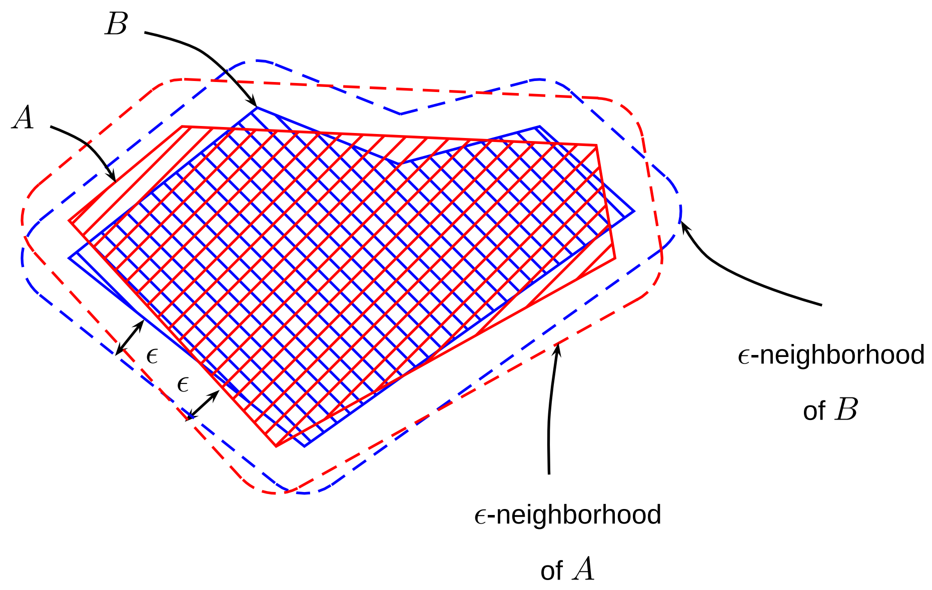


$\mathcal{H}(A|B)$ measures how much A is *contained* in B.

(Two-sided) Hausdorff Distance

A, B : compact sets in $\mathbb{R}^n$.

$$\begin{aligned}\mathcal{H}(A,B) &= \max\{\max_{p\in A} d(p,B), \max\{\max_{q\in B} d(q,A)\}\} \\ &= \max\{\mathcal{H}(A|B), \mathcal{H}(B|A)\}.\end{aligned}$$



$\mathcal{H}(A,B)$ measures how *similar* A and B are.

One-sided Stability of MAT

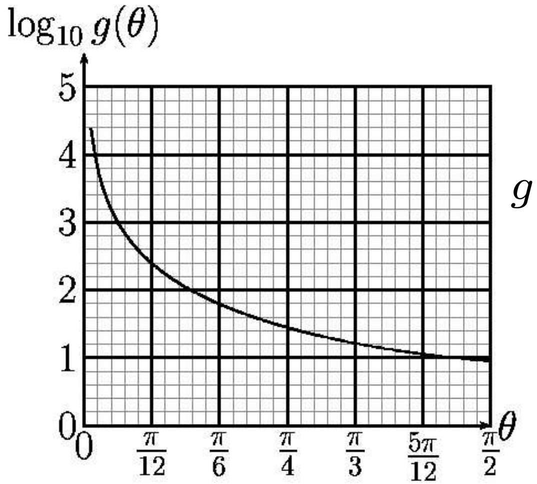
Theorem (*ICPR* 2000, *DAGM* 2001, *ACM Symp. Solid Modeling* 2002, *J. Math. Imaging & Vision* 2002)

Let Ω be a *weakly injective* domain (2D & 3D). Then we have

$$\begin{aligned} \mathcal{H}(\mathbf{MAT}(\Omega)|\mathbf{MAT}(\Omega')) &\leq g(\theta_\Omega)\cdot\epsilon + o(\epsilon), \\ \mathcal{H}(\mathbf{MA}(\Omega)|\mathbf{MA}(\Omega')) &\leq g(\theta_\Omega)\cdot\epsilon + o(\epsilon), \end{aligned}$$

for every pseudonormal domain Ω' such that

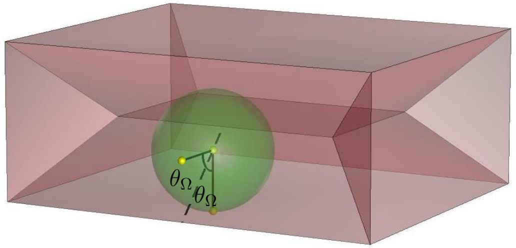
$\max\{\mathcal{H}(\Omega,\Omega'),\mathcal{H}(\partial\Omega,\partial\Omega')\} \leq \epsilon$.



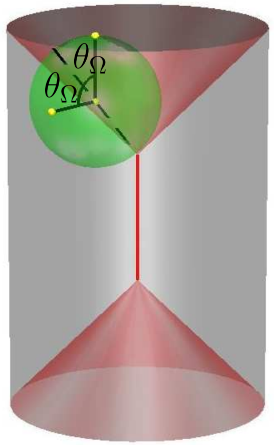
$$g(\theta) = 3\left(1+\frac{2\sqrt{1+\cos^2\theta}}{1-\cos\theta}\right) \sim \frac{1}{1-\cos\theta} \sim \frac{1}{\theta^2}$$

$$(0 < \theta \leq \pi/2)$$

Examples



$\theta_\Omega = 45°$, $g(\theta_\Omega) = 28.089243\ldots$



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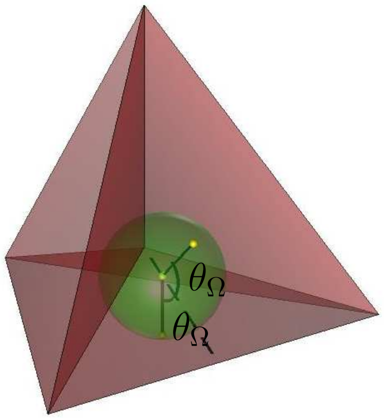
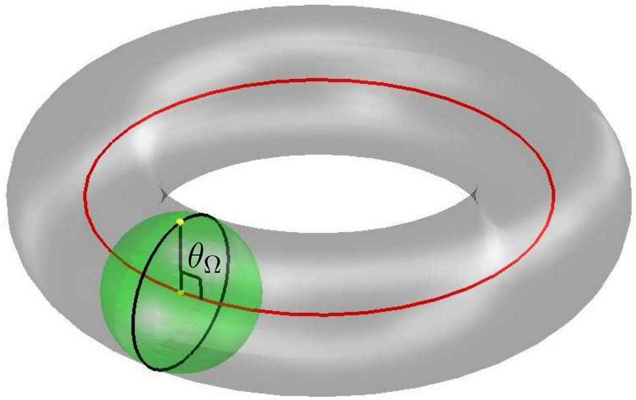
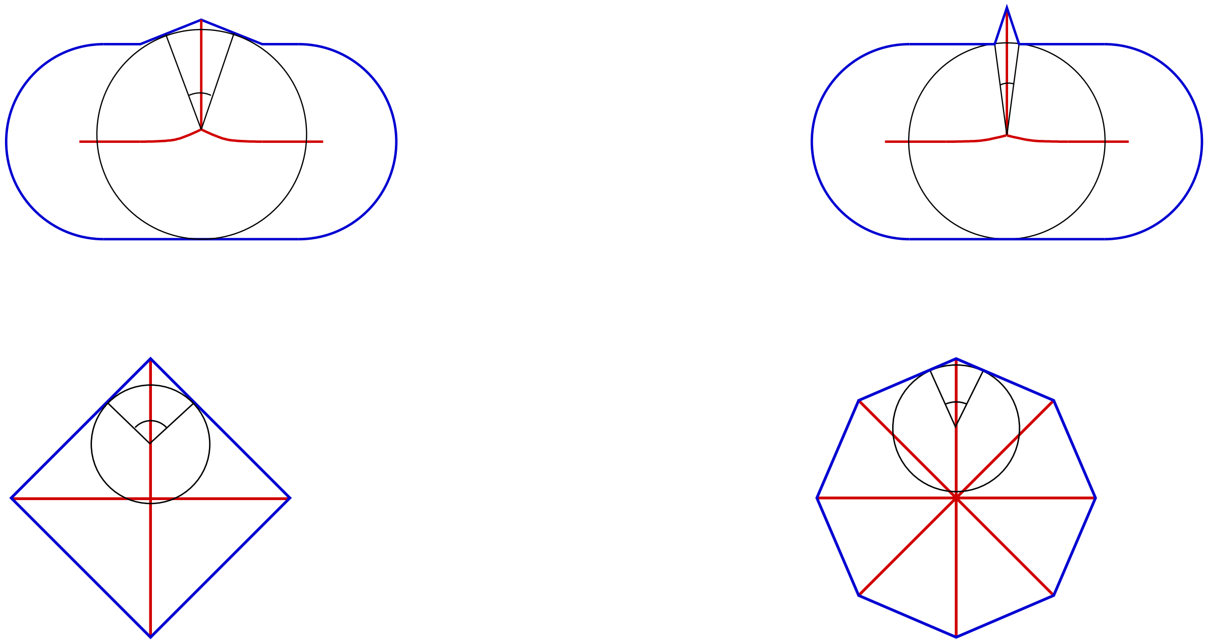


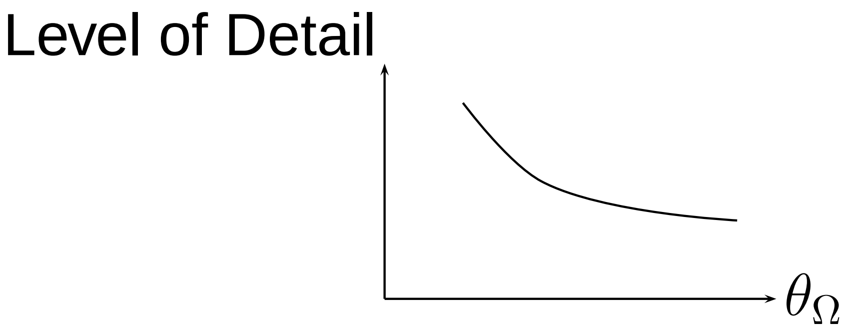
$\theta_\Omega = 54.73561\ldots°$, $g(\theta_\Omega) = 19.3923\ldots$



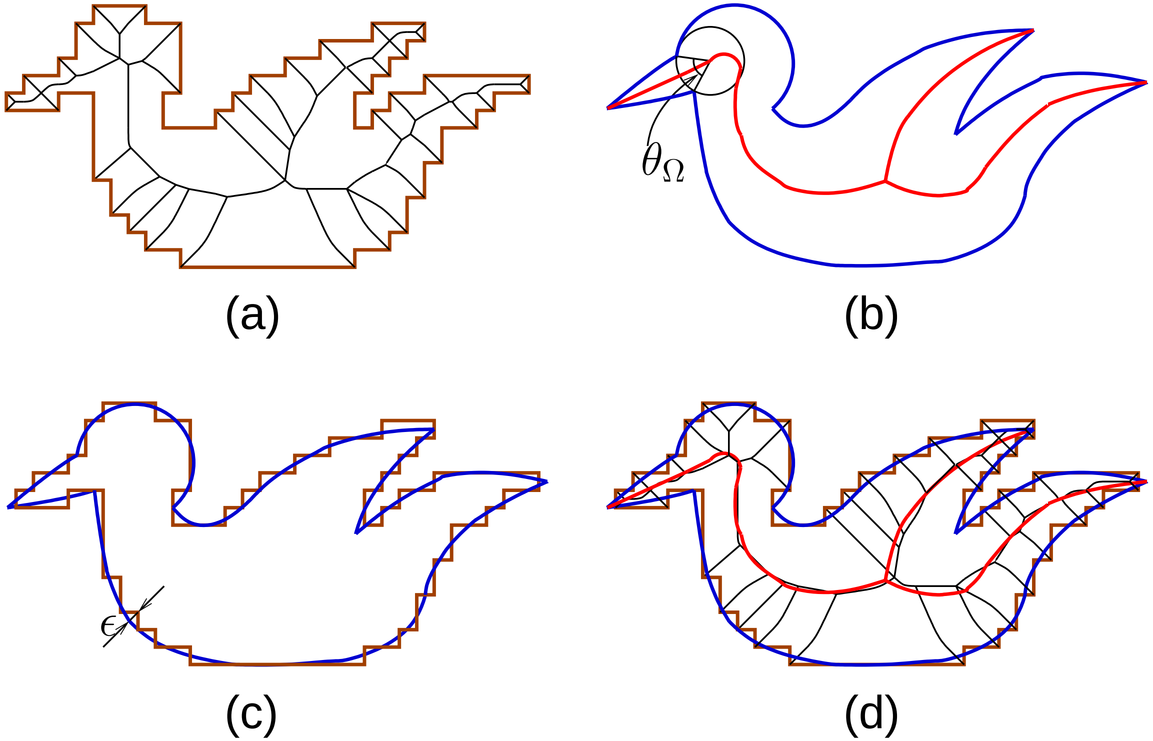
$\theta_\Omega = 90°$, $g(\theta_\Omega) = 9$

θ_Ω and the Level of Detail





Pruning Scheme



- no need to compute the whole complex noisy MAT
- precise error analysis

Independence of Approximations

Corollary

Let Ω be a pseudonormal 3D domain, and let Ω_1 and Ω_2 be two weakly injective 3D domains such that

$$\max\{\mathcal{H}(\Omega_i,\Omega),\mathcal{H}(\partial\Omega_i,\partial\Omega)\}\leq\epsilon$$

for $i=1,2$. Let $\theta=\min\{\theta_{\Omega_1},\theta_{\Omega_2}\}$. Then we have

$$\begin{aligned}\mathcal{H}\left(\mathbf{MAT}(\Omega_1),\mathbf{MAT}(\Omega_2)\right) &\leq 2g(\theta)\cdot\epsilon+o(\epsilon),\\ \mathcal{H}\left(\mathbf{MA}(\Omega_1),\mathbf{MA}(\Omega_2)\right) &\leq 2g(\theta)\cdot\epsilon+o(\epsilon).\end{aligned}$$

General Case of Normal Domains

Theorem (*VMV* 2001)

Let Ω be a normal domain (2D) which is not weakly injective. Then we have

$$\begin{aligned} &\mathcal{H}(\mathbf{MA}(\Omega)|\mathbf{MA}(\Omega')) \\ &\qquad \leq \quad K_\Omega \cdot \epsilon^{\frac{N_\Omega - 1}{N_\Omega + 1}} + o\left(\epsilon^{\frac{N_\Omega - 1}{N_\Omega + 1}}\right), \\ &\mathcal{H}(\mathbf{MAT}(\Omega)|\mathbf{MAT}(\Omega')) \\ &\qquad \leq \quad K_\Omega \cdot \epsilon^{\frac{N_\Omega - 1}{N_\Omega + 1}} + o\left(\epsilon^{\frac{N_\Omega - 1}{N_\Omega + 1}}\right), \end{aligned}$$

for every normal domain Ω' such that $\max\{\mathcal{H}(\Omega, \Omega'), \mathcal{H}(\partial\Omega, \partial\Omega')\} \leq \epsilon$.

- nonlinear bound: $\frac{N_\Omega - 1}{N_\Omega + 1} = \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \cdots \nearrow 1$, $N_\Omega = 2, 3, 4, \cdots$
- K_Ω, N_Ω can be explicitly calculated from the boundary.

Asymptotic Constants

$P=(p,r)\in \mathbf{MAT}(\Omega)$: non-degenerate 1-prong point
$f(t)=\sum_{n=2} a_n t^n$, $a_2=\frac{1}{2r}=\frac{k}{2}$.
$A_n=k^{n-1}Q_n-a_n,\ n=2,3,\cdots$.
N_P: the largest integer such that $A_2=A_3=\cdots=A_{N_P}=0$.

$$\Longrightarrow N_P\geq 2 \text{ and } A_{N_P+1}>0.$$

$$N_\Omega=\min\{N_P : P \text{ is a non-degenerate } 1\text{-prong of } \mathbf{MAT}(\Omega)\}.$$

$$K_P=\begin{cases} \dfrac{\sqrt{2}\cdot 12^{\frac{N_P-1}{N_P+1}}(N_P+1)}{(N_P-1)^{\frac{N_P-1}{N_P+1}}}\cdot r^2 A_{N_P+1}^{\frac{2}{N_P+1}}, & P\text{: type (i)}\\ \dfrac{\sqrt{2}\cdot 2^{\frac{3N_P-1}{N_P+1}}\cdot 6^{\frac{N_P-1}{N_P+1}}\cdot N_P^{\frac{2N_P}{N_P+1}}}{(N_P-1)^{\frac{2(N_P-1)}{N_P+1}}}\cdot r^2 A_{N_P+1}^{\frac{2}{N_P+1}}, & P\text{: type (ii), (iii)}\end{cases}$$

$$K_\Omega=\max\{K_P : P \text{ is a non-degenerate } 1\text{-prong point of } \mathbf{MAT}(\Omega) \textit{ s.t. } N_P=N_\Omega\}.$$

Conclusions – Instability

- Analyzed *quantitatively* the effect of boundary perturbation on MAT (weakly injective).
- Proved the stability of MAT under *one-sided* Hausdorff distance.
- Introduced the indicators θ_Ω, K_Ω, N_Ω for *level of detail*.
- Suggested a new pruning strategy with *error analysis*.

Problems:

- Systematic method for approximation
- Too large bounds : bounds for specific applications, introduction of probability measure
- Other distances : topological information, normal information

Directions

- Extension to 3D
  - shape, algorithm, instability
  - need deeper mathematics : differential geometry, (real) algegraic/analytic geometry, differential topology, singularity theory, *etc.*
- Relation with Subdivision
- Real-world applications

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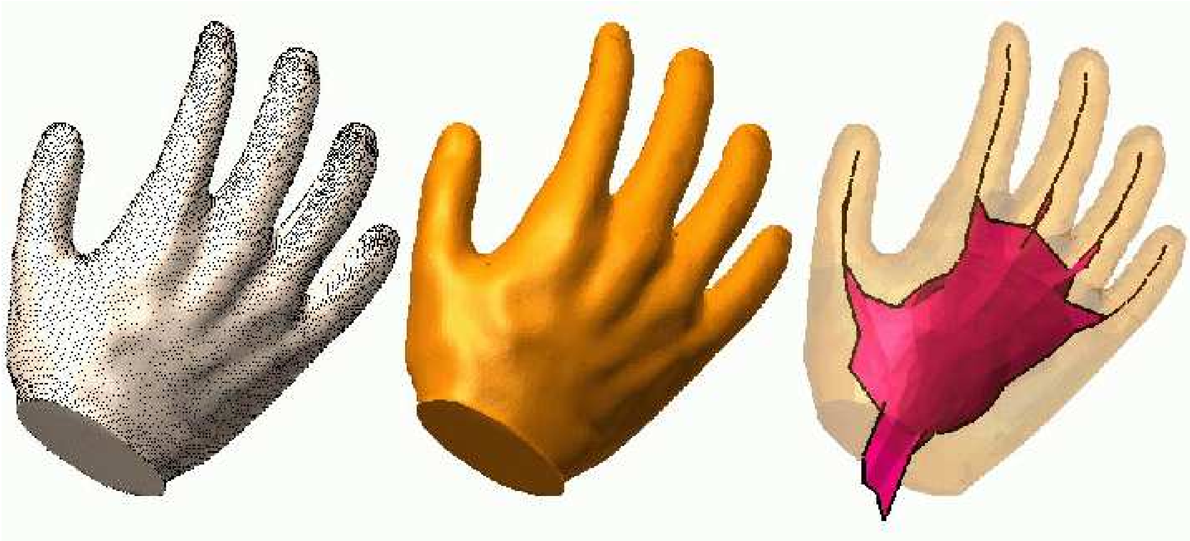
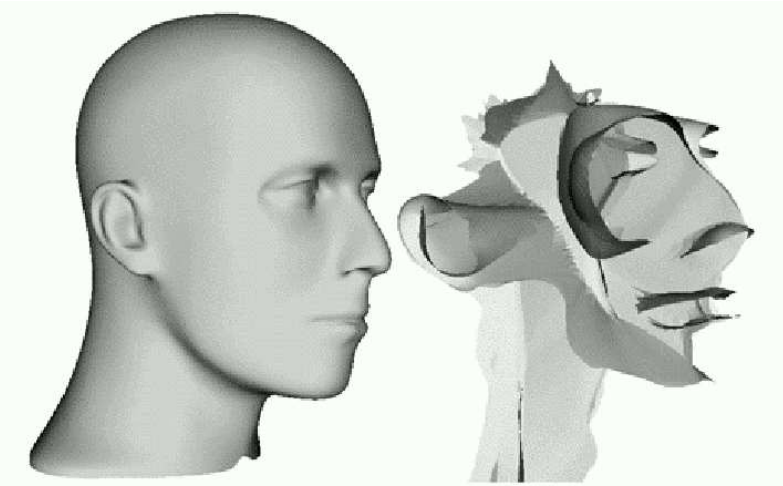
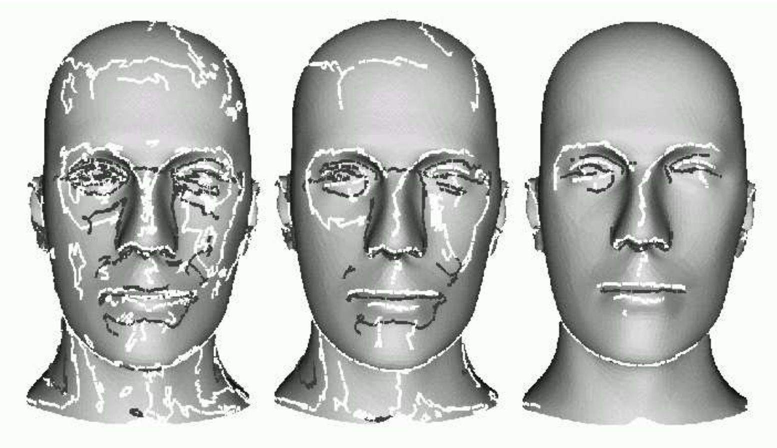
Surface Reconstruction



“Power Crust”, N. Amenta, *et al.*

MAT and Feature Extraction





“Ridges & Ravines”

A. G. Belyaev, *et al.*

Global Shape Deformation