Linear Independence of Finite Gabor Systems

Yoon Gang-Joon

School of Mathematics
Korea Institute for Advanced Study

Short trip to Frame Area

Frames were introduced in 1952 by Duffin and Schaeffer in the context of nonharmonic Fourier series as an alternative to orthonormal or Riesz bases in Hilbert spaces.

Definition. A sequence $\{f_n\}$ in a Hilbert space H is a frame for H if there exist $A, B > 0$ such that

$$A||f||^2 \leq \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \leq B||f||^2, \quad \forall f \in H.$$

Constants A and B are called lower and upper frame bounds.

By providing a stable reconstruction formula for a L^2-function, frames play an important role in Wavelet analysis, Fourier analysis, signal analysis and physics.

Theorem. Given a sequence $\{f_n\}_{n\in\mathbb{N}}$ in a Hilbert space H, the following two statements are equivalent:

(1) $\{f_n\}_{n\in\mathbb{N}}$ is a frame with bounds A, B.

(2) The operator $S : H \to H$ given by

$$S(f) = \sum_{n\in\mathbb{N}} \langle f, f_n\rangle f_n \quad \forall f \in H$$

is a bounded operator with $AI \leq S \leq BI$, called the frame operator for $\{f_n\}_{n\in\mathbb{N}}$.

In this case, S is invertible and satisfies the followings:

(3) $\{S^{-1}f_n\}$ is also a frame with bounds B^{-1}, A^{-1} called the canonical dual frame of $\{f_n\}_{n\in\mathbb{N}}$.

(4) Every $f \in H$ can be written

$$f = \sum_{n\in\mathbb{N}} \langle f, S^{-1}f_n\rangle f_n = \sum_{n\in\mathbb{N}} \langle f, f_n\rangle S^{-1}f_n.$$

For a given sequence $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ in a Hilbert space H, we introduce some concepts and related equivalence conditions to the definitions.

- An order collection $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ is a *Schauder basis* for H if for each $f \in H$ there exist unique scalars $c_n(f)$ such that

$$f = \sum_{n\in\mathbb{N}} c_n(f) f_n$$

Then we have several equivalent definitions of a Schauder basis:

(1) $\mathcal{F}$ is a Schauder basis.

(2) There exists a biorthogonal sequence $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ such that the partial sum operators $S_N : H \to H$ by

$$S_N(f) = \sum_{n=0}^{N} \langle f, f_n \rangle f_n, \quad \forall f \in H$$

are uniformly bounded, i.e., $\exists K > 0$ such that

$$\sup_N ||S_N|| \leq K.$$

(3) There exists a bi-orthogonal sequence $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ such that for every $f \in H$,

$$f = \sum_{n=0}^{\infty} \langle f, \tilde{f}_n \rangle f_n.$$

- $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ is a *Bessel sequence* there exists a constant $B > 0$ such that

$$\sum_{n=0}^{\infty} |\langle f, f_n\rangle|^2 \leq B||f||^2, \quad \forall f \in H.$$

  Equivalently, $\mathcal{F}$ is a Bessel sequence if and only if the operator T defined on $\ell^2(\mathbb{N})$ by

$$T : \{c_n\} \rightarrow \sum_{n\in\mathbb{N}} c_n f_n$$

  is a well-defined bounded operator into H.

- $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ is a *Riesz basis* if it is isomorphic to an orthonomal basis for H, which is equivalent to each of the followings:

  (1) $\mathcal{F}$ is a Schauder basis, the basis expansions converges unconditionally, and $0 < \inf ||f_n|| \leq \sup ||f_n|| < \infty$.

  (2) $\mathcal{F}$ is a frame and is biorthogonal to its canonical dual frame.

  (3) $\mathcal{F}$ is complete and there exists $A, B > 0$ such that

$$\forall c_0, \ldots, c_N, \quad A\sum_{n=1}^{N} |c_n|^2 \leq ||\sum_{n=1}^{N} c_n f_n||^2 \leq B\sum_{n=1}^{N} |c_n|^2.$$

Gabor Frame

The Mathematical theory for Gabor analysis in $L^2(\mathbb{R})$ is based on two operators,

- Translation by $a \in \mathbb{R}$, T_a, defined by $T_a(f)(x) = f(x-a)$.
- Modulation by $b \in \mathbb{R}$, M_b, defined by $M_a(f)(x) = e^{2\pi ibx} f(x)$.

Gabor analysis aims at representing functions $f \in L^2(\mathbb{R})$ as superpositions of time-frequency shifts $M_b T_a g$ of a fixed function $g \in L^2(\mathbb{R})$ with $||g||_2 = 1$ in two ways:

- The short-time Fourier transform $V_g f$ of f:

$$V_g f(x,w) = \int_{\mathbb{R}} f(t)\overline{g(t-x)}e^{-2\pi iwt}dt$$

  And an integral representation of f is given as

$$f(y) = \int\int V_g f(x,w) M_w T_x g(y) dw dx.$$

- Gabor expansions : a discrete subset $\Lambda \subset \mathbb{R}^2$ and a function g,

$$f(x) = \sum_{(a,b)\in\Lambda} c_{a,b}(f) M_b T_a g(x).$$

Gabor proposed using the Gaussian function $g(t) = e^{-t^2}$ as a window function along the unit time-frequency shift lattice $\mathbb{Z}^2$. However, while this Gaussian Gabor system is complete in $L^2(\mathbb{R})$, it is not an orthogonal basis, Riesz basis, a frame, or a Schauder basis.

Density Theorem. Let $g \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$ be given. For the Gabor system $(\mathcal{G}, \alpha, \beta) = \{e^{2\pi i\beta mx} g(x-n)\}_{(n,m)\in\mathbb{Z}^2}$, we have the followings:

(a) If $\{e^{2\pi i\beta mx} g(x-n)\}_{(n,m)\in\mathbb{Z}^2}$ is a complete, then $\alpha\beta \leq 1$. In particular, if the system is a frame, $\alpha\beta \leq 1$.

(b) If $\{e^{2\pi i\beta mx} g(x-n)\}_{(n,m)\in\mathbb{Z}^2}$ is a Riesz basis, then $\alpha\beta = 1$.

Classical Balian-Low Theorem. If $g \in L^2(\mathbb{R})$ and

$$\int |t|^2 |g(t)|^2 dt < 0 \quad \text{and} \quad \int |\xi|^2 |\hat{g}(\xi)|^2 d\xi < 0$$

for the Fourier transform $\hat{g}$, then $(\mathcal{G}, 1, 1)$ is not a Riesz basis.

Theorem. Let $g \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$ be such that

(1) there exists $A, B > 0$ such that

$$0 < A \leq \sum_{n \in \mathbb{N}} |g(x - n\alpha)|^2 \leq B, \quad a.e,$$

(2) g has compact support with $supp(g) \subset I \subset \mathbb{R}$, where I is some interval of length $1/\beta$.

Then $\{e^{2\pi i\beta mx} g(x-n)\}_{(n,m)\in\mathbb{Z}^2}$ is a Gabor frame with frame bounds $\beta^{-1}A, \beta^{-1}B$.

Corollary Suppose g is a continuous function supported on an interval I of length $L > 0$ which does not vanish in the interior of I. Then $\{e^{2\pi i\beta mx} g(x-n)\}_{(n,m)\in\mathbb{Z}^2}$ is a frame for any $0 < \alpha < L$ and $0 < \beta \leq 1/L$.

Linear Independence Conjecture for Time-Frequency Shifts

If $g \in L^2(\mathbb{R})$ is nonzero and $\{(\alpha_k, \beta_k)\}_{k=1}^N$ is any set of finitely many distinct points in $\mathbb{R}^2$, then $\{e^{2\pi i \beta_k} g(x - \alpha_k)\}_{k=1}^N$ is linearly independent in $L^2(\mathbb{R})$.

Partial answers: The independence conclusion holds in the cases when

- $g \in L^2(\mathbb{R})$ is compactly supported;
- $g(x) = p(x)e^{-x^2}$ for a nonzero polynomial $p(x)$;
- $N \leq 3$;
- $\{(\alpha_k, \beta_k)\}_{k=1}^N$ is collinear;
- $\{(\alpha_k, \beta_k)\}_{k=1}^N$ is a subset of some lattice $A(\mathbb{Z}^2)$.

Theorem (Perturbations) Assume that $g \in L^2(\mathbb{R})$ and $\{(\alpha_k, \beta_k)\}_{k=1}^N$ are such that $\{e^{2\pi i \beta_k} g(x - \alpha_k)\}_{k=1}^N$ is linearly independent. Then the following statements hold.

(a) There exists $\varepsilon > 0$ such that $\{e^{2\pi i \beta_k} h(x - \alpha_k)\}_{k=1}^N$ is independent for any $h \in L^2(\mathbb{R})$ with $||h - g||_2 < \varepsilon$.

(b) There exists $\varepsilon > 0$ such that $\{e^{2\pi i \beta'_k} g(x - \alpha'_k)\}_{k=1}^N$ is linearly independent for any set $\{(\alpha'_k, \beta'_k)\}_{k=1}^N$ such that $|\alpha_k - \alpha'_k|, |\beta_k - \beta'_k| < \varepsilon$ for $k = 1, 2, \ldots, N$.

Conjecture. If $g \in L^2(\mathbb{R})$ is nonzero, then each of the following is linearly independent in $L^2(\mathbb{R})$:

(a) g is of Schwarz class, that is, $g \in C^\infty$ and for any n and m,

$$\lim_{|x|\to\infty} x^n g^{(m)}(x) = 0$$

(b) $\{g(x), g(x-1), e^{2\pi i x} g(x), e^{2\pi i \sqrt{2} x} g(x-\sqrt{2})\}$

(c) $\{g(x), g(x-1), e^{2\pi i x} g(x), g(x-\pi)\}$