n-term Rational approximation and Franklin bases

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26th August 2006

Outline

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- **Multilevel triangulations.**

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(e) **No hanging vertices condition:** No vertex of any triangle $\Delta \in \mathcal{T}_m$ lies in the interior of an edge of another triangle from $\mathcal{T}_m$.
(f) For any compact $K \subset \mathbb{R}^2$ and any fixed $m \in \mathbb{Z}$, there is a finite collection of triangles from $\mathcal{T}_m$ which cover K.

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- *There exists a constant* $0 < \delta_1 \le 1$ *s.t. for any* $\Delta', \Delta'' \in \mathcal{T}_m$ $(m \in \mathbb{Z})$ *with a common edge,*

$$\delta_1 \le |\Delta'|/|\Delta''| \le \delta_1^{-1}.$$

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[**Flexibility**]

These conditions, however, allow the triangles in $\mathcal{T}$ to change in size, shape, and orientation quickly when moving around at a given level or through the levels. In particular, triangles with arbitrarily sharp angles are allowed in any location and at any level.

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[**Affine transform angle condition:**]

There exists a constant $\beta = \beta(\mathcal{T})$, $0 < \beta \leq \pi/3$, such that if $\Delta_0 \in \mathcal{T}_m$, $m \in \mathbb{Z}$, and $A : \mathbb{R}^2 \to \mathbb{R}^2$ is an affine transform that maps Δ_0 one-to-one onto an equilateral reference triangle, then for every $\Delta \in \mathcal{T}_m$ which has at least one common vertex with Δ_0, we have

$$\min \operatorname{angle}(A(\Delta)) \geq \beta,$$

where $A(\Delta)$ is the image of Δ by the affine transform A.

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- *Let $\mathcal{T}$ be an LR-triangulation of $\mathbb{R}^2$. Suppose that $\Delta', \Delta'' \in \mathcal{T}_m, m \in \mathbb{Z}$, and Δ' and Δ'' can be connected by n edges from $\mathcal{T}_m$. Then*

$$c_1^{-1} n^{-s} \leq |\Delta'|/|\Delta''| \leq c_1 n^s.$$

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- *Let $\mathcal{T}$ be an SLR-triangulation of $\mathbb{R}^2$. Suppose that $\Delta', \Delta'' \in \mathcal{T}_m, m \in \mathbb{Z}$, and Δ' and Δ'' can be connected by n edges from $\mathcal{T}_m$. Then*

$$c_2^{-1} n^{-u} \leq \frac{|\max \mathrm{edge}\,(\Delta')|}{|\max \mathrm{edge}\,(\Delta'')|} \leq c_2 n^u.$$

1. Hierarchical families of bases. (Multiresolution analysis)

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Existence of $\mathrm{supp}\varphi_\theta$ is guaranteed by No hanging vertices condition.

Muliresolution Analysis

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Problem: Characterize the approximation spaces generated by n-term approximation from Φ such as
$\{f \in L_p : \sigma_n(f, \Phi) = O(n^{-\alpha})\} \quad (0 < \alpha < \infty)$.

Characterization of approximation space

Approximation spaces.

$A_q^\gamma := A_q^\gamma(\Phi, L_p)$, $\alpha > 0$, $0 < q \le \infty$,
the space of all $f \in L_p$ such that

$$\|f\|_{A_q^\gamma} := \|f\|_p + \left(\sum_{n=1}^{\infty} (n^\gamma \sigma_n(f,\Phi)_p)^q \frac{1}{n}\right)^{1/q} < \infty.$$

1. Interpolation space

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Definition. (Real method of interpolation by Peetre K-functional)

For $0 < \theta < 1$, $0 < q \leq \infty$, *the* **interpolation space** $(X, Y)_{\theta,q}$*:*
the set of all functions $f \in X$ *s.t*

$$|f|_{(X,Y)_{\theta,q}} := \begin{cases} \left(\int_0^\infty (t^{-\theta} K(f;t))^q \frac{dt}{t}\right)^{1/q}, & 0 < q < \infty \\ \sup_{t>0} t^{-\theta} K(f;t), & q = \infty \end{cases}$$

is finite.

Smoothness space

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Definition. (Besov Spaces)

Let $\alpha > 0$, $0 < p, q \leq \infty$, take $k := [\alpha] + 1$. The Besov space $B_q^\alpha(L_p)$ is defined as the set of all f such that

$$|f|_{B_q^\alpha(L_p)} := \begin{cases} \left(\int_0^\infty (t^{-\alpha}\omega_k(f,t)_p)^q \frac{dt}{t}\right)^{1/q}, & 0 < q < \infty \\ \sup_{t>0} t^\alpha \omega_k(f,t)_p, & q = \infty \end{cases}$$

is finite, where $\omega_k(f,t)_p := \sup_{|h| \leq t} \|\Delta_h^k(f,\cdot)\|_p$ is the k-th modulus of smoothness of f in L_p.

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where $E_k(f,\Delta)_\tau$ is the error of L_τ-approximation on Δ from Π_k and $\omega_k(f,\Delta)_\tau$ is the L_τ-modulus of smoothness of f on Δ. •

Similarity

$$\text{Spaces :} \qquad A_q^\alpha \qquad (X,Y)_{\theta,q} \qquad B_{q,p}^\alpha$$

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- **Similarity**

Spaces :	A^{α}_{q}	$(X,Y)_{\theta,q}$	$B^{\alpha}_{q,p}$
Tools :	σ_n	$k(f,t)$	$w_k(f,t)_p$

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Let $0 < p < \infty$ and $\alpha > 0$ or $p = \infty$ and $\alpha \geq 1$, and $1/\tau := \alpha + 1/p$.

Spline approximation (general approach)

Theorem.

[Jackson] *If* $f \in B_\tau^{\alpha k}(\Phi)$, *then*

$$\sigma_n(f,\Phi)_p \leq cn^{-\alpha}\|f\|_{B_\tau^\alpha(\Phi)}.$$

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[Bernstein] *If* $s \in \Sigma_n(\Phi)$, *then*

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One specific case:

Theorem.

Suppose $0 < p < \infty$ *and* $\alpha > 0$ *or* $p = \infty$ *and* $\alpha > 1$, *and let* $\tau := (\alpha + 1/p)^{-1}$.

Characterization

Approximation spaces.

Theorem.

If $0 < \gamma < \alpha$ *and* $0 < q \le \infty$, *then*

$$A_q^\gamma(\Phi, L_p) = (L_p, B_\tau^{\alpha k}(\Phi))_{\frac{\gamma}{\alpha},q}$$

with equivalent norms.

One specific case:

Theorem.

Suppose $0 < p < \infty$ *and* $\alpha > 0$ *or* $p = \infty$ *and* $\alpha > 1$, *and let* $\tau := (\alpha + 1/p)^{-1}$. *Then we have*

$$A_\tau^{\alpha k}(\Phi, L_p) = B_\tau^\alpha(\Phi)$$

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where each r_m is of the form

$$r(x) = \prod_{j=1}^{K} \frac{a_j x_1 + b_j x_2 + c_j}{(\alpha_j x_1 + \beta_j x_2 + \gamma_j)^2 + 1},$$

where $a_j, \ldots, \gamma_j \in \mathbb{R}$, $x := (x_1, x_2) \in \mathbb{R}^2$, and K is fixed. (Every $R \in \mathcal{R}_n$ depends on $< cn$ parameters.)

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$R_n(f)_p$: the error of L_p-approximation to f from $\mathcal{R}_n$:

$$R_n(f)_p := \inf_{R \in \mathcal{R}_n} \|f - R\|_p.$$

n-tern rational approximation

Theorem

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Let $f \in L_p(\mathbb{R}^2)$, $0 < p < \infty$, $\alpha > 0$, and $k \geq 1$. Then

$$R_n(f)_p \leq cn^{-\alpha}\Big(\sum_{m=1}^{n} \frac{1}{m}(m^\alpha \sigma_m(f, \mathcal{T})_p)^{p^*} + \|f\|_p^{p^*} \Big)^{1/p^*}, \quad n = 1, \dots, \tag{1}$$

where $p^* = \min\{1, p\}$ and c depends only on α, p, k, and the parameters of $\mathcal{T}$.

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Corollary

If $\sigma_n(f,\mathcal{T})_p = O(n^{-\gamma})$ *for an arbitrary SLR-triangulation* $\mathcal{T}$, $0 < p < \infty$, *and* $\gamma > 0$, *then* $R_n(f)_p = O(n^{-\gamma})$.

Proof.

- Construct a n-term rational function which approximate a spline over a triangle.

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- Construct a n-term rational function which approximate a spline over a triangle.
- Estimate the error by using a **geometric vector valued maximal function**.

□

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Suppose $\inf_{\mathcal{T}} \|f\|_{B_\tau^{\alpha k}(\Phi_{\mathcal{T}})} < \infty$ *with* $\alpha > 0$, $k \geq 1$, *and* $1/\tau := \alpha + 1/p$, $0 < p < \infty$. *Then*

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$$R_n(f)_p \leq c n^{-\alpha} \inf_{\mathcal{T}} \|f\|_{B_\tau^{\alpha k}(\Phi_{\mathcal{T}})}.$$

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- K. Park
  Bivariate n-term rational approximation (2005)

Franklin Bases

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We obtain an orthonormal system $\mathcal{F}_{\mathcal{T}} := \{f_\theta\}_{\theta\in\Theta^*}$ in $L_2(E)$ **consisting of continuous piecewise linear functions**. Each Franklin function f_θ is uniquely determined (up to a multiple ± 1) by the conditions:

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(a) $f_\theta \in \mathrm{span}\,\{\varphi_{\theta'} : \theta' \preceq \theta\}$.

(b) $\langle f_\theta, \varphi_{\theta'}\rangle = 0$ for all $\theta' \prec \theta$,

(c) $\|f_\theta\|_2 = 1$.

Note that $f_{\theta_0} = \pm\tilde{\mathbf{1}}_{\theta_0} := \pm|E|^{-1/2}\,\mathbf{1}_E$.

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- **In fact, unconditional bases for the corresponding (anisotropic) Hardy spaces**
- **thus characterizes the corresponding (anisotropic) BMO.**

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In one specific case, $A_q^\alpha(\mathcal{F}_\mathcal{T}, L_p)$ can be identified as a B-space:

Theorem

Assuming that $1 < p < \infty$, $\alpha > 0$, *and* $1/\tau := \alpha + 1/p$, *we have*

$$A_\tau^\alpha(\mathcal{F}_\mathcal{T}, L_p) = B_\tau^\alpha(\mathcal{T})$$

with equivalent norms.

Perspectives

Characterize the approximation space generated by R_n.

Characterize the Hardy space by more smooth functions.

Develop rational bases on this setting.

etc

Thank you