Non- stationary Biorthogonal Wavelet Systems Based on Exponentenetical B- spline

Jungho Yoon, Sang Soo Park, Yeon Ju Lee,

Ewha Univ.
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B- spline and Exponential B- spline
**B- spline**

\[
B_0(x) = \begin{cases} 
1, & 0 \leq x < 1, \\
0, & \text{otherwise}
\end{cases}
\]

\[
B_n(x) = \int_0^1 B_{n-1}(x-t) dt = B_0(x) \otimes B_{n-1}(x) \quad p(x) \otimes q(x) = \int_{-\infty}^{\infty} p(t)q(x-t) dt
\]
Exponential B-spline

\[ \beta_\alpha(t) = \rho_\alpha(t) - e^\alpha \rho_\alpha(t - 1) = \begin{cases} 
  e^{\alpha t}, & 0 \leq t < 1, \\
  0, & \text{otherwise}
\end{cases} \]

\[ \beta_{\bar{\alpha}}(t) = (\beta_{\alpha_1} \otimes \beta_{\alpha_2} \otimes \cdots \otimes \beta_{\alpha_n})(t) \]
Linear Exponential B-spline

\[ \alpha_1 = -\frac{1}{8}, \alpha_2 = 0 \]

\[ (\beta_{\alpha_1} \otimes \beta_{\alpha_2})(t) \]
Linear Exponential B-spline

\[(\beta_{\alpha_1} \otimes \beta_{\alpha_2})(t), \alpha_2 = -1 \ldots 1\]
Approximation using Splines
Non-stationary MRA
An MRA of $L^2(\mathbb{R})$ is a sequence of closed subspaces \( \{V_j\}_{j \in \mathbb{Z}} \) of $L^2(\mathbb{R})$ satisfying the following properties:

(a) \( V_j \subset V_{j+1} \quad \forall j \in \mathbb{Z} \)

(b) \( \bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}) \)

(c) \( \bigcap_{j \in \mathbb{Z}} V_j = \{0\} \)

(d) \( f(x) \in V_j \iff f(2x) \in V_{j+1} \quad \forall j \in \mathbb{Z} \)

(e) \( f(x) \in V_0 \Rightarrow f(x-n) \in V_0 \quad \forall n \in \mathbb{Z} \)

(f) \( \exists \phi_j \in V_j \quad \text{s.t.} \)

\( \{\phi_{j,n}\}_{n \in \mathbb{Z}} \) is a Riesz basis of $V_j$

where \( \phi_{j,k}(x) = \phi_j(x-k) \quad j, k \in \mathbb{Z} \)
For a scaling function \( \phi \),
\[ \exists \text{ dual } \tilde{\phi} \text{ s.t. } \langle \phi(\cdot - k), \tilde{\phi}(\cdot - m) \rangle = \delta_{k,m} \]
with \( \phi = \sum h_n \phi_{1,n} \) and \( \tilde{\phi} = \sum \tilde{h}_n \tilde{\phi}_{1,n} \).
The corresponding wavelet functions \( \psi, \tilde{\psi} \) are constructed by
\[
\psi = \sum g_n \phi_{1,n} \text{ with } g_n = (-1)^n \tilde{h}_{1-n} \\
\tilde{\psi} = \sum \tilde{g}_n \tilde{\phi}_{1,n} \text{ with } \tilde{g}_n = (-1)^n h_{1-n}
\]
Then \( V_j \perp \tilde{W}_j \) and
\[ V_{j+1} = V_j + W_j \]
Wavelet Decomposition

Let the projection $P_j$ and $Q_j$ be

\[ P_j : \ L^2(\mathbb{R}) \rightarrow V_j , \]
\[ Q_j : \ L^2(\mathbb{R}) \rightarrow W_j . \]

\[ P_j f = \sum_k < f, \tilde{\phi}_{j,k} > \phi_{j,k} , \]
\[ Q_j f = \sum_k < f, \tilde{\psi}_{j,k} > \psi_{j,k} . \]

Let $c^j_k = < f, \tilde{\phi}_{j,k} >$ and $d^j_k = < f, \tilde{\psi}_{j,k} >$. 
Wavelet Decomposition

\[ \cdots \quad V_{j-1} \subset V_j \subset V_{j+1} \quad \cdots \]

\[ \cdots \quad c_j^{j-1} \leftarrow c_j^j \leftarrow c_j^{j+1} \quad \cdots \]

\[ \downarrow \quad \downarrow \]

\[ d_j^{j-1} \quad d_j^j \]
Refinable relation of Exponential B-spline

\[ \phi_j(x) = \sum a_n^{[j]} \phi_{j+1}(2x - n). \]

\[ a^{[j-1]}(z) = 2 \prod_{n=1}^{2N} \frac{1 + e^{2^{-j} \alpha_n} z}{1 + e^{2^{-j} \alpha_n}}. \]

\[ a(z) = \frac{(1 + z)^{2N}}{2^{2N-1}}. \text{(B-spline)} \]
Approximation order
For convenience, in the sequel, we use the notation

\[ \Gamma := \Gamma_{\tilde{N}} := \{ \gamma_n \in \mathbb{C} : n = 1, \ldots, \tilde{N} \}. \]

(Generalized Strang-Fix condition) Let \( j \in \mathbb{Z}_+ \). We say that a function \( \phi_j \) satisfies the \( N \)-th order Generalized Strang-Fix condition with respect to \( \Gamma \) if the Fourier transform \( \hat{\phi}_j \) satisfies the conditions: There exist \( \beta_1, \ldots, \beta_n \in \mathbb{Z}_+ \) such that

(a)  \( \hat{\phi}_j(-i2^{-j}\gamma_n) \neq 0, \quad n = 0, \ldots, \tilde{N}, \quad (1) \),

(b)  \( \hat{\phi}_j(\beta)(2k\pi - i2^{-j}\gamma_n) = 0, \quad \beta = 0, \ldots, \beta_{\tilde{N}} \in \mathbb{N}, \)

where \( \beta_1 + \cdots + \beta_{\tilde{N}} = N \).
[Lemma] TFAE:

(i) The function $\phi_j$ satisfies the $N$th-order Generalized Strang-Fix condition with respect to $\{\gamma_1, \ldots, \gamma_N\} \subset \mathbb{C}$.

(ii) For each $j \in \mathbb{Z}_+$, the Laurent polynomials corresponding to $\phi$ is for the form

$$a[j](z) = 2 \prod_{n=1}^{\bar{N}} \frac{1}{2}(1 + e^{\gamma n 2^{-j-1} z})^{\beta_n} b_j(z),$$

with some Laurent polynomial $b_j(z)$, where $\beta_0 + \cdots + \beta_{\bar{N}} = N$. 
[Theorem] Suppose that the dual functions $\phi_j$ and $\tilde{\phi}_j$, $j \in \mathbb{Z}_+$, satisfy the Generalized Strang-Fix condition of order $N$ with respect to $\Gamma$. Then the approximation error of the projection $P_j f$ for any $f \in W^N(\Omega)$ can be estimated as follows:

$$\|f - P_j f\|_{L^2(\mathbb{R})} \leq c 2^{-jN} \|f\|_{N,2}$$

with a constant $c$ independent of $f$. 
Define the \((\beta, \gamma)\)-moment of \(\phi_j(x)\) as follows:

\[
\mathcal{M}^{\phi_j}_{\beta, \gamma} := \int_{\mathbb{R}} x^\beta e^{-2^{-j} \gamma x} \phi_j(x) \, dx = i^\beta \tilde{\phi}_j^{(\beta)}(-i2^{-j} \gamma),
\]

and define \(\eta_j(y) := \langle \phi_j(\cdot - y), \tilde{\phi}_j(\cdot) \rangle\). Then

\[
\mathcal{M}^{\eta_j}_{\beta, \gamma} = \sum_{k=0}^{\beta} \binom{\beta}{k} (-1)^{\beta-k} \mathcal{M}^{\phi_j}_{\beta-k, \gamma} \mathcal{M}^{\tilde{\phi}_j}_{\beta-k, \gamma}
\]

\[
= \begin{cases} 
C_{j, \gamma} & \text{if } \beta = 0, \\
0 & \text{if } \beta = 1, \ldots, \gamma_n.
\end{cases}
\]

where

\[
C_{j, \gamma} := C^{[\phi_j, \tilde{\phi}_j]}_{j, \gamma} := \tilde{\phi}_j(-i2^{-j} \gamma) \tilde{\phi}_j(-i2^{-j} \gamma).
\]
For the given functions $\phi_j$ and $\tilde{\phi}_j$ with $j \in \mathbb{Z}_+$, define the kernel $K_j(x, y)$ by

$$K_j(x, t) := 2^j \sum_{\ell \in \mathbb{Z}} \phi_j(2^j t - \ell) \tilde{\phi}_j(2^j x - \ell).$$

[Proposition] Assume that $\phi$ satisfies the Generalized Strang-Fix condition of order $N$ with respect to $\Gamma_{\tilde{N}}$. Let $S = \text{span}\{x^\beta e^{\gamma_n x} : \beta = 0, \ldots, \beta_n - 1, \ n = 0, \ldots, \tilde{N}\}$. Then, for any fixed $\gamma_n$, we have the exponential polynomial reproducing property in the following sense:

$$C^{-1}_{j, \gamma_n} \int_{\mathbb{R}} K_j(x, t)t^\beta e^{-\gamma_n t} \, dt = x^\beta e^{-\gamma_n x},$$

$$\beta = 0, \ldots, \beta_n - 1.$$
[Theorem] Let $\Omega$ be a compact set in $\mathbb{R}$, and assume that $f \in W^N(\mathbb{R})$. Then,

$$|f(x) - P_j f(x)| \leq c_N 2^{-Nj} \|f|_{r2^{-j}B(x)}\|_{N,\infty}$$

for some $r > 0$, where $B(x)$ indicate the unit ball centered at $x$.

[Theorem] Let $j \in \mathbb{N}$, and assume that $\tilde{\psi}_j$ is orthogonal to the integer translates of $\phi_j$, i.e., $<\psi_j(2^j \cdot), \phi_j(2^j \cdot - \ell) >= 0$ for any $\ell \in \mathbb{Z}$. Then, for any element $x^\beta e^{\gamma_n x}$ in $S$,

$$\int_{\mathbb{R}} \psi_j(x) x^\beta e^{\gamma_n x} 2^{-j} \, dx = 0, \quad j \in \mathbb{N}.$$
Signal Compression using Exponential B- spline
\[ f(t) = \cos(2\pi Ft + \beta \sin(2\pi F_t)), \quad \text{on} \quad [0, 32] \]

with \( F = 1.2, F_s = 0.0062 \) and \( \beta = 5.75 \).
$$f(t) = \cos(2\pi F t + \beta \sin(2\pi F_s t)), \text{ on } [0, 32]$$

with $F = 1.2$, $F_s = 0.0062$ and $\beta = 5.75$. 

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2006. 4. 21

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Image Compression using Exponential B- spline
Decomposition using 97-tab with r=[0,0,0], threshold=20

PSNR=12.2912
Decomposition using 97-tab with \( r = [0+0.5i, 0+1i, 0+2i] \), threshold=20

PSNR=12.2918