Why Exponential Wavelet?

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Discreet Signal Processing

- Decomposition

$$[S^0 := \{s_k\}_{k=1}^n] \longrightarrow [S^{-1}]$$

- Reconstruction

$$[S^0 := \{s_k\}_{k=1}^n] \longleftarrow [S^{-1}]$$

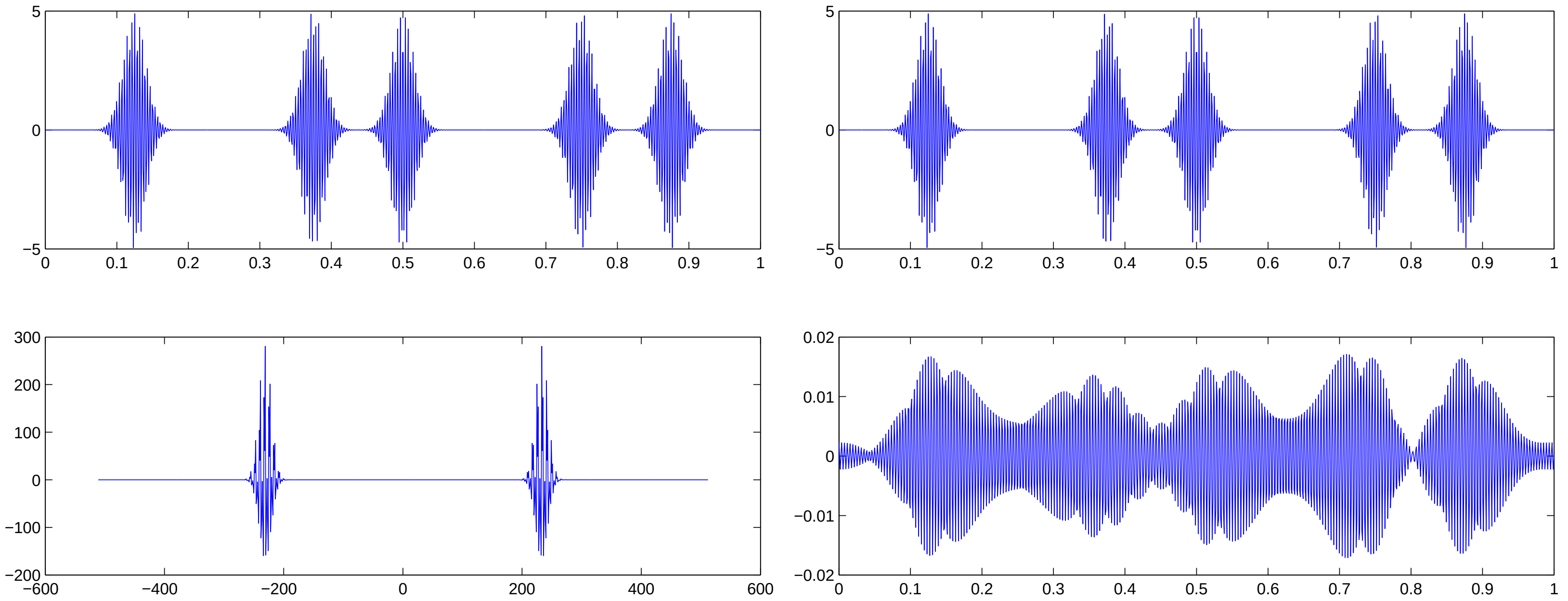
Example : Discreet Fourier Matrix

$$D := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{-1\pi i/n} & e^{-2\pi i/n} & \cdots & e^{-n\pi i/n} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-(n-1)\pi i/n} & e^{-2(n-1)\pi i/n} & \cdots & e^{-n(n-1)\pi i/n} \end{bmatrix}, \qquad R := \frac{1}{n} D^*$$

$$f(x) = (5cos2\pi\nu x)[e^{-640\pi(x-1/8)^2} + e^{-640\pi(x-3/8)^2} + e^{-640\pi(x-4/8)^2} + e^{-640\pi(x-6/8)^2} + e^{-640\pi(x-7/8)^2}]$$

We consider the sample

$$F_{1024} = \{f(\frac{i}{1024})\}_{i=0}^{1023}$$



theorem

$\forall f \in L^2([0, 2\pi))$,

$$f(x) = \sum_{i \in \mathbb{Z}} f_n e^{int}, \qquad \text{where } f_n = \int f(t) e^{-int} dt$$

Remark

The **Fourier Analysis** has also fatal drawbacks such as Gibbs Phenomenon, approximation order, $\cdots$. They originate from the **global support of the bases functions**.

Now, what do we want?

ψ : compact support such that

$$f(x) = \sum_{n,j\in\mathbb{Z}} f_{n,j}\psi(2^n x - j) \qquad \forall f \in L^2(\mathbb{R})$$

Notation

- $a(\xi) = \sum_k \frac{1}{2} a_k e^{-ik\xi}$
- $a(z) = \sum_k \frac{1}{2} a_k z^k$
- $\hat{f}(\xi) = \int f(x) e^{-ix\xi}\, dx$
- $V_n := \{\phi(2^n \cdot -j) | j \in n\mathbb{Z}\}$
- $W_n,\ \tilde{V}_n,\ \tilde{W}_n$: similar way

Definition

Biorthogoanl Wavelets System $\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\}$

- two scaling relations

$$\phi(x) = \sum_k a_k \phi(2x-k) \quad \tilde{\phi}(x) = \sum_k \tilde{a}_k \tilde{\phi}(2x-k)$$
$$\psi(x) = \sum_k b_k \phi(2x-k) \quad \tilde{\psi}(x) = \sum_k \tilde{b}_k \tilde{\phi}(2x-k)$$

- biorthogonal relations

$$\begin{aligned}
\left\langle \phi, \tilde{\phi}(\cdot - j) \right\rangle &= \delta_{0,j} &\Leftarrow\quad & a(\xi)\overline{\tilde{a}(\xi)} + a(\xi+\pi)\overline{\tilde{a}(\xi+\pi)} = 1 \\
\left\langle \phi, \tilde{\psi}(\cdot - j) \right\rangle &= 0 &\Leftarrow\quad & a(\xi)\overline{\tilde{b}(\xi)} + a(\xi+\pi)\overline{\tilde{b}(\xi+\pi)} = 0 \\
\left\langle \tilde{\phi}, \psi(\cdot - j) \right\rangle &= 0 &\Leftarrow\quad & \tilde{a}(\xi)\overline{b(\xi)} + \tilde{a}(\xi+\pi)\overline{b(\xi+\pi)} = 0 \\
\left\langle \psi, \tilde{\psi}(\cdot - j) \right\rangle &= \delta_{0,j} &\Leftarrow\quad & b(\xi)\overline{\tilde{b}(\xi)} + b(\xi+\pi)\overline{\tilde{b}(\xi+\pi)} = 1
\end{aligned}$$

- Theoretical things : completeness, stability

Well-Known

- $V_0 \subset V_1, \quad i.e., \quad \phi(\cdot - j) \in Span\{\phi(2\cdot - k) | j \in \mathbb{Z}\}$
- $V_1 \subset V_0 + W_0, \quad i.e., \quad \phi(2\cdot - j) \in Span\{\phi(\cdot - k), \psi(\cdot - k) | j \in \mathbb{Z}\}$

How to construct the B.W.S.

Find $a, \tilde{a}$ and $b, \tilde{b}$ satisfying

$$\begin{bmatrix} a(\xi) & a(\xi+\pi) \\ b(\xi) & b(\xi+\pi) \end{bmatrix} \begin{bmatrix} \tilde{a}(\xi) & \tilde{a}(\xi+\pi) \\ \tilde{b}(\xi) & \tilde{b}(\xi+\pi) \end{bmatrix}^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Most popular two methods

- Stating with a, b having finite supports : Unser's Method
- Stating with $a, \tilde{a}$ having finite supports: Our Method

Algorithm of Our Method

1. Start with given a
2. To find $\tilde{a}$, solve the linear equation

$$a(\xi)\overline{\tilde{a}(\xi + \pi)} + a(\xi + \pi)\overline{\tilde{a}(\xi + \pi)} = 1$$

3. Put

$$b(\xi) = (-1)^N \tilde{a}(\xi + \pi), \quad \text{and} \quad \tilde{b}(\xi) = (-1)^N a(\xi + \pi)$$

Stationary Subdivision Scheme

Definition Subdivision Operator $S_a : \ell(\mathbb{Z}) \to \ell(\mathbb{Z})$ is defined by

$$(S_a v)_i = \sum_{j \in \mathbb{Z}} a_{i-2j} v_j.$$

example: $a = [1/2, 1, 1/2]$

Construct $\delta \to S_a\delta \to S_a^2\delta \to S_a^3\delta \cdots$

- $\delta =[\ \cdots,\ 1.0000\ \cdots]$
- $S_a\delta =[\ \cdots,\ 0.5000\ 1.0000\ 0.5000\ \cdots]$
- $S_a^2\delta =[\ \cdots,\ 0.2500\ 0.5000\ 0.7500\ 1.0000\ 0.7500\ 0.5000\ \cdots]$
- $S_a^3\delta =[\cdots,\ 0.125\ 0.250\ 0.375\ 0.500\ 0.625.\ 0.750\ \cdots]$

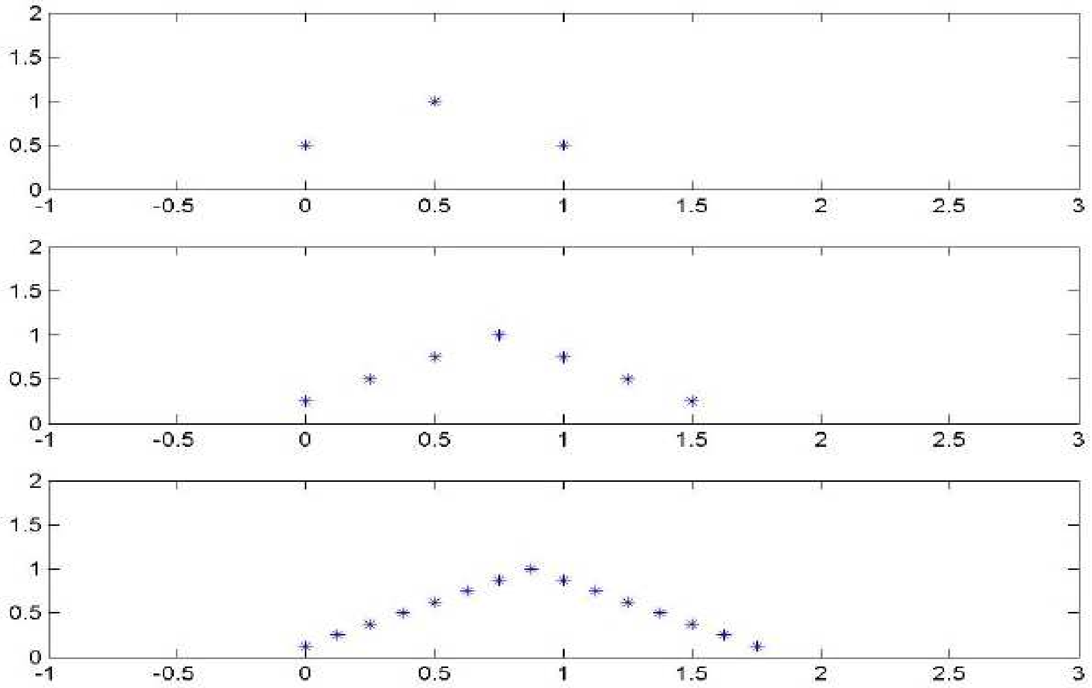


Figure 1: Top : $S_a\delta$, Middle:$S_a^2\delta$, Bottom : $S_a^3\delta$, where a=[1/2,1,1/2],

Continuing this process, we put

$$\phi = S_a^{\infty}\delta$$

Non-Stationary Subdivision Scheme

Definition Subdivision Operator $S_{a^{[j]}} : \ell(\mathbb{Z}) \to \ell(\mathbb{Z})$ is defined by

$$(S_{a^{[j]}})_i = \sum_{j \in \mathbb{Z}} a^{[j]}{}_{i-2j} v_j .$$

Construct $\delta \to S_{a^{[0]}} \delta \to S_{a^{[1]}} S_{a^{[0]}} \delta \to S_{a^{[2]}} S_{a^{[1]}} S_{a^{[0]}} \delta \cdots$

We put

$$\phi^{[j]} := \lim_{n \to \infty} S_{a^{[j+n]}} \cdots S_{a^{[j+1]}} S_{a^{[j]}} \delta$$

Important things : How to choose a

- $a \sim \tilde{b}$
- $a \sim \phi$ and $\tilde{b} \sim \tilde{\psi}$
- If we determine $\tilde{b}$ satisfying some properties such as vanishing moments, then a can be constructed.

Advantage

If a has a compact support and symmetry, then we can choose $\tilde{a}$ with a compact support. So, all of functions in BWS have compact supports

Discreet Signal Processing via B.W.S

- Decomposition

$$[S^0 := \{s_k\}] \rightarrow [S^{-1} : D^{-1}] \rightarrow \cdots \rightarrow [S^{-n} : D^{-n} : \cdots : D^{-1}]$$

Algorithm
$$S^{-k} = (\downarrow 2)\tilde{a}(-) * S^{-k+1}$$
$$D^{-k} = (\downarrow 2)\tilde{b}(-) * S^{-k+1}$$

- Reconstruction

$$[S^0 := \{s_k\}] \leftarrow [S^{-1} : D^{-1}] \leftarrow \cdots \leftarrow [S^{-n} : D^{-n} : \cdots : D^{-1}]$$

Algorithm $S^{-k+1} = a * (\uparrow 2)S^{-k} + b * (\uparrow 2)d^{-k}$

Decomposition

$$[S^0 := \{s_k^0\}] \to [S^{-1} : D^{-1}] \to \cdots \to [S^{-n} : D^{-n} : \cdots : D^{-1}]$$

Algorithm
$$S^{-k} = (\downarrow 2)\tilde{a}(-) * S^{-k+1}$$
$$D^{-k} = (\downarrow 2)\tilde{b}(-) * S^{-k+1}$$

Analysis Recall : $V_0 \subset V_{-1} + W_{-1}$

$$\sum_k s_k^0 \phi(x-k) = \sum_j s_j^{-1} \phi(\frac{x}{2} - j) + \sum_j d_j^{-1} \psi(\frac{x}{2} - j)$$

where
$$s_j^{-1} = \sum_k \tilde{a}_{k-2j} s_k^0$$
$$d_j^{-1} = \sum_k \tilde{b}_{k-2j} s_k^0 = \langle \sum_k s_k^0 \phi(x-k), \tilde{\psi}(\frac{x}{2} - j) \rangle$$

Polynomial Vanishing Moments

- $\int x^n \tilde{\psi}(x)dx = 0 \quad \Leftrightarrow \quad (1-z)^{n+1} \mid \tilde{b}(z) \quad \Leftrightarrow \quad (1+z)^{n+1} \mid a(z)$

Exponential Vanishing Moments

- $\int e^{\gamma x} \tilde{\psi}(x)dx = 0 \quad \Leftrightarrow \quad (1-e^{i\gamma}z) \mid \tilde{b}(z) \quad \Leftrightarrow \quad (1+e^{i\gamma}z) \mid a(z)$

Summary

1. Start with

$$a^{[j]}(z) = 2\prod_{n=1}^{N} \frac{(1+e^{\frac{\gamma}{2^{j+1}}}z)(1+e^{-\frac{\gamma}{2^{j+1}}}z)}{(1+e^{\frac{\gamma}{2^{j+1}}})(1+e^{-\frac{\gamma}{2^{j+1}}})}$$

2. Solve the linear equation

$$a^{[j]}(\xi)\overline{\tilde{a}^{[j]}(\xi)} + a^{[j]}(\xi+\pi)\overline{\tilde{a}^{[j]}(\xi+\pi)} = 1$$

3. Put

$$b^{[j]}(\xi) = (-1)^N \tilde{a}^{[j]}(\xi+\pi), \quad \text{and} \quad \tilde{b}^{[j]}(\xi) = (-1)^N a^{[j]}(\xi+\pi)$$

4. we have the non-stationary B.W.S.

$$\phi^{[j]}(x) = \textstyle\sum_k a_k^{[j]}\phi^{[j+1]}(2x-k) \quad \tilde{\phi}^{[j]}(x) = \textstyle\sum_k \tilde{a}_k^{[j]}\tilde{\phi}^{[j+1]}(2x-k)$$
$$\psi^{[j]}(x) = \textstyle\sum_k b_k^{[j]}\phi^{[j+1]}(2x-k) \quad \tilde{\psi}^{[j]}(x) = \textstyle\sum_k \tilde{b}_k^{[j]}\tilde{\phi}^{[j+1]}(2x-k)$$

Main Tools: Dyn, Levin,Yoon's result

Let $c^{[j]}(z) = \frac{1}{2}(1 + r_j z)d^{[j]}(z)$, and

$$f^{[j]}(x) = \sum_{k\in\mathbb{Z}} c_k^{[j]} f^{[j+1]}(2x - k)$$

$$g^{[j]}(x) = \sum_{k\in\mathbb{Z}} d_k^{[j]} g^{[j+1]}(2x - k)$$

If

$$|1 - r_j| \le C2^{-j}, \qquad k \ge K \in \mathbb{N}$$

Then, the Hölder exponent $f^{[j]}$ =1+the Hölder exponent $g^{[j]}$

Example

$$a^{[j]}(z) = 2\frac{(1 + e^{\frac{\gamma}{2^{j+1}}} z)(1 + e^{-\frac{\gamma}{2^{j+1}}} z)}{(1 + e^{\frac{\gamma}{2^{j+1}}})(1 + e^{-\frac{\gamma}{2^{j+1}}})}(1 + z)^2$$

Advantage beating Unser's Method

- $a^{[j]}, \ \tilde{a}^{[j]}, b^{[j]}, \tilde{b}^{[j]}$: all finite support

Numerical Experiment

$$f(t) = cos\left(2\pi Ft + \beta sin(2\pi F_s t)\right), \quad \text{where } F = 1.2,\ F_s = 0.0062,\ \beta = 5.75$$