Non-statinary Biorthogonal Wavelet Systems Based on Exponenetial B-spline

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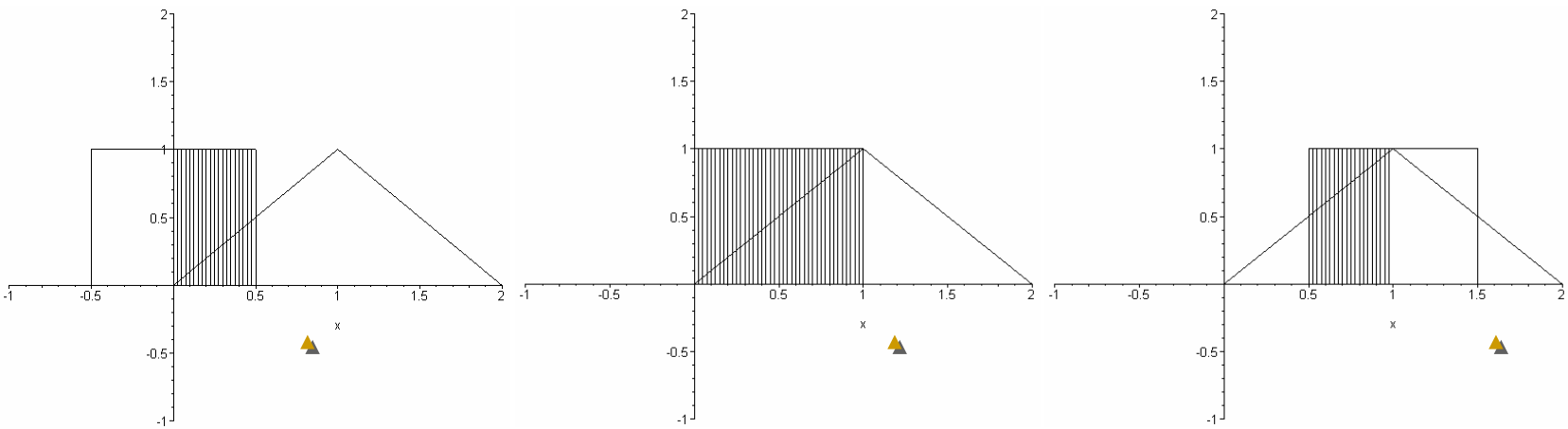
B-spline and Exponential B-spline

B-spline

$$B_0(x) = \begin{cases} 1, & 0 \le x < 1, \\ 0, & otherwise \end{cases}$$

$$B_n(x) = \int_0^1 B_{n-1}(x-t)dt$$

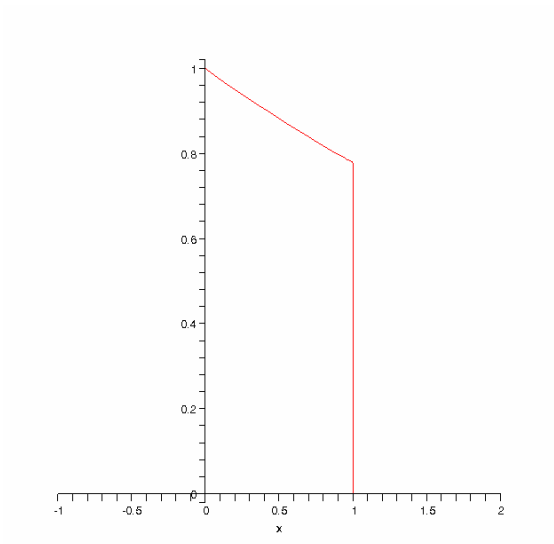
$$= B_0(x) \otimes B_{n-1}(x) \qquad p(x) \otimes q(x) = \int_{-\infty}^{\infty} p(t)q(x-t)dt$$

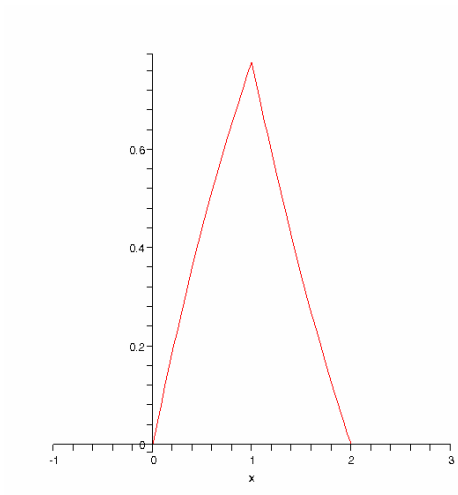


Exponential B-spline

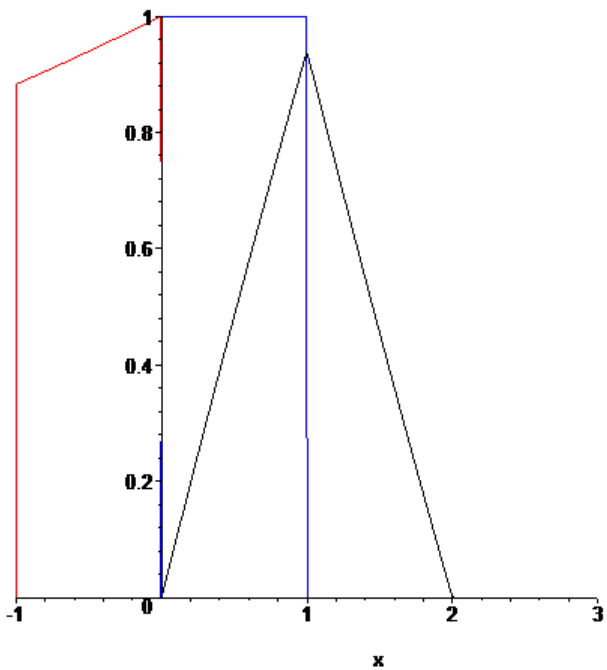
$$\beta_\alpha(t) = \rho_\alpha(t) - e^\alpha \rho_\alpha(t-1) = \begin{cases} e^{\alpha t}, & 0 \le t < 1, \\ 0, & otherwise \end{cases}$$

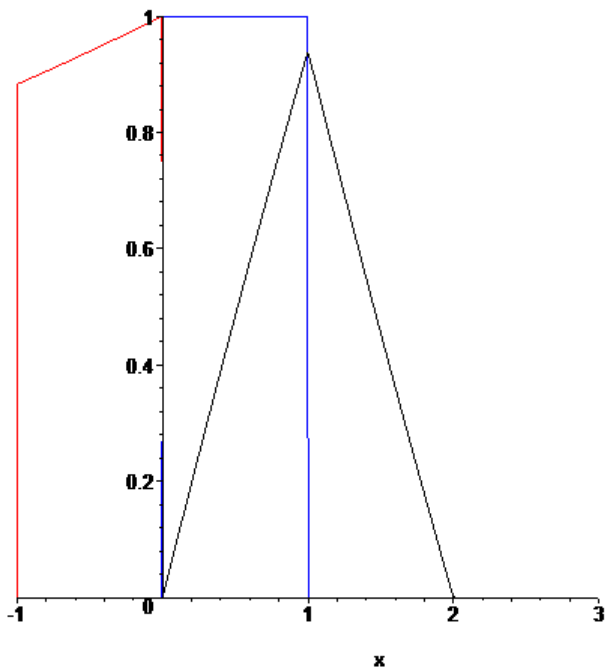
$$\beta_{\vec{\alpha}}(t) = (\beta_{\alpha_1} \otimes \beta_{\alpha_2} \otimes \cdots \otimes \beta_{\alpha_n})(t)$$





Linear Exponential B-spline

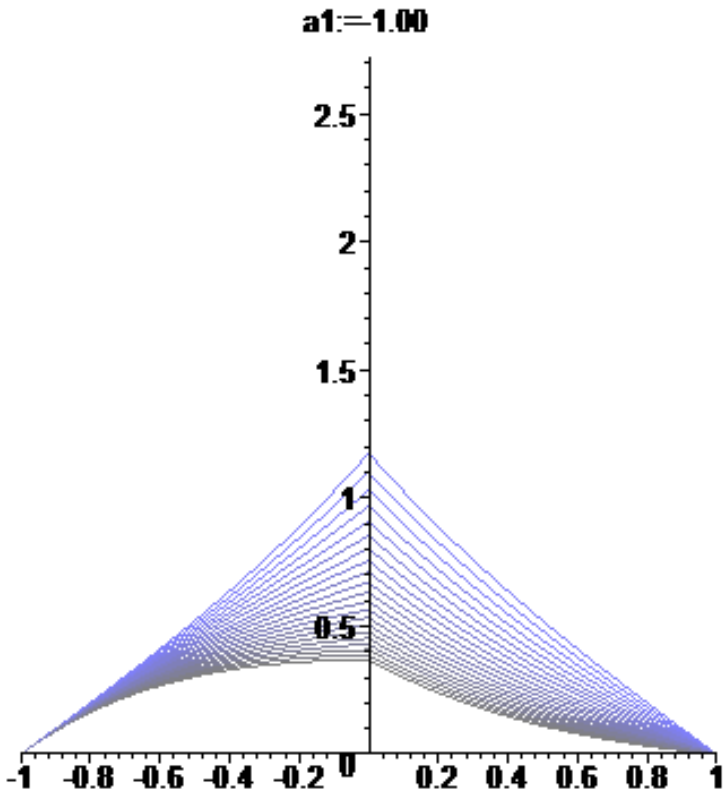




$\alpha_1 = -\frac{1}{8}, \alpha_2 = 0$

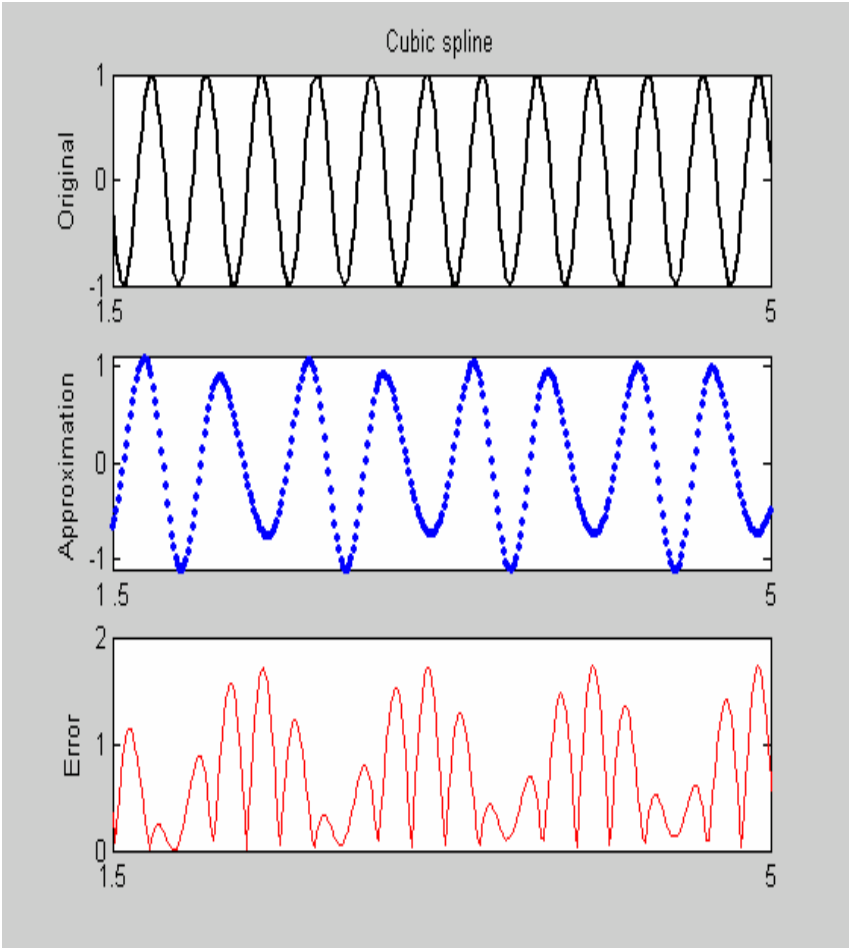
$(\beta_{\alpha_1} \otimes \beta_{\alpha_2})(t)$

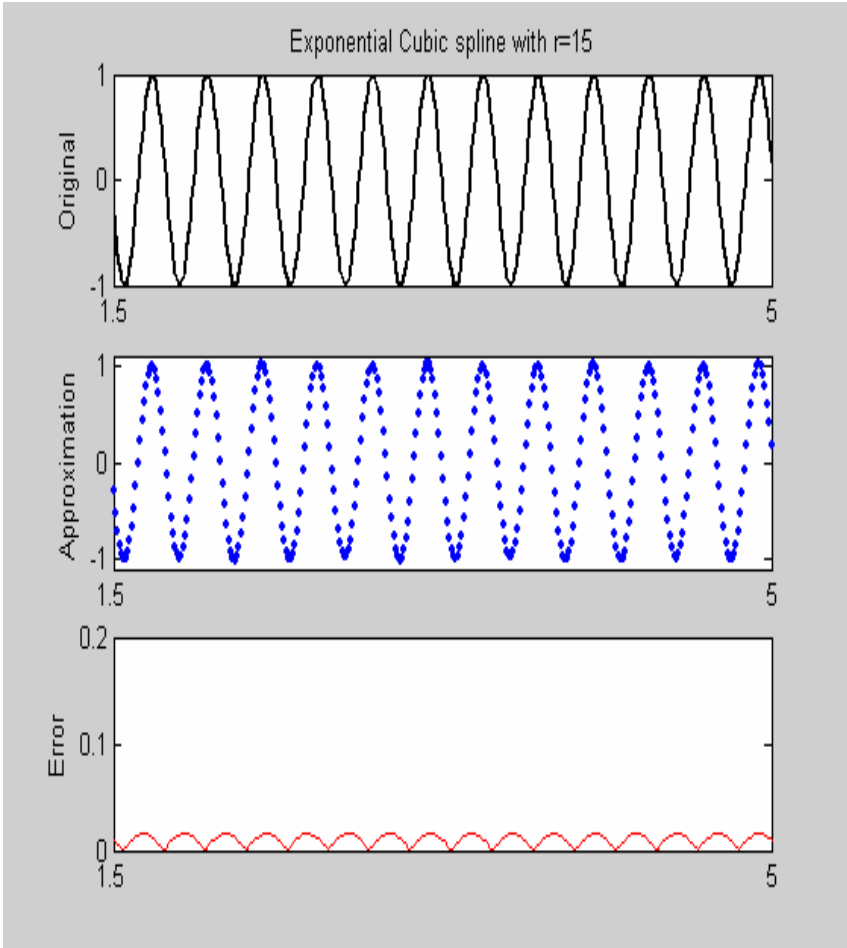
Linear Exponential B-spline



$(\beta_{\alpha_1} \otimes \beta_{\alpha_2})(t), \alpha_2 = -1 \ldots 1$

Approximation using Splines





Non-stationary MRA

An MRA of $L^2(\mathbf{R})$ is a sequence of closed subspaces $\{V_j\}_{j\in\mathbf{Z}}$ of $L^2(\mathbf{R})$ satisfying the following properties :

(a) $V_j \subset V_{j+1} \quad \forall j \in \mathbf{Z}$

(b) $\overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R})$ $\qquad (c)$ $\bigcap_{j\in\mathbf{Z}} V_j = \{0\}$

(d) $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1} \quad \forall j \in \mathbf{Z}$

(e) $f(x) \in V_0 \Rightarrow f(x-n) \in V_0 \quad \forall n \in \mathbf{Z}$

(f) $\exists \phi_j \in V_j \quad s.t.$

$\{\phi_{j,n}\}_{n\in\mathbf{Z}}$ is a Riesz basis of V_j

where $\phi_{j,k}(x) = \phi_j(x-k) \quad j,k \in \mathbf{Z}$

For a scaling function ϕ,
$\exists$ dual $\tilde{\phi}$ s.t. $< \phi(\cdot - k), \tilde{\phi}(\cdot - m) >= \delta_{k,m}$
with $\phi = \sum h_n \phi_{1,n}$ and $\tilde{\phi} = \sum \tilde{h}_n \tilde{\phi}_{1,n}$.
The corresponding wavelet functions $\psi, \tilde{\psi}$ are constructed by

$$\begin{aligned} \psi &= \sum g_n \phi_{1,n} \text{ with } g_n = (-1)^n \tilde{h}_{1-n} \\ \tilde{\psi} &= \sum \tilde{g}_n \tilde{\phi}_{1,n} \text{ with } \tilde{g}_n = (-1)^n h_{1-n} \end{aligned}$$

Then $V_j \perp \tilde{W}_j$ and

$$V_{j+1} = V_j + W_j$$

Wavelet Decomposition

Let the projection P_j and Q_j be

$$
\begin{aligned}
P_j : \quad & L^2(\mathbf{R}) \longrightarrow V_j \quad , \\
Q_j : \quad & L^2(\mathbf{R}) \longrightarrow W_j \quad . \\
P_j f \quad & = \quad \sum_k < f, \tilde{\phi}_{j,k} > \phi_{j,k} \quad , \\
Q_j f \quad & = \quad \sum_k < f, \tilde{\psi}_{j,k} > \psi_{j,k} \quad .
\end{aligned}
$$

Let $c_k^j =< f, \tilde{\phi}_{j,k} >$ and $d_k^j =< f, \tilde{\psi}_{j,k} >$.

Wavelet Decomposition

$$\cdots \quad V_{j-1} \subset V_j \subset V_{j+1} \quad \cdots$$

$$\cdots \quad c^{j-1} \leftarrow c^j \leftarrow c^{j+1} \quad \cdots$$

$$\swarrow \qquad \swarrow$$

$$d^{j-1} \qquad d^j$$

Refinable relation of Exponential B-spline

$$\phi_j(x) = \sum a_n^{[j]} \phi_{j+1}(2x - n).$$

$$a^{[j-1]}(z) = 2 \prod_{n=1}^{2N} \frac{1 + e^{2^{-j}\alpha_n} z}{1 + e^{2^{-j}\alpha_n}}.$$

$$a(z) = \frac{(1+z)^{2N}}{2^{2N-1}}.\text{(B-spline)}$$

Approximation order

For convenience, in the sequel, we use the notation

$$\Gamma := \Gamma_{\tilde{N}} := \{\gamma_n \in \mathbb{C} : n = 1, \ldots, \tilde{N}\}.$$

(Generalized Strang-Fix condition) Let $j \in \mathbb{Z}_+$. We say that a function ϕ_j satisfies the N-th order Generalized Strang-Fix condition with respect to Γ if the Fourier transform $\widehat{\phi}_j$ satisfies the conditions: There exist $\beta_1, \ldots, \beta_n \in \mathbb{Z}_+$ such that

$$(a) \quad \widehat{\phi}_j(-i2^{-j}\gamma_n) \neq 0, \quad n = 0, \ldots, \bar{N}, \qquad (1) \quad ,$$

$$(b) \quad \widehat{\phi}_j^{(\beta)}(2k\pi - i2^{-j}\gamma_n) = 0, \quad \beta = 0, \ldots, \beta_{\bar{N}} \in \mathbb{N},$$

where $\beta_1 + \cdots + \beta_{\bar{N}} = N$.

[Lemma]TFAE:

(i) The function ϕ_j satisfies the Nth-order Generalized Strang-Fix condition with respect to $\{\gamma_1, \ldots, \gamma_{\bar{N}}\} \subset \mathbb{C}$.

(ii)For each $j \in \mathbb{Z}_+$, the Laurent polynomials corresponding to ϕ is for the form

$$a^{[j]}(z) = 2 \prod_{n=1}^{\bar{N}} \frac{1}{2}(1 + e^{\gamma_n 2^{-j-1}} z)^{\beta_n} b_j(z),$$

with some Laurent polynomial $b_j(z)$, where $\beta_0 + \cdots + \beta_{\bar{N}} = N$.

[Theorem] Suppose that the dual functions ϕ_j and $\tilde{\phi}_j$, $j \in \mathbb{Z}_+$, satisfy the Generalized Strang-Fix condition of order N with respect to Γ. Then the approximation error of the projection $P_j f$ for any $f \in W^N(\Omega)$ can be estimated as follows:

$$\|f - P_j f\|_{L_2(\mathbb{R})} \leq c 2^{-jN} \|f\|_{N,2}$$

with a constant c independent of f.

Define the (β,γ)-moment of $\phi_j(x)$ as follows:

$$\mathcal{M}^{\phi_j}_{\beta,\gamma} := \int_{\mathbb{R}} x^\beta e^{-2^{-j}\gamma x}\phi_j(x)dx = i^\beta \hat{\phi}_j^{(\beta)}(-i2^{-j}\gamma),$$

and define $\eta_j(y) :=< \phi_j(\cdot - y),\ \tilde{\phi}_j(\cdot) >$. Then

$$\mathcal{M}^{\eta_j}_{\beta,\gamma} = \sum_{k=0}^{\beta} \binom{\beta}{k}(-1)^{\beta-k}\mathcal{M}^{\phi_j}_{k,\gamma}\mathcal{M}^{\tilde{\phi}_j}_{\beta-k,-\gamma}$$

$$= \begin{cases} C_{j,\gamma} & \text{if} \quad \beta = 0, \\ 0 & \text{if} \quad \beta = 1, \cdots, \gamma_n. \end{cases}$$

where

$$C_{j,\gamma} := C_{j,\gamma}^{[\phi_j,\tilde{\phi}_j]} := \hat{\phi}_j(-i2^{-j}\gamma)\hat{\tilde{\phi}}_j(-i2^{-j}\gamma).$$

For the given functions ϕ_j and $\tilde{\phi}_j$ with $j \in \mathbb{Z}_+$, define the kernel $K_j(x,y)$ by

$$K_j(x,t) := 2^j \sum_{\ell \in \mathbb{Z}} \phi_j(2^j t - \ell)\tilde{\phi}_j(2^j x - \ell).$$

[Proposition] Assume that ϕ satisfies the Generalized Strang-Fix condition of order N with respect to $\Gamma_{\tilde{N}}$. Let $\mathbf{S} = \text{span}\{x^\beta e^{\gamma_n x} : \beta = 0, \ldots, \beta_n - 1,\ n = 0, \ldots, \bar{N}\}$. Then, for any fixed γ_n, we have the exponential polynomial reproducing property in the following sense:

$$C_{j,\gamma_n}^{-1} \int_{\mathbb{R}} K_j(x,t) t^\beta e^{-\gamma_n t} dt = x^\beta e^{-\gamma_n x},$$
$$\beta = 0, \ldots, \beta_n - 1.$$

[Theorem] Let Ω be a compact set in $\mathbb{R}$, and assume that $f \in W^N(\mathbb{R})$. Then,

$$|f(x) - P_j f(x)| \leq c_N 2^{-Nj} \|f_{|_{r2^{-j}\bar{B}(x)}}\|_{N,\infty}$$

for some $r > 0$, where $B(x)$ indicate the unit ball centered at x.

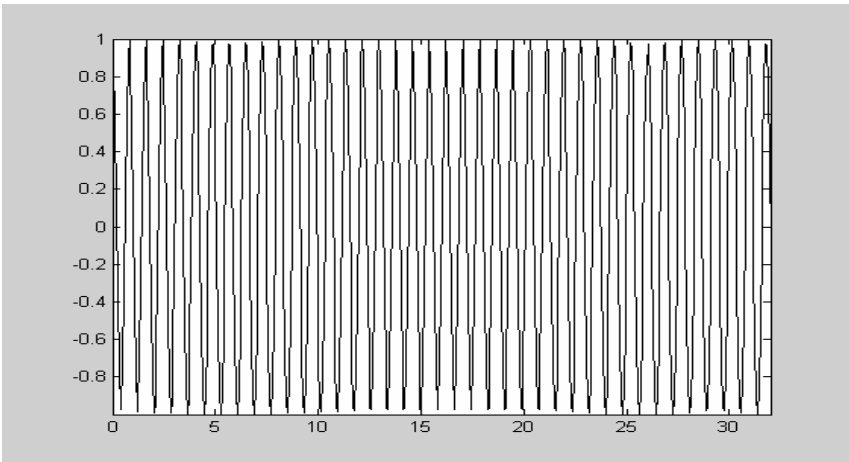
[Theorem] Let $j \in \mathbb{N}$, and assume that $\tilde{\psi}_j$ is orthogonal to the integer translates of ϕ_j, i.e., $< \psi_j(2^j \cdot), \phi_j(2^j \cdot - \ell) >= 0$ for any $\ell \in \mathbb{Z}$. Then, for any element $x^\beta e^{\gamma_n x}$ in $\mathbf{S}$,

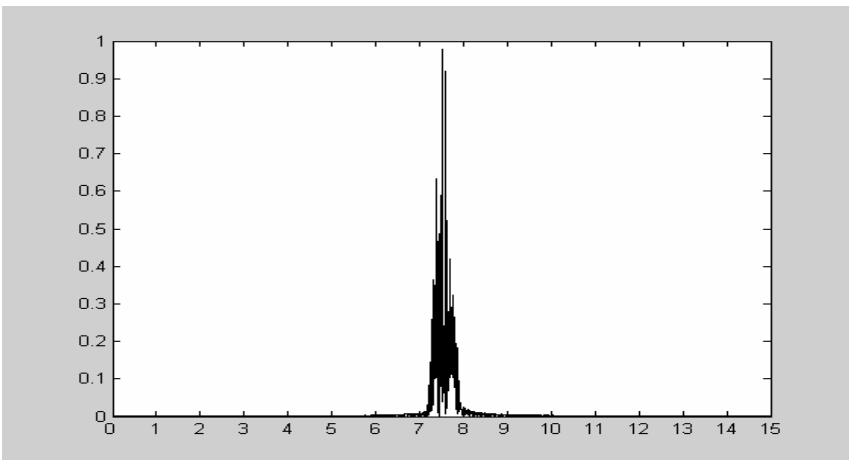
$$\int_{\mathbb{R}} \psi_j(x) x^\beta e^{\gamma_n x 2^{-j}} dx = 0, \quad j \in \mathbb{N}.$$

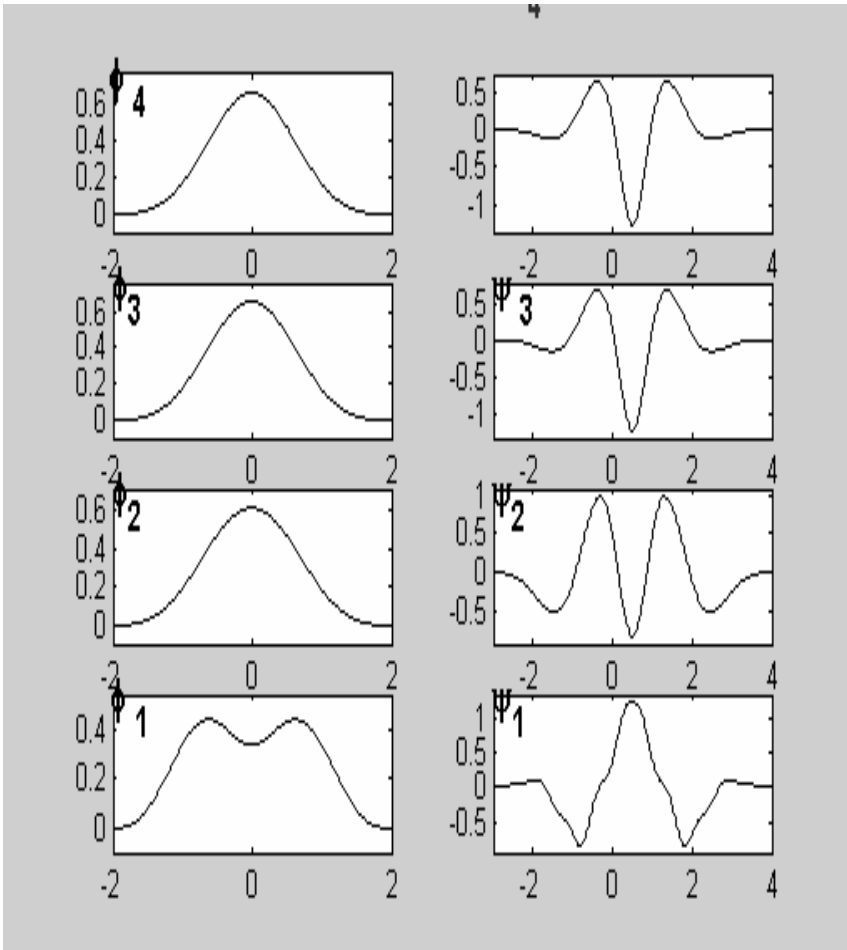
Signal Compression using Exponential B-spline

$$f(t) = \cos(2\pi F t + \beta \sin(2\pi F_s t)), \text{on} \quad [0, 32]$$

with $F = 1.2, F_s = 0.0062$ and $\beta = 5.75$.

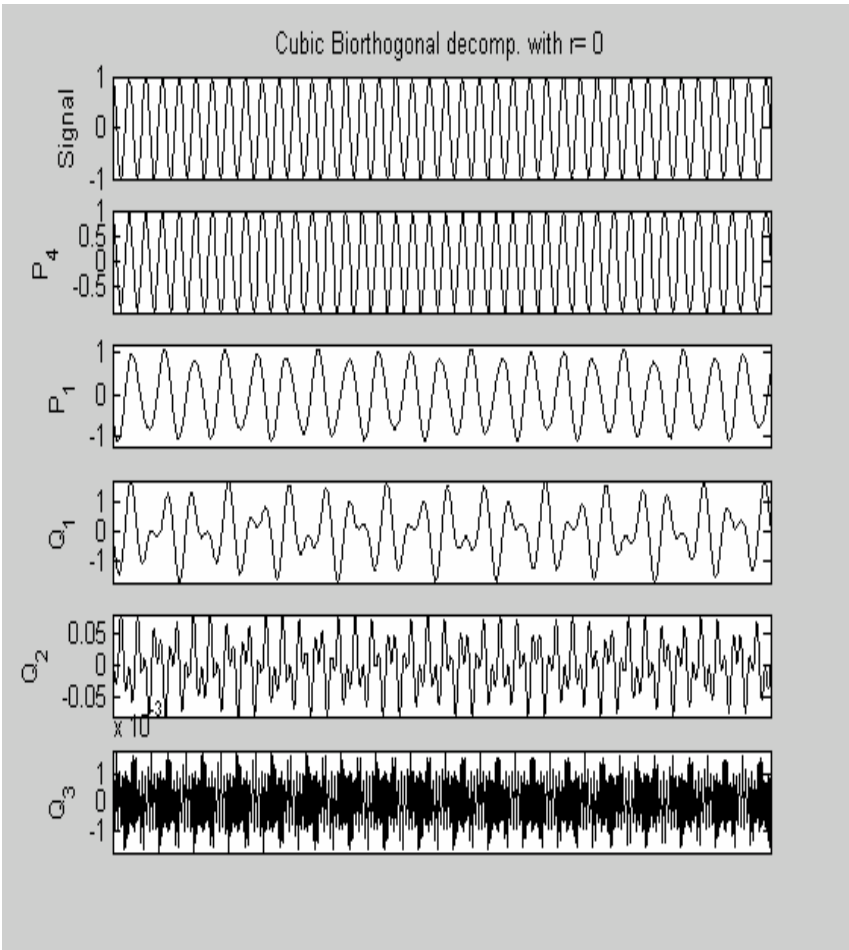






$$f(t) = \cos(2\pi F t + \beta \sin(2\pi F_s t)), \text{on} \quad [0, 32]$$

with $F = 1.2, F_s = 0.0062$ and $\beta = 5.75$.



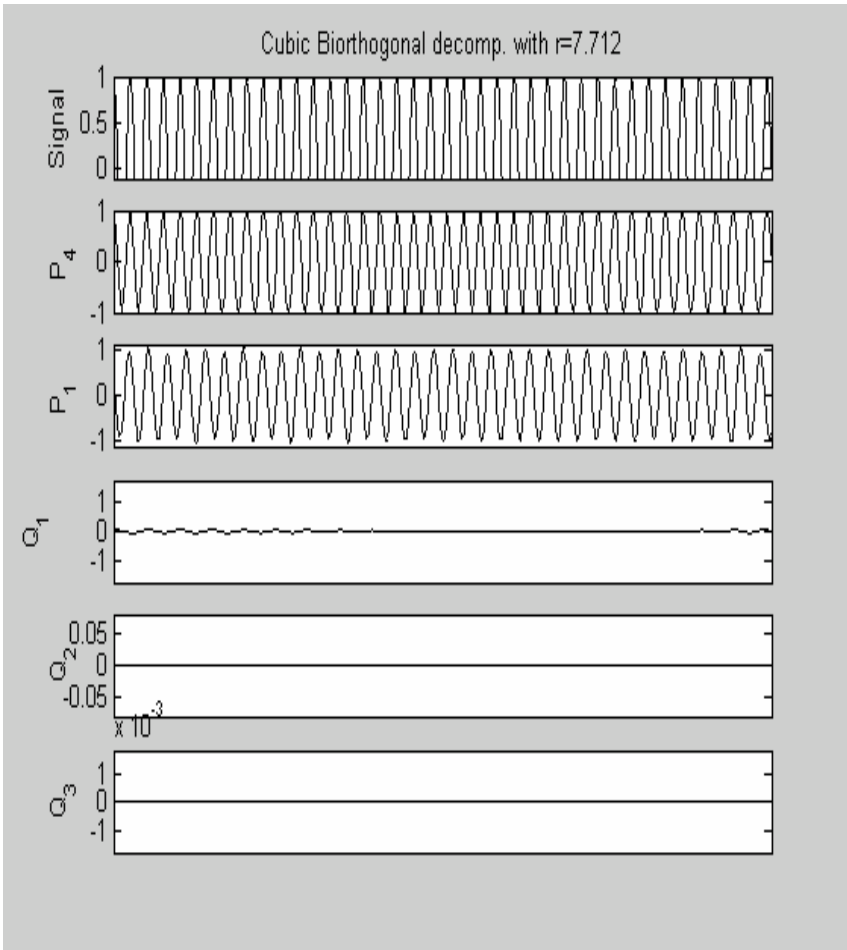
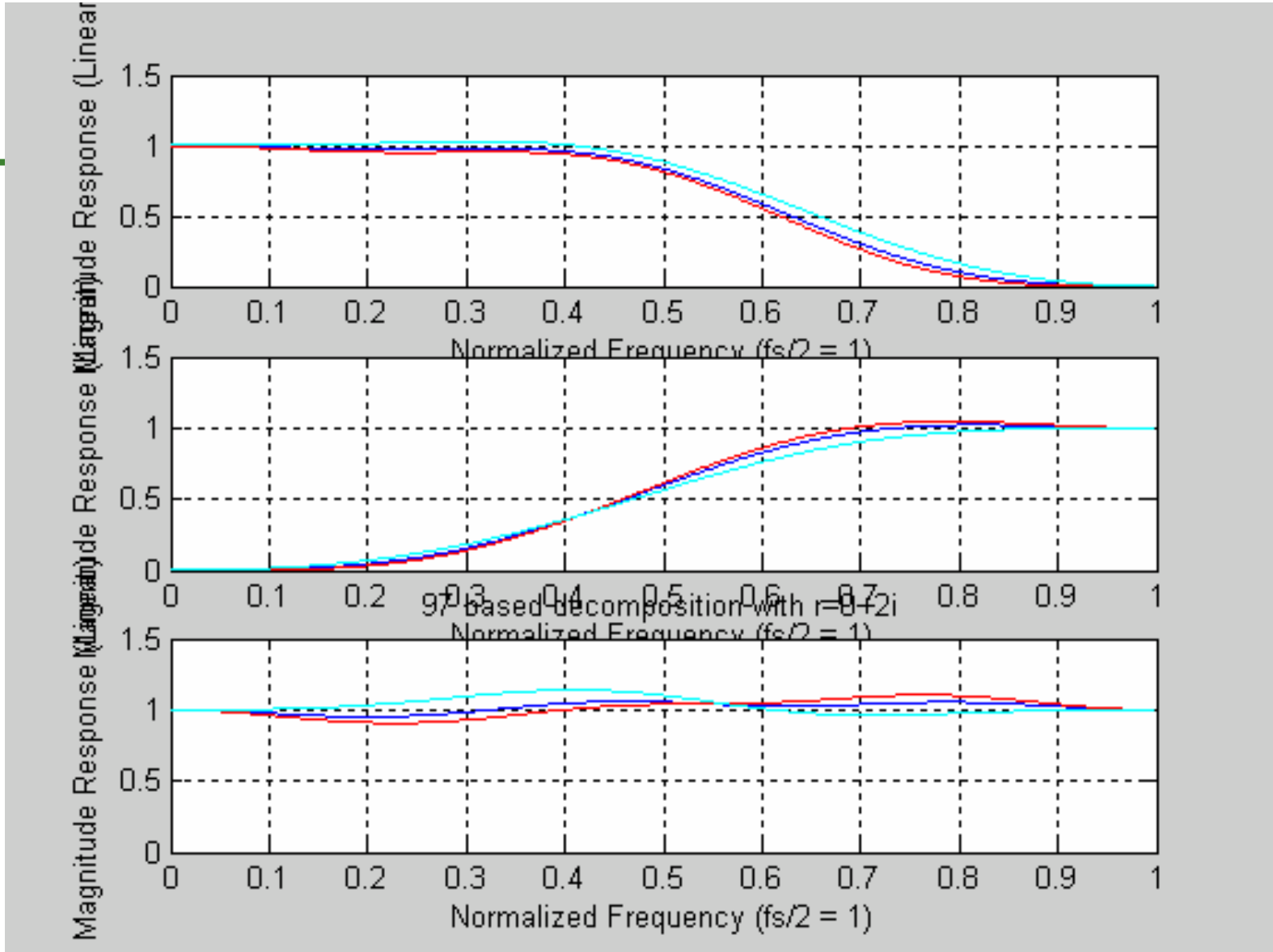
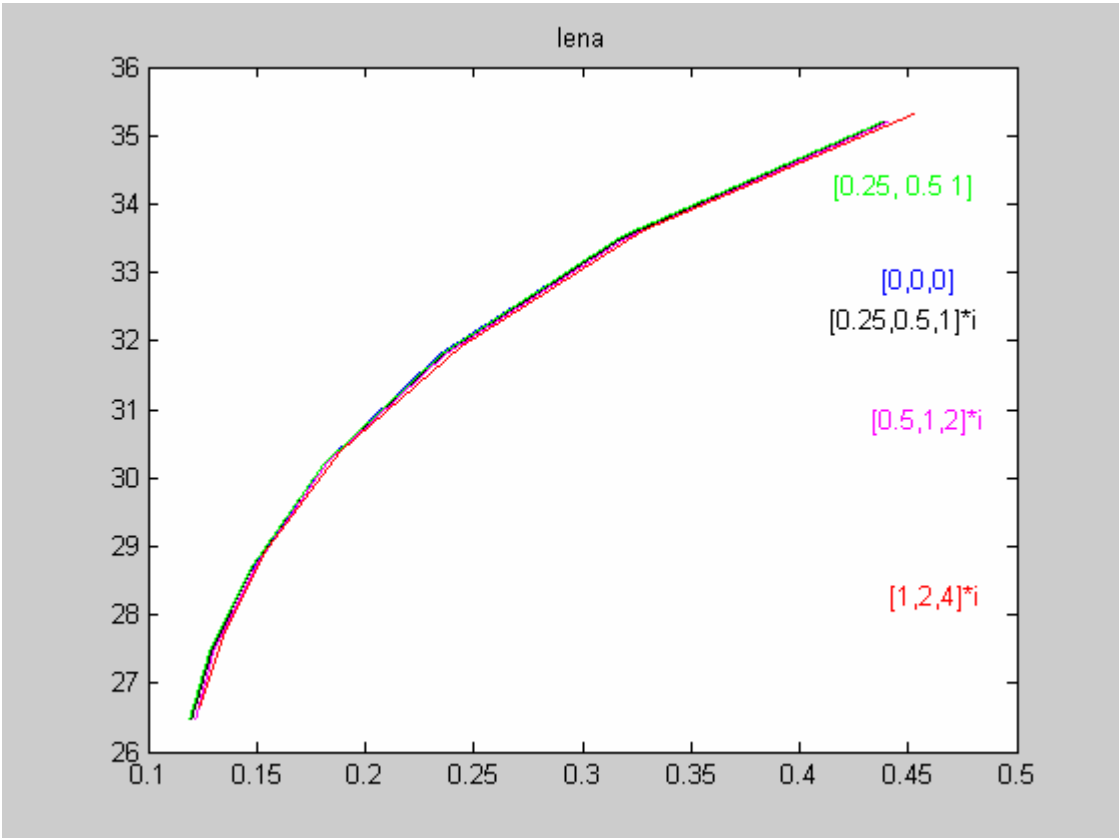
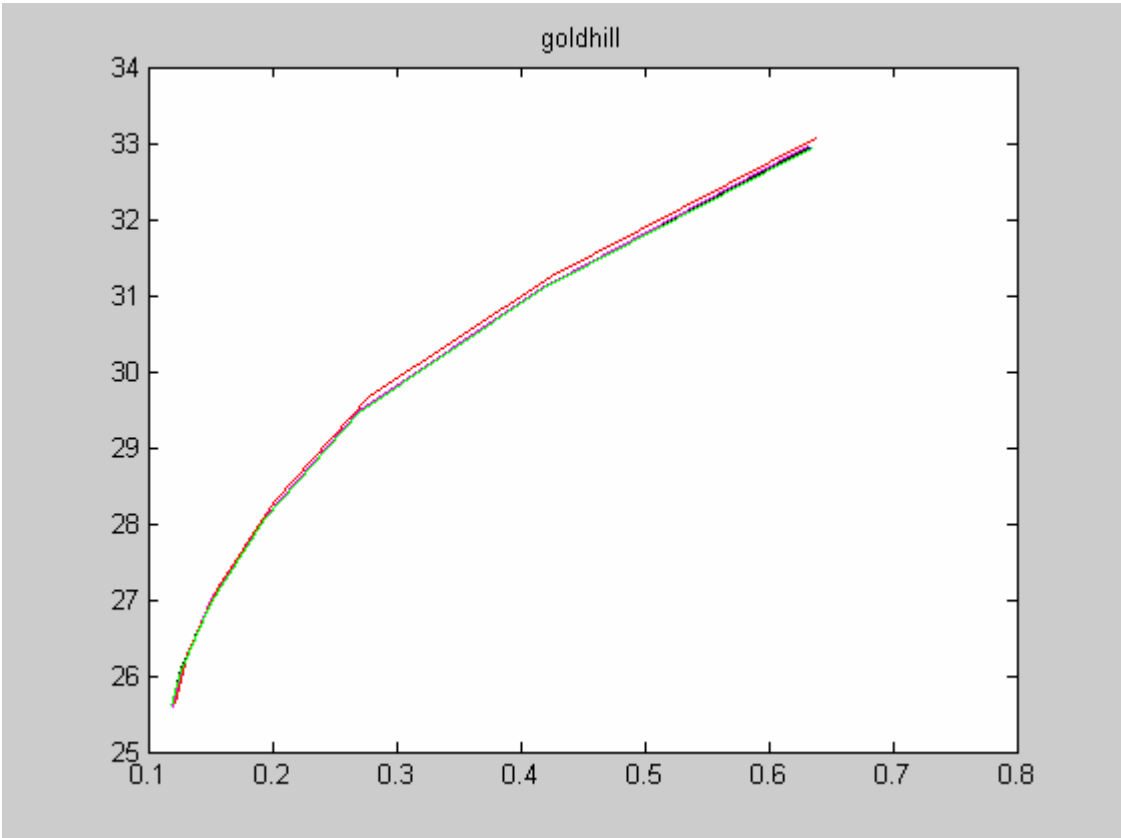


Image Compression using Exponential B-spline


Magnitude Response (Linear)
1.5
1
0.5
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
Normalized Frequency (fs/2 = 1)
Magnitude Response (Linear)
1.5
1
0.5
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
Normalized Frequency (fs/2 = 1)
Magnitude Response (Linear)
1.5
1
0.5
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
Normalized Frequency (fs/2 = 1)


lena
36
35
34
33
32
31
30
29
28
27
26
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
[0.25, 0.5 1]
[0,0,0]
[0.25,0.5,1]*i
[0.5,1,2]*i
[1,2,4]*i


goldhill
34
33
32
31
30
29
28
27
26
25
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8

Dedomposition using 97-tab with r=[0,0,0], threshold=20

PSNR=12.2912

Dedomposition using 97-tab with r=[0+0.5i,0+1i,0+2i], threshold=20

PSNR=12.2918