Mathematical and Computational Issues on Medial Axis Transform

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What is Medial Axis Transform?

\[
\begin{align*}
\text{MAT}(\Omega) &= \{(p, r) \in \mathbb{R}^n \times \mathbb{R} \mid B_r(p) \text{ is a maximal ball in } \Omega\} \\
\text{MA}(\Omega) &= \{p \in \mathbb{R}^n \mid \exists r \geq 0 \text{ s.t. } (p, r) \in \text{MAT}(\Omega)\}
\end{align*}
\]
Other Definitions

- Grass Fire Model
  - points where wave starting from the boundary meet

- Cut Locus
  - points where the distance to the boundary stop minimizing

- Set of Singularities
  - points where the distance function to the boundary is not differentiable
Examples
More Examples
Good & Bad Aspects

- Homotopy equivalence: Preserves topological structure of domains.
- Compact representation (graph structure): Easy to store and process in computer.
- Wide range of applications (Shape Analysis) pattern recognition, computer vision, computer graphics, CAGD, mechanical engineering, biological/medical applications

Problems:
- Difficult to compute
- Sensitive to noise
- Poor mathematical analyses so far
Pathological Shapes

- MA (MAT) can exhibit pathological behaviours even for domains with $C^\infty$ boundary.
Normal Domain

Ω: compact & connected in \( \mathbb{R}^2 \) is a normal domain, if

- \( \partial \Omega \): finite number of simple closed curves
- each boundary component: finite number of real analytic curve pieces.

Theorem \((\text{Pacific. J. Math. } 1997)\)

Ω: a normal domain. \( \implies \)

- \( \text{MA}(\Omega) \) and \( \text{MAT}(\Omega) \) have finite graph structures.
- \( \text{MA}(\Omega) \) is a strong deformation retract of \( \Omega \).

Almost every domain in applications is normal.
**Generic Points – 2D**

$G(\Omega)$: the set of MA points with two contact points $\text{MA}(\Omega)$ except for finitely many points

$$\theta_\Omega = \inf \{ \theta(p) : p \in G(\Omega) \}$$

$$(0 \leq \theta_\Omega \leq \frac{\pi}{2})$$

Definition: $\Omega$ is **weakly injective**, if $\theta_\Omega > 0$. 
Three Types of 1-Prong Points – 2D

\[ \theta_\Omega = 0 \iff \text{MA}(\Omega) \text{ has a 1-prong point of type (c).} \]

\[ \Omega \text{ is weakly injective} \iff \text{only (a) and (b)} \]
Examples

weakly injective

not weakly injective (normal)
Definition: \( \Omega \) is \textit{weakly injective}, if \( \theta_\Omega > 0 \).
Types of $1$-Prong Points – 3D

(a1)  (b1)  (c1)

(a2)  (b2)  (c2)
Examples

weakly injective

not weakly injective (pseudonormal)
Algorithm

Essentially the first for domains with free-form boundary and non-trivial homology (GMIP 1997)

- approximation/interpolation (transcendental curves)
- domain decomposition (fast)
- updating tree data structure (topologically correct)

♣ 3D extension – need extensive theoretical analysis
Example
Example
Instability of MAT

Very sensitive to boundary perturbation.

- Introduces many hairy prongs.

- No explicit and quantitative error analyses before.
One-sided Hausdorff Distance

$A, B$: compact sets in $\mathbb{R}^n$

$$\mathcal{H}(A|B) = \max_{p \in A} d(p, B)$$

$\mathcal{H}(A|B)$ measures how much $A$ is contained in $B$. 
(Two-sided) Hausdorff Distance

$A, B :$ compact sets in $\mathbb{R}^n$.

$$
\mathcal{H}(A, B) = \max \left\{ \max_{p \in A} d(p, B), \max_{q \in B} d(q, A) \right\}
= \max \{ \mathcal{H}(A|B), \mathcal{H}(B|A) \}.
$$

$\mathcal{H}(A, B)$ measures how similar $A$ and $B$ are.
One-sided Stability of 
MAT


Let \( \Omega \) be a weakly injective domain (2D & 3D). Then we have

\[
\mathcal{H}(\text{MAT}(\Omega)|\text{MAT}(\Omega')) \leq g(\theta_\Omega) \cdot \epsilon + o(\epsilon),
\]

\[
\mathcal{H}(\text{MA}(\Omega)|\text{MA}(\Omega')) \leq g(\theta_\Omega) \cdot \epsilon + o(\epsilon),
\]

for every pseudonormal domain \( \Omega' \) such that

\[
\max \{ \mathcal{H}(\Omega, \Omega'), \mathcal{H}(\partial \Omega, \partial \Omega') \} \leq \epsilon.
\]

\[
g(\theta) = 3 \left( 1 + \frac{2\sqrt{1 + \cos^2 \theta}}{1 - \cos \theta} \right) \sim \frac{1}{1 - \cos \theta} \sim \frac{1}{\theta^2}
\]

\((0 < \theta \leq \pi/2)\)
Examples

$\theta_\Omega = 45^\circ, g(\theta_\Omega) = 28.089243\ldots$

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$\theta_\Omega = 54.73561\ldots^\circ, g(\theta_\Omega) = 19.3923\ldots$

$\theta_\Omega = 90^\circ, g(\theta_\Omega) = 9$
Pruning Scheme

- no need to compute the whole complex noisy MAT
- precise error analysis
Independence of Approximations

Corollary
Let $\Omega$ be a pseudonormal 3D domain, and let $\Omega_1$ and $\Omega_2$ be two weakly injective 3D domains such that

$$\max \{ \mathcal{H}(\Omega_i, \Omega), \mathcal{H}(\partial\Omega_i, \partial\Omega) \} \leq \epsilon$$

for $i = 1, 2$. Let $\theta = \min \{ \theta_{\Omega_1}, \theta_{\Omega_2} \}$. Then we have

$$\mathcal{H}(\text{MAT}(\Omega_1), \text{MAT}(\Omega_2)) \leq 2g(\theta) \cdot \epsilon + o(\epsilon),$$

$$\mathcal{H}(\text{MA}(\Omega_1), \text{MA}(\Omega_2)) \leq 2g(\theta) \cdot \epsilon + o(\epsilon).$$
General Case of Normal Domains

**Theorem (VMV 2001)**

Let $\Omega$ be a normal domain (2D) which is not weakly injective. Then we have

$$
\mathcal{H}(\text{MA}(\Omega)|\text{MA}(\Omega'))
\leq K_\Omega \cdot \epsilon \frac{N_\Omega - 1}{N_\Omega + 1} + o \left( \epsilon \frac{N_\Omega - 1}{N_\Omega + 1} \right),
$$

$$
\mathcal{H}(\text{MAT}(\Omega)|\text{MAT}(\Omega'))
\leq K_\Omega \cdot \epsilon \frac{N_\Omega - 1}{N_\Omega + 1} + o \left( \epsilon \frac{N_\Omega - 1}{N_\Omega + 1} \right),
$$

for every normal domain $\Omega'$ such that

$$
\max \{ \mathcal{H}(\Omega, \Omega'), \mathcal{H}(\partial \Omega, \partial \Omega') \} \leq \epsilon.
$$

- **nonlinear bound**: $\frac{N_\Omega - 1}{N_\Omega + 1} = \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \cdots \uparrow 1$, $N_\Omega = 2, 3, 4, \cdots$
- $K_\Omega, N_\Omega$ can be explicitly calculated from the boundary.
Asymptotic Constants

\[ P = (p, r) \in \text{MAT} (\Omega) : \text{non-degenerate 1-prong point} \]

\[ f(t) = \sum_{n=2} a_n t^n, \quad a_2 = \frac{1}{2r} = \frac{k}{2}. \]

\[ A_n = k^{n-1} Q_n - a_n, \quad n = 2, 3, \ldots. \]

\[ N_P : \text{the largest integer such that } A_2 = A_3 = \cdots = A_{N_P} = 0. \]

\[ \quad \implies N_P \geq 2 \text{ and } A_{N_P+1} > 0. \]

\[ N_{\Omega} = \min \{ N_P : P \text{ is a non-degenerate 1-prong of MAT(} \Omega) \}. \]

\[ K_P = \begin{cases} \\
\sqrt{2} \cdot 12 \frac{N_P - 1}{N_P + 1} (N_P + 1) \cdot r^2 A \frac{2}{N_P + 1}, & P: \text{type (i)} \\
\frac{3}{2} \cdot 2 \cdot \frac{N_P - 1}{N_P + 1} \cdot 6 \cdot \frac{N_P - 1}{N_P + 1} \cdot \frac{2N_P}{N_P + 1} \cdot r^2 A \frac{2}{N_P + 1}, & P: \text{type (ii), (iii)} \\
\end{cases} \]

\[ K_{\Omega} = \max \{ K_P : P \text{ is a non-degenerate 1-prong point of MAT}(\Omega) \text{ s.t. } N_P = N_{\Omega} \}. \]
Conclusions – Instability

- Analyzed *quantitatively* the effect of boundary perturbation on MAT (weakly injective).
- Proved the stability of MAT under *one-sided* Hausdorff distance.
- Introduced the indicators $\theta_\Omega$, $K_\Omega$, $N_\Omega$ for *level of detail*.
- Suggested a new pruning strategy with *error analysis*.

Problems:

- Systematic method for approximation
- Too large bounds: bounds for specific applications, introduction of probability measure
- Other distances: topological information, normal information
Directions

- Extension to 3D
  - shape, algorithm, instability
  - need deeper mathematics: differential geometry, (real) algebraic/analytic geometry, differential topology, singularity theory, etc.

- Relation with Subdivision

- Real-world applications
Surface Reconstruction

MAT and Feature Extraction

“Ridges & Ravines”
A. G. Belyaev, et al.
Global Shape Deformation