PIECEWISE BILINEAR PRECONDITIONING ON HIGH-ORDER FINITE ELEMENT METHODS *

SANG DONG KIM†

Abstract. The bounds of eigenvalues which are independent of both degrees of high-order elements and mesh sizes are shown for the preconditioned system by bilinear elements for the high-order finite elements discretizations applied to a model uniformly elliptic operator.

Key words. Multigrid, high-order finite element methods, piecewise linear preconditioning

AMS subject classifications. 65F10, 65M30

1. Introduction. High-order finite element methods for discretizing a second-order uniformly elliptic partial differential equation lead to a linear equation $\widetilde{L}_{N^2}U = F$ which requires efficient iterative methods such as Schwarz-based methods (see [5] [13] and [16]), preconditioning methods related to multilevel methods, multigrid methods (see [6], [7] and [8]) and etc. This is because such linear systems have large condition numbers which depend on the order of the elements used and the mesh spacing. In particular, an algebraic multigrid (AMG) method is useful in the case of irregular grids. However it was reported that a direct application of AMG to $\widetilde{L}_{N^2}U = F$ is not so efficient (see [8] and [15]). The convergence factor degrades rapidly as the order of the elements is increased. For the case of Stokes and elasticity equations, the complexity from the high-order finite element discretizations for AMG is even worse than that of a simple elliptic partial differential equation.

In [8], a preconditioning was constructed by using the Legendre-Gauss-Lobatto quadrature points in each cell as mesh points for a bilinear discretization. The preconditioning was approximately inverted by one AMG V-cycle. This approach has several advantages, including the possibility to avoid assembly of the high-order stiffness matrix. Numerical results show that this preconditioning was very effective, especially when accelerated by a conjugate gradient method. It has also an advantage of a straightforward matrix-free implementation for the fine grid high-order element matrix.

In order to show that such a bilinear preconditioning is effective, we will consider a uniformly elliptic boundary value problem like

$$L_p u := -\nabla \cdot p(x,y)\nabla u + q(x,y)u \quad \text{in} \quad \Omega = (-1,1)\times(-1,1) \tag{1.1}$$

with boundary conditions

$$u = 0 \quad \text{on} \quad \Gamma_D, \quad \mathbf{n}\cdot\nabla u = 0 \quad \text{on} \quad \Gamma_N \tag{1.2}$$

where $\Gamma = \Gamma_D \cup \Gamma_N$ with a nonempty Γ_D and $p(x,y)$ and $q(x,y)$ are nonnegative smooth bounded functions on Ω. The piecewise bilinear finite element preconditioner will be constructed by another uniformly elliptic boundary operator B like

$$Bv := -\nabla\cdot\nabla v + 2v \quad \text{in} \quad \Omega = (-1,1)\times(-1,1) \tag{1.3}$$

*This work was supported by KOSEF C00006(R02-2004-000-10109-0)

†Department of Mathematics, Kyungpook National University, Daegu 702-701, Korea (skim@knu.ac.kr)

with the same boundary (1.2). This operator B yields a matrix $\widehat{B}_{h^2}$ to reduce the condition numbers of a matrix $\widetilde{L}_{N^2}$ induced by high-order elements applied to (1.1)

For a convenience, we assume throughout this paper that the Dirichlet part of the boundary Γ is

$$\Gamma_D = \{-1\} \times [-1,1] \cup [-1,1] \times \{-1\}. \tag{1.4}$$

The main object in this article is to prove that the eigenvalues $(\widehat{B}_{h^2})^{-1}\widetilde{L}_{N^2}$ are independent of the degrees of high-order elements and the mesh sizes. As a result the condition numbers of the preconditioned systems are fixed and small so that the complexity is no longer problem when the AMG algorithm is applied. These make one to employ multigrid algorithms for solving problems like (1.1) with high-order elements discretizations, which guarantee convergence of the strategy of preconditioning the high-order matrix with a bilinear or trilinear matrix based on Legendre-Gauss-Lobatto quadrature nodes well suited to a solution by multigird methods. For a single spectral element, this kind of preconditioning was analyzed for Legendre spectral collocation methods in [3], [11], [12] and [14], for example.

The goal of this paper can be achieved by extending the results of [11] to high-order elements and by applying H^1, L_2 estimates in [17] of a local interpolation operator $\mathcal{I}_{N_j^t}$ to a global interpolation operator $\mathcal{I}^h_{N_j^t}$. Further, we note that such a H^1 semi-norm estimate of the local interpolation operator defined on a space of piecewise linear can be extended to the space of H^1 by modifying the relevant results in [1]. We also note that the discussions here can be extended to singular value results for a general elliptic operators which are not positive definite. For this, one had better refer to [11].

This paper consists of as follows. In next section, we recall some known results, piecewise polynomial basis, interpolation operators and etc. In section 3, we extend the results in [11] and [17] which lead to one and two dimensional preconditioning results for the constant coefficients case in section 4. The variable coefficients case is dealt with in section 5 using the tensor representation appeared in [4]. In section 6, we will provide some numerical results which support the developed theories. Finally we mention some conclusions in last section.

2. Preliminary. With the direction notation $t = x$ or y, we assume that M^t and N_j^t are natural numbers. Let $\{t_k\}_{k=0}^{M^t}$ be the knots in the interval $I = [-1,1]$ such that

$$-1 =: t_0 < t_1 < \cdots < t_{M^t-1} < t_{M^t} := 1. \tag{2.1}$$

Let $\{\eta_k\}_{k=0}^{N_j^t}$ and $\{\omega_k\}_{k=0}^{N_j^t}$ be the Legendre-Gauss-Lobatto (=:LGL) points in I arranged by

$$-1 =: \eta_0 < \eta_1 < \cdots < \eta_{N_j^t-1} < \eta_{N_j^t} := 1 \tag{2.2}$$

and its corresponding LGL weights respectively. Here M^t denotes the number of subintervals of $I = [-1,1]$ and N_j^t denotes the number of LGL points on a j^{th} subinterval by a translation of I. By the translation from I to a j^{th} subinterval $I_j^t := [t_{j-1}, t_j]$ we denote $\mathcal{G} := \{\xi_{j,k}^t\}_{j=1,k=0}^{M^t, N_j^t}$ as the $k^{th}-$ LGL points in each subinterval I_j^t for $j = 1, 2, \cdots, M^t$ and enumerate them as

$$\xi_{j,0}^t := t_{j-1} < \xi_{j,1}^t < \cdots < \xi_{j,N_j^t-1}^t < t_j =: \xi_{j,N_j^t}^t \tag{2.3}$$

where

$$\xi^t_{j,k} = \frac{h^t_j}{2}\eta_k + \frac{1}{2}(t_{j-1} + t_j), \quad h^t_j = t_j - t_{j-1} \tag{2.4}$$

and the corresponding LGL weights $\{\rho^t_{j,k}\}_{k=0}^{N^t_j}$ are given by

$$\rho^t_{j,k} = \frac{h^t_j}{2}\omega_k, \quad j = 1, 2, \cdots, M^t. \tag{2.5}$$

With $\xi^t_{M^t+1,0} := t_{M^t}$, note that in (2.3) and (2.5)

$$\xi^t_{j-1,N^t_j} = \xi^t_{j,0}, \ \rho^t_{j-1,N^t_j} = \rho^t_{j,0}, \quad j = 2, \cdots, M^t + 1. \tag{2.6}$$

Let $\mathcal{P}_k$ be the space of all polynomials $p_k(t)$ defined on I whose degrees are less than or equal to k and let $\mathcal{P}^h_{N^t_j}$ be the subspace of $C[-1, 1]$ which consists of piecewise polynomials $p_{N^t_j}(t)$ with support $I^t_j = [t_{j-1}, t_j]$ whose degree is less than or equal to N^t_j. For the space $\mathcal{P}^h_{N^t_j}$, we describe two types of Lagrangian basis functions with respect to $\mathcal{G}$, one of which are *internal-Lagrange basis functions* denoted as $\{\phi^t_{j,k}(t)\}_{j=1,k=1}^{M^t,N^t_j-1}$ supported in I^t_j and the other of which are *knot-Lagrange basis functions* denoted as $\{\phi^t_{j,N^t_j}(t)\}_{j=1}^{M^t-1}$ with support on $[t_{j-1}, t_{j+1}]$, and $\phi^t_{1,0}(t)$ and $\phi^t_{M^t,N^t_j}(t)$ with support on $[t_1, t_2]$ and $[t_{M^t-1}, t_{M^t}]$ respectively. For two dimensional high-order space, let

$$[\mathcal{P}^h_N]^2 := \mathcal{P}^h_{N^x_j} \otimes \mathcal{P}^h_{N^y_j}, \tag{2.7}$$

whose basis functions are given by tensor products of one dimensional piecewise Lagrange polynomials. Let $\mathcal{V}_{N^t_j}$ be the space of all piecewise Lagrange linear functions $\hat{\psi}_k(x)$ with respect to $\{\eta_k\}_{k=0}^{N^t_j}$ on I. Define $\mathcal{V}^h_{N^t_j}$ as the space of all piecewise Lagrange linear functions $\{\psi^t_{j,k}(t)\}_{j=1,k=0}^{M^t,N^t_j}$ with respect to $\mathcal{G}$. For two dimensional piecewise linear space, let

$$[\mathcal{V}^h_N]^2 := \mathcal{V}^h_{N^x_j} \otimes \mathcal{V}^h_{N^y_j}, \tag{2.8}$$

whose basis functions are given by tensor products of one dimensional piecewise Lagrange linear functions. Define two interpolation operators $\mathcal{I}_{N^t_j} : C[-1, 1] \to \mathcal{P}_{N^t_j}(I)$ such that

$$(\mathcal{I}_{N^t_j} u)(\eta_k) = u(\eta_k), \quad u \in C[-1, 1] \tag{2.9}$$

and $\mathcal{I}^h_{N^t_j} : C[-1, 1] \to \mathcal{P}^h_{N^t_j}(I)$ such that

$$(\mathcal{I}^h_{N^t_j} v)(\xi^t_{j,k}) = v(\xi^t_{j,k}), \quad v \in C[-1, 1]. \tag{2.10}$$

Define a discrete inner product $\langle u, v\rangle_N$ on $C[-1, 1] \times C[-1, 1]$ as

$$\langle u, v\rangle_N := \sum_{j=1}^{M^t} \sum_{k=0}^{N^t_j-1} u(\xi^t_{j,k})v(\xi^t_{j,k})\rho^t_{j,k} + u(\xi^t_{M^t,N^t_j})v(\xi^t_{M^t,N^t_j})\rho^t_{M^t,N^t_j} \tag{2.11}$$

and its corresponding norm is given by

$$\|u\|_N = \langle u, u\rangle_N^{\frac{1}{2}}, \quad \text{for} \quad u \in C[-1,1]. \tag{2.12}$$

Finally, the notation $a \sim b$ for any two real quantities a and b is meant by that there are two positive constants which do not depend on mesh sizes and degrees of polynomials such that $0 < c \le \frac{a}{b} < C < \infty$. The notation (U, V) stands for $\sum u_i v_i$ for any two vectors $U = (u_1, \cdots, u_d)^T$ and $V = (v_1, \cdots, v_d)^T$ where the superscript T denotes the transpose of a vector. The standard spaces H^1 and L^2 will be used.

3. Basic estimates. In this section, we will discuss some estimates of global interpolation operator $\mathcal{I}^h_{N^t_j}$ in terms of H^1 and L^2 norms. For $t \in [-1,1]$ and $s_t \in [t_{j-1}, t_j]$, let

$$\hat{\phi}(t) := \phi(s_t) = \phi\Big(\frac{h^t_j}{2}t + \frac{1}{2}(t_{j-1}+t_j)\Big). \tag{3.1}$$

Then for $k = 0, 1, \cdots, N^t_j$ we have

$$(\mathcal{I}_{N^t_j}\hat{\phi})(\eta_k) = \hat{\phi}(\eta_k) = \phi\Big(\frac{h^t_j}{2}\eta_k + \frac{1}{2}(t_{j-1}+t_j)\Big) = \phi(\xi^t_{j,k}) = (\mathcal{I}^h_{N^t_j}\phi)(\xi^t_{j,k}), \tag{3.2}$$

which yields

$$(\mathcal{I}_{N^t_j}\hat{\phi})(t) = (\mathcal{I}^h_{N^t_j}\phi)(s_t). \tag{3.3}$$

Also, we have

$$\|\phi(s)\|_1^2 = \sum_{j=1}^{M}\Big[\frac{h^t_j}{2}\int_{-1}^{1}|\hat{\phi}(t)|^2\,dt + \frac{2}{h^t_j}\int_{-1}^{1}|\hat{\phi}'(t)|^2\,dt\Big] \tag{3.4}$$

and

$$\|(\mathcal{I}^h_{N^t_j}\phi)(s)\|_1^2 = \sum_{j=1}^{M}\Big[\frac{h^t_j}{2}\int_{-1}^{1}|(\mathcal{I}_{N^t_j}\hat{\phi})(t)|^2\,dt + \frac{2}{h^t_j}\int_{-1}^{1}|(\mathcal{I}_{N^t_j}\hat{\phi})'(t)|^2\,dt\Big], \tag{3.5}$$

where $\|\cdot\|_1$ denotes the standard Sobolev H^1 norm. From now on, we will use $|\cdot|_1$ as Sobolev H^1 seminorm and $\|\cdot\|$ as usual L^2 norm. In order to discuss the piecewise linear finite elements preconditioner, it may be required to analyze the relations between $\mathcal{I}_{N^t_j}\hat{\phi}$ and $\hat{\phi}$ in the sense of H^1- and L^2- norm. For this purpose, we recall the following lemma (see Lemma 7.2 in [17]) which can be also extended to higher dimensions (see Theorem 7.3 in [17]).

LEMMA 3.1. *It follows that for all* $\hat{\phi} \in \mathcal{V}_{N^t_j}$

$$|\hat{\phi}|_1 \sim |\mathcal{I}_{N^t_j}\hat{\phi}|_1, \quad \textit{and} \quad \|\mathcal{I}_{N^t_j}\hat{\phi}\| \sim \|\hat{\phi}\|. \tag{3.6}$$

Note that the result $|\mathcal{I}_{N^t_j}\hat{\phi}|_1 \le C|\hat{\phi}|_1$ in (3.6) can be verified for any function $\hat{\phi} \in H^1(I)$ by modifying Theorem 1.7, Corollary 1.9 in Chapter II and Corollary 1.16, Theorem 1.19 in Chapter III with usages of Theorem 1.15, Lemma 1.18 and

Proposition 1.17 in Chapter III therein in [1] where $\|\mathcal{I}_{N_j^t}\hat{\phi}^h\|_1 \leq C\|\hat{\phi}^h\|_1$ is found. For reader's convenient, we put the statement here.

Proposition 3.2. *For all $\hat{\phi} \in H^1(I)$, there is a positive constant C such that*

$$|\mathcal{I}_{N_j^t}\hat{\phi}|_1 \leq C\,|\hat{\phi}|_1.$$

Now the extension of Lemma 3.1 to the global interpolation operator $\mathcal{I}^h_{N_j^t}$ can be done easily by combining (3.4), (3.5) with Lemma 3.1. Here we put it as theorem.

Theorem 3.3. *It follows that for all $u \in \mathcal{V}^h_{N_j^t}$*

$$|\mathcal{I}^h_{N_j^t}u|_1 \sim |u|_1. \tag{3.7}$$

Lemma 3.4. *For $\phi(t) \in \mathcal{P}^h_{N_j^t}[-1,1]$, it follows that*

$$\|\phi\| \sim \|\phi\|_N. \tag{3.8}$$

Proof. First note that $\phi(t)$ is a polynomial of degree N_j^t on $[t_{j-1}, t_j]$. Then the equivalence of LGL numerical quadrature (see [1]) and (2.6) yield

$$\begin{aligned}\|\phi(t)\|^2 &= \int_{-1}^{1}|\phi(t)|^2\,dt = \sum_{j=1}^{M^t}\int_{t_{j-1}}^{t_j}|\phi(t)|^2\,dt\\ &\sim \sum_{j=1}^{M^t}\sum_{k=0}^{N_j^t}|\phi(\xi^t_{j,k})|^2\rho^t_{j,k}\\ &= \sum_{j=1}^{M^t}\sum_{k=0}^{N_j^t-1}|\phi(\xi^t_{j,k})|^2\rho^t_{j,k} + |\phi(\xi^t_{M^t,N_j^t})|^2\rho^t_{M^t,N_j^t} + \sum_{j=1}^{M^t-1}|\phi(\xi^t_{j,N_j^t})|^2\rho^t_{j,N_j^t}\\ &\sim \langle\phi,\phi\rangle_N = \|\phi\|_N^2.\end{aligned} \tag{3.9}$$

In the last equivalence in (3.9), the observations (2.6) were used. □

Theorem 3.5. *For all $u \in \mathcal{V}^h_{N_j^t}$, we have*

$$\|u\| \sim \|\mathcal{I}^h_{N_j^t}u\|, \quad \text{and} \quad \|u\| \sim \|\mathcal{I}^h_{N_j^t}u\|_N. \tag{3.10}$$

Proof. Since $\mathcal{I}^h_{N_j^t}u \in \mathcal{P}^h_{N_j^t}[-1,1]$, Lemma 3.4 yields $\|\mathcal{I}^h_{N_j^t}u\| \sim \|\mathcal{I}^h_{N_j^t}u\|_N$. Hence for the proof of (3.10) it is enough to show that for $u \in \mathcal{V}^h_{N_j^t}$

$$\|u\| \sim \|\mathcal{I}^h_{N_j^t}u\|_N. \tag{3.11}$$

Using (3.1) and (3.2) for functions $\hat{u}$ and u, we have

$$\|u\|^2 = \sum_{j=1}^{M^t}\frac{h_j}{2}\|\hat{u}\|^2, \quad \sum_{j=1}^{M^t}\frac{h_j}{2}\|\mathcal{I}_{N_j^t}\hat{u}\|_N^2 = \sum_{j=1}^{M^t}\sum_{k=0}^{N_j^t}\left|(\mathcal{I}^h_{N_j^t}u)(\xi^j_k)\right|^2\rho^j_k. \tag{3.12}$$

Since (see Theorem 3.1 in[11])

$$\|\hat{u}\| \sim \|\mathcal{I}_{N_j^t}\hat{u}\|_N, \quad \text{for all} \quad \hat{u} \in \mathcal{V}_{N_j^t}, \tag{3.13}$$

using (3.12), it follows that

$$\|u\|^2 \sim \sum_{j=1}^{M^t}\sum_{k=0}^{N_j^t}\Big|(\mathcal{I}_{N_j^t}^h u)(\xi_k^j)\Big|^2 \rho_k^j.$$

Therefore, it is enough to show

$$\sum_{j=1}^{M^t}\sum_{k=0}^{N_j^t}\Big|(\mathcal{I}_{N_j^t}^h u)(\xi_k^j)\Big|^2 \rho_k^j \sim \|\mathcal{I}_{N_j^t}^h u\|_N^2. \tag{3.14}$$

Actually, because of (2.6) we can rewrite the left term of (3.14) as

$$\begin{aligned}
\sum_{j=1}^{M^t}\sum_{k=0}^{N_j^t}\Big|(\mathcal{I}_{N_j^t}^h u)(\xi_k^j)\Big|^2 \rho_k^j &= \sum_{j=1}^{M^t}\sum_{k=0}^{N_j^t-1}\Big|(\mathcal{I}_{N_j^t}^h u)(\xi_k^j)\Big|^2 \rho_k^j + \Big|(\mathcal{I}_{N_j^t}^h u)(\xi_k^{M^t})\Big|^2 \rho_k^{M^t} \\
&\quad + \sum_{j=1}^{M^t-1}\Big|(\mathcal{I}_{N_j^t}^h u)(\xi_{N_j^t}^j)\Big|^2 \rho_{N_j^t}^j \\
&= \|\mathcal{I}_{N_j^t}^h u\|_{N_j^t}^2 + \sum_{j=1}^{M^t-1}\Big|(\mathcal{I}_{N_j^t}^h u)(\xi_{N_j^t}^j)\Big|^2 \rho_{N_j^t}^j \\
&= \|\mathcal{I}_{N_j^t}^h u\|_{N_j^t}^2 + \sum_{j=2}^{M^t}\Big|(\mathcal{I}_{N_j^t}^h u)(\xi_0^j)\Big|^2 \rho_0^j.
\end{aligned} \tag{3.15}$$

Then one may see that (3.14) holds. These arguments complete the proof of (3.11) and consequently (3.10). □

4. Case of constant coefficients: $1D$ and $2D$.

First we discuss one dimensional case. For convenience, let $M := M^t$ and $N_j := N_j^t$ for one dimensional case only. Consider two uniformly positive definite elliptic operators defined in $I = (-1, 1)$ such that

$$Lu = -(a_1 u')' + a_2\, u, \quad \text{in} \quad I, \quad u(-1) = u'(1) = 0 \tag{4.1}$$

and

$$Bv = -(b_1 v')' + b_2\, v, \quad \text{in} \quad I, \quad v(-1) = v'(1) = 0 \tag{4.2}$$

where a_1, b_1 are positive constants and a_2, b_2 are nonnegative constants, which lead to two bilinear forms on $\mathbf{V} \times \mathbf{V}$ where $\mathbf{V} := \{u \in H^1(I), \quad u(-1) = u'(1) = 0\}$ as

$$l_1(u, v) = \int_{-1}^{1} a_1 u'v' + a_2 uv\, dt \quad \text{and} \quad b_1(u, v) = \int_{-1}^{1} b_1 u'v' + b_2 uv\, dt. \tag{4.3}$$

For the high-order and piecewise linear approximations to (4.1) and (4.2), let

$$\mathcal{P}_{N_j}^{h,m} := \{v \in \mathcal{P}_{N_j}^h, \quad v(-1) = v'(1) = 0\},$$
$$\mathcal{V}_{N_j}^{h,m} := \{u \in \mathcal{V}_{N_j}^h, \quad u(-1) = u'(1) = 0\}$$

whose suitable basis functions $\{\phi_\mu\}_{\mu=1}^d$ and $\{\psi_\nu\}_{\nu=1}^d$ can be given respectively where

$$d := \dim(\mathcal{P}_{N_j}^{h,m}) = \dim(\mathcal{V}_{N_j}^{h,m}). \tag{4.4}$$

Then the stiffness matrix $\widehat{L}_N$ with high-order elements based on $\mathcal{G}$ of (4.1) is given by

$$\widehat{L}_N(\mu,\nu) = l_1(\phi_\mu,\phi_\nu), \quad \mu,\nu = 1,2,\cdots,d, \tag{4.5}$$

and the stiffness matrix $\widehat{B}_h$ associated with piecewise linear elements based on $\mathcal{G}$ corresponding to (4.2) is given by

$$\widehat{B}_h(\mu,\nu) = b_1(\psi_\mu,\psi_\nu), \quad \mu,\nu = 1,2,\cdots,d. \tag{4.6}$$

Denote $\widehat{M}_N$ and $\widehat{M}_h$ by mass matrices with respect to $\{\phi_\mu\}_{\mu=1}^d$ and $\{\psi_\mu\}_{\mu=1}^d$ respectively, that is, $\mu,\nu = 1,2,\cdots,d$,

$$\widehat{M}_N(\mu,\nu) = (\phi_\mu,\phi_\nu), \quad \widehat{M}_h(\mu,\nu) = (\psi_\mu,\psi_\nu). \tag{4.7}$$

Since all the stiffness and mass matrices are symmetric and positive definite, the preconditioned matrix below also has all positive real eigenvalues.

Theorem 4.1. *For every $U = (u_1,u_2,\cdots,u_d)^T$, we have*

$$(\widehat{B}_h U,U) \sim (\widehat{L}_N U,U), \quad \textit{and} \quad (\widehat{M}_h U,U) \sim (\widehat{M}_N U,U).$$

Hence, the eigenvalues of the preconditioned matrix $\widehat{B}_h^{-1}\widehat{L}_N$ has all positive real eigenvalues $\{\lambda_\mu\}_{\mu=1}^d$ independent of mesh sizes h_j and degrees N_j of polynomials, that is, there is absolute positive constants c and C such that

$$0 < c \le \lambda_\mu \le C < \infty. \tag{4.8}$$

Proof. Let $u(t) \in \mathcal{V}_{N_j}^{h,m}$ be represented as $u(t) = \sum_{\mu=1}^d u_\mu \psi(t)$. Then its piecewise polynomial interpolation can be written as $(\mathcal{I}_{N_j}^h u)(t) = \sum_{\mu=1}^d u_\mu \phi(t)$. The definitions of bilinear forms yield that

$$(\widehat{L}_N U,U) = l_1(\mathcal{I}_{N_j}^h u,\mathcal{I}_{N_j}^h u) \sim \|\mathcal{I}_{N_j}^h u\|_1^2, \quad (\widehat{B}_N U,U) = b_1(u,u) \sim \|u\|_1^2,$$

and

$$(\widehat{M}_N U,U) = (\mathcal{I}_{N_j}^h u,\mathcal{I}_{N_j}^h u) = \|\mathcal{I}_{N_j}^h u\|^2, \quad \text{and} \quad (\widehat{M}_h U,U) = (u,u) = \|u\|^2.$$

Then using Theorem 3.3 and 3.5 completes the proofs. □

For actual computations, the bilinear form $l_1(\mathcal{I}_{N_j}^h u,\mathcal{I}_{N_j}^h v)$ and $(\mathcal{I}_{N_j}^h u,\mathcal{I}_{N_j}^h v)$ will be calculated at LGL points. Define two matrices $\widetilde{L}_N$ and $\widetilde{M}_N$ as

$$\widetilde{L}_N(\mu,\nu) = l_{1,N}(\phi_\mu,\phi_\nu), \quad \widetilde{M}_N(\mu,\nu) = \langle\phi_\mu,\phi_\nu\rangle_N, \tag{4.9}$$

where

$$l_{1,N}(u,v) = a_1\langle u', v'\rangle_N + a_2\langle u,v\rangle_N. \tag{4.10}$$

Note that $\widetilde{M}_N$ is the diagonal matrix which consists of LGL weights, that is

$$\widetilde{M}_N = \mathrm{diag}(\rho^t_{j,k}) \tag{4.11}$$

and the equivalence of numerical quadrature leads to

$$(\widehat{L}_N U, U) \sim (\widetilde{L}_N U, U), \quad \text{and} \quad (\widehat{M}_N U, U) \sim (\widetilde{M}_N U, U) \tag{4.12}$$

and these matrices $\widetilde{L}_N$ and $\widetilde{M}_N$ are symmetric and positive definite.

Corollary 4.2. *The eigenvalues of the preconditioned matrix $\widehat{B}_h^{-1}\widetilde{L}_N$ has all positive real eigenvalues $\{\lambda_\mu\}_{\mu=1}^d$ independent of mesh sizes h_j and degrees N_j of polynomials, that is, there is absolute positive constants c and C such that*

$$0 < c \le \lambda_\mu \le C < \infty. \tag{4.13}$$

Proof. Let $U = (u_1, u_2, \cdots, u_d)^T$ be any nonzero vector. Since

$$\frac{(\widetilde{L}_N U, U)}{(\widehat{B}_h U, U)} = \frac{(\widetilde{L}_N U, U)}{(\widehat{L}_N U, U)}\frac{(\widehat{L}_N U, U)}{(\widehat{B}_h U, U)},$$

using the min-max theorem, Theorem 4.1 and (4.12) we have the conclusion. This argument completes the proof because all involved matrices are symmetric and positive definite. □

We now turn to two dimensional case. For this, we consider the model elliptic operator L such that

$$Lu = -[u_{xx} + u_{yy}] + 2u, \quad u = 0 \quad \text{on} \quad \Gamma_D, \quad \mathbf{n}\cdot\nabla u = 0 \quad \text{on} \quad \Gamma_N, \tag{4.14}$$

where Γ_D is the boundary described in (1.4), which leads to the bilinear form

$$l(u,v) = (\nabla u, \nabla v) + 2(u,v), \quad \text{for} \quad u, v \in H^1_D(\Omega), \tag{4.15}$$

where

$$H^1_D(\Omega) := \{u \in H^1(\Omega) \,|\, u = 0 \quad \text{on} \quad \Gamma_D\}.$$

Let

$$[\mathcal{P}_N^{h,m}]^2 := \mathcal{P}^{h,m}_{N_j^x} \otimes \mathcal{P}^{h,m}_{N_j^y}, \quad [\mathcal{V}_N^{h,m}]^2 := \mathcal{V}^{h,m}_{N_j^x} \otimes \mathcal{V}^{h,m}_{N_j^y}.$$

Let us order the LGL points by horizontal lines and we list all LGL points $\{\Xi_P\}_{P=1}^{d^2}$ as

$$\Xi_P = (\xi_\mu, \xi_\nu), \quad \text{where} \quad P = \mu + d(\nu - 1), \quad \mu, \nu = 1, 2, \cdots, d,$$

where d is defined in (4.4). Accordingly, we order the basis vectors $\Phi_P(x,y) \in [\mathcal{P}_N^{h,m}]^2$ and $\Psi_P(x,y) \in [\mathcal{V}_N^{h,m}]^2$ in the same order. Let $\widehat{L}^s_{N^2}$ and $\widehat{B}^s_{h^2}$ be the stiffness matrices induced by (4.15) on the space $[\mathcal{P}_N^{h,m}]^2$ and $[\mathcal{V}_N^{h,m}]^2$ respectively. From now on, assume that

$$a_i = b_i = 1, \quad i = 1, 2$$

in the operators L_1 and B_1 in (4.1) and (4.2). Then using the one dimensional stiffness matrices $\widehat{L}_{N_j^t}$, $\widehat{B}_{h_j^t}$ and mass matrices $\widehat{M}_{N_j^t}$, $\widehat{M}_{h_j^t}$, we have

$$\widehat{L}^s_{N^2} = \widehat{M}_{N_j^y} \otimes \widehat{L}_{N_j^x} + \widehat{L}_{N_j^y} \otimes \widehat{M}_{N_j^x}, \tag{4.16}$$

$$\widehat{B}^s_{h^2} = \widehat{M}_{h_j^y} \otimes \widehat{B}_{h_j^x} + \widehat{B}_{h_j^y} \otimes \widehat{M}_{h_j^x}. \tag{4.17}$$

Lemma 4.3. *For every vector $U = (u_1, \cdots, u_{d^2})^T$, we have*

$$\Big((\widehat{M}_{N_j^y} \otimes \widehat{L}_{N_j^x})U, U\Big) \sim \Big((\widehat{M}_{h_j^y} \otimes \widehat{B}_{h_j^x})U, U\Big) \tag{4.18}$$

and

$$\Big((\widehat{L}_{N_j^y} \otimes \widehat{M}_{N_j^x})U, U\Big) \sim \Big((\widehat{B}_{h_j^y} \otimes \widehat{M}_{h_j^x})U, U\Big). \tag{4.19}$$

Proof. First note that all the matrices here are symmetric and positive definite. Hence it is enough to estimate (4.18) and (4.19) in terms of eigenvalues. Now because of Theorem 4.1 the conclusions (4.18) and (4.19) can be verified by following Lemma 5.4 in [11]. The details are as follows: Let $U^1 = (u_1, u_2, \cdots, u_d)^T$ and $V^1 = (v_1, v_2, \cdots, v_d)^T$. Theorem 4.1 implies that

$$(\widehat{L}_{N_j^t} U^1, U^1) \sim (\widehat{B}_{h_j^t} U^1, U^1),\ (\widehat{M}_{N_j^t} V^1, V^1) \sim (\widehat{M}_{h_j^t} V^1, V^1), \text{ where } t = x, y. \tag{4.20}$$

Now consider eigenvalue problems

$$\widehat{L}_{N_j^t} U^1 = \kappa \widehat{B}_{h_j^t} U^1, \quad \text{and} \quad \widehat{M}_{N_j^t} V^1 = \lambda \widehat{M}_{h_j^t} V^1. \tag{4.21}$$

From (4.20) we know that κ and λ are uniformly bounded in terms of mesh sizes h_j^x, h_j^y and degrees N_j^x, N_j^y. Note that each in (4.21) has a complete set of eigenvectors U^1_μ and V^1_ν, $\mu, \nu = 1, \cdots, d$. Therefore the vectors and eigenvalues

$$Z_{\mu\nu} = U^1_\mu \otimes V^1_\nu, \quad X_{\mu\nu} = V^1_\nu \otimes U^1_\mu, \quad \Lambda_{\mu\nu} = \kappa_\mu \lambda_\nu$$

are complete set of eigenvectors and eigenvalues of the eigenvalue problem

$$(\widehat{M}_{N_j^y} \otimes \widehat{L}_{N_j^x})U = \Lambda(\widehat{M}_{h^y} \otimes \widehat{B}_{h^x})U, \quad \text{and} \quad (\widehat{L}_{N_j^y} \otimes \widehat{M}_{N_j^x})U = \Lambda(\widehat{B}_{h^y} \otimes \widehat{M}_{h^x})U.$$

Hence one can see the uniform bounds of eigenvalues $\Lambda_{\mu\nu}$ because of the uniform bounds of κ and λ in terms of mesh sizes h_j^x, h_j^y and degrees N_j^x, N_j^y. ∎

Proposition 4.4. *For every $U = (u_1, \cdots, u_{d^2})^T$, it follows that*

$$(\widehat{B}^s_{h^2} U, U) \sim (\widehat{L}^s_{N^2} U, U). \tag{4.22}$$

Hence the eigenvalues of $(\widehat{B}^s_{h^2})^{-1} \widehat{L}^s_{N^2}$ are all positive and bounded. The bounds are independent of the mesh sizes h_j^x, h_j^y and the degrees N_j^x, N_j^y of polynomials.

Proof. It follows from Lemma 4.3 with (4.16) and (4.17). ☐

For actual computations of (4.15), that is, for computations of $\widehat{L}^s_{N^2}$, we use LGL quadrature formula. For this, consider

$$l_N(u,v) = \langle \nabla u, \nabla v\rangle_N + \langle u, v\rangle_N, \tag{4.23}$$

which can be written as, for $u, v \in [\mathcal{P}_N^{h,m}]^2$

$$l_N(u,v) = V^T(\widetilde{M}_{N_j^y} \otimes \widetilde{L}_{N_j^x} + \widetilde{L}_{N_j^y} \otimes \widetilde{M}_{N_j^x})U \tag{4.24}$$

where the vectors $U = (u_1, \cdots, u_{d^2})^T$ and $V = (v_1, \cdots, v_{d^2})^T$ are vector representations of

$$u(x,y) = \sum_{P=1}^{d^2} u_P \Phi_P(x,y) \quad \text{and} \quad v(x,y) = \sum_{P=1}^{d^2} v_P \Phi_P(x,y).$$

Now we will use the matrix $\widehat{B}^s_{h^2}$ in (4.17) as the preconditioner for

$$\widetilde{L}^s_{N^2} U = F \tag{4.25}$$

where

$$\widetilde{L}^s_{N^2} := \widetilde{M}_{N_j^y} \otimes \widetilde{L}_{N_j^x} + \widetilde{L}_{N_j^y} \otimes \widetilde{M}_{N_j^x}. \tag{4.26}$$

Then we can show that the eigenvalues of $(\widehat{B}^s_{h^2})^{-1}\widetilde{L}^s_{N^2}$ are bounded well in terms of mesh sizes h_j^x, h_j^y and degrees N_j^x, N_j^y.

LEMMA 4.5. *For every vector $U = (u_1, \cdots, u_{d^2})^T$, we have*

$$((\widetilde{M}_{N_j^y} \otimes \widetilde{L}_{N_j^x})U, U) \sim ((\widehat{M}_{h_j^y} \otimes \widehat{B}_{h_j^x})U, U) \tag{4.27}$$

and

$$((\widetilde{L}_{N_j^y} \otimes \widetilde{M}_{N_j^x})U, U) \sim ((\widehat{B}_{h_j^y} \otimes \widehat{M}_{h_j^x})U, U). \tag{4.28}$$

Proof. Recall that $A \otimes B$ is symmetric and positive definite if the matrices A and B are symmetric, positive definite. Note that

$$\frac{((\widetilde{M}_{N_j^y} \otimes \widetilde{L}_{N_j^x})U, U)}{((\widehat{M}_{h_j^y} \otimes \widehat{B}_{h_j^x})U, U)} = \frac{((\widetilde{M}_{N_j^y} \otimes \widetilde{L}_{N_j^x})U, U)}{((\widehat{M}_{N_j^y} \otimes \widehat{L}_{N_j^x})U, U)} \frac{((\widehat{M}_{N_j^y} \otimes \widehat{L}_{N_j^x})U, U)}{((\widehat{M}_{h_j^y} \otimes \widehat{B}_{h_j^x})U, U)},$$

and

$$\frac{((\widetilde{L}_{N_j^y} \otimes \widetilde{M}_{N_j^x})U, U)}{((\widehat{B}_{h_j^y} \otimes \widehat{M}_{h_j^x})U, U)} = \frac{((\widetilde{L}_{N_j^y} \otimes \widetilde{M}_{N_j^x})U, U)}{((\widehat{L}_{N_j^y} \otimes \widehat{M}_{N_j^x})U, U)} \frac{((\widehat{L}_{N_j^y} \otimes \widehat{M}_{N_j^x})U, U)}{((\widehat{B}_{h_j^y} \otimes \widehat{M}_{h_j^x})U, U)}.$$

Therefore, due to min-max theorem and Lemma 4.3, it is enough to show that

$$\frac{((\widetilde{M}_{N_j^y} \otimes \widetilde{L}_{N_j^x})U, U)}{((\widehat{M}_{N_j^y} \otimes \widehat{L}_{N_j^x})U, U)}, \quad \text{and} \quad \frac{((\widetilde{L}_{N_j^y} \otimes \widetilde{M}_{N_j^x})U, U)}{((\widehat{L}_{N_j^y} \otimes \widehat{M}_{N_j^x})U, U)} \tag{4.29}$$

are bounded independently of degrees N_j^x, N_j^y and mesh sizes h_j^x, h_j^y. Due to (4.12), this can be done by following the similar arguments of Lemma 4.3. These arguments complete the proof. □

THEOREM 4.6. *For every vector $U = (u_1, \cdots, u_{d^2})^T$, it follows that*

$$(\widehat{B}^s_{h^2}U, U) \sim (\widetilde{L}^s_{N^2}U, U). \tag{4.30}$$

Hence the eigenvalues of $(\hat{B}^s_{h^2})^{-1}\widetilde{L}^s_{N^2}$ are all positive and bounded. The bounds are independent of the mesh sizes h_j^x, h_j^y and the degrees N_j^x N_j^y.

Proof. It comes from (4.26) and (4.17) and Lemma 4.5 □

5. Case of variable coefficients: $2D$.

Consider the bilinear form corresponding to (1.1) and (1.2) as

$$l_p(u, v) = (p(x, y)\nabla u, \nabla v) + (q(x, y)u, v) \quad \text{for} \quad u, v \in H^1_D(\Omega). \tag{5.1}$$

As suggested in [4], by expanding the coefficients $p(x, y)$ and $q(x, y)$ in terms of $2D$ tensor Lagrange basis, we approximate (5.1) on the space $[\mathcal{P}^{h,m}_N]^2$ by

$$l_{p,N}(u, v) = \langle p(x, y)\nabla u, \nabla v\rangle_N + \langle q(x, y)u, v\rangle_N, \quad u, v \in [\mathcal{P}^{h,m}_N]^2. \tag{5.2}$$

With the differentiation matrix D at LGL points (see [4] for example) and diagonal matrices

$$P := \mathrm{diag}(p_\alpha := p_{\mu\nu}),\ Q := \mathrm{diag}(q_\alpha := q_{\mu\nu}), \quad \alpha = \mu + d(\nu - 1), \tag{5.3}$$

the matrix representation of (5.2) used in [4] is

$$\begin{aligned}\widetilde{L}_{N^2} := {}& (E_{N_j^y} \otimes \widetilde{D}^T_{N_j^x})\widetilde{W}^P_N(E_{N_j^y} \otimes \widetilde{D}_{N_j^x}) + (\widetilde{D}^T_{N_j^y} \otimes E_{N_j^x})\widetilde{W}^P_N(\widetilde{D}_{N_j^y} \otimes E_{N_j^x}),\\ &+ W^Q_N\end{aligned} \tag{5.4}$$

where $E_{N_j^t}$ is the identity matrix of order d

$$\widetilde{W}^S_N := S(\widetilde{M}_{N_j^y} \otimes \widetilde{M}_{N_j^x}), \quad \text{where} \quad S = P \,\text{or} \quad Q. \tag{5.5}$$

Note that the matrix $\widetilde{W}^S_N$ is diagonal whose elements are positive because the matrices $S(= P, Q)$ and $(\widetilde{M}_{N_j^y} \otimes \widetilde{M}_{N_j^x})$ are diagonal with positive elements. Further if the matrix P is identity and the matrix Q is $2E$, then the matrix $\widetilde{L}_{N^2}$ is the same as $\widetilde{L}^s_{N^2}$ in (4.26) with

$$\widetilde{L}_{N_j^t} = \widetilde{D}^T_{N_j^t}\widetilde{M}_{N_j^t}\widetilde{D}_{N_j^t}, \quad t = x, y.$$

LEMMA 5.1. *For any vector $U = (u_1, \cdots, u_{d^2})^T$, it follows that*

$$(E_{N_j^y} \otimes \widetilde{D}^T_{N_j^x})\widetilde{W}^P_N(E_{N_j^y} \otimes \widetilde{D}_{N_j^x}) \sim (E_{N_j^y} \otimes \widetilde{D}^T_{N_j^x})(\widetilde{M}_{N_j^y} \otimes \widetilde{M}_{N_j^x})(E_{N_j^y} \otimes \widetilde{D}_{N_j^x}) \tag{5.6}$$

$$(\widetilde{D}^T_{N_j^y} \otimes E_{N_j^x})\widetilde{W}^P_N(\widetilde{D}_{N_j^y} \otimes E_{N_j^x}) \sim (\widetilde{D}^T_{N_j^y} \otimes E_{N_j^x})(\widetilde{M}_{N_j^y} \otimes \widetilde{M}_{N_j^x})(\widetilde{D}_{N_j^y} \otimes E_{N_j^x}) \tag{5.7}$$

$$\widetilde{W}^Q_N \sim 2(\widetilde{M}_{N_j^y} \otimes \widetilde{M}_{N_j^x}), \tag{5.8}$$

where the matrix equivalence $A \sim B$ should be understood as

$$(AU, U) \sim (BU, U).$$

Proof. Note that the variable coefficients $p(x, y)$ and $q(x, y)$ are positive bounded functions and the matrices $\widetilde{W}^S_{N^{ys}}$ and $(\widetilde{M}_{N^y_j} \otimes \widetilde{M}_{N^x_j})$ are diagonal with positive elements. Hence for any vector U it follows immediately that

$$\Big(\widetilde{W}^S_N U, U\Big) \sim \Big((\widetilde{M}_{N^y_j} \otimes \widetilde{M}_{N^x_j})U, U\Big), \quad \text{where} \quad S = P,\, Q. \tag{5.9}$$

This argument complete (5.8). Let $V = (E_{N^y_j} \otimes \widetilde{D}_{N^x_j})U$. Since

$$\Big((E_{N^y_j} \otimes \widetilde{D}^T_{N^x_j})\widetilde{W}^P_N (E_{N^y_j} \otimes \widetilde{D}_{N^x_j})U, U\Big) = \Big(\widetilde{W}^P_N V, V\Big),$$

we have the conclusion (5.6) with help of (5.9). The similar arguments complete (5.7). ∎

The main goal of this section is to show that the eigenvalues of the preconditioned matrix

$$(\widehat{B}^s_{h^2})^{-1} \widetilde{L}_{N^2}$$

are real and bounded as follows.

Theorem 5.2. *For any vector $U = (u_1, \cdots, u_{d^2})^T$, we have*

$$(\widehat{B}^s_{h^2} U, U) \sim (\widetilde{L}_{N^2} U, U). \tag{5.10}$$

In the sense of eigenvalues, it follows that all eigenvalues of the matrix $(\widehat{B}^s_{h^2})^{-1} \widetilde{L}_{N^2}$ are real positive and bounded. The bounds are independent of the mesh sizes h^x_j, h^y_j and the degrees N^x_j, N^y_j of piecewise polynomials.

Proof. Note that

$$(E_{N^y_j} \otimes \widetilde{D}^T_{N^x_j})(\widetilde{M}_{N^y_j} \otimes \widetilde{M}_{N^x_j})(E_{N^y_j} \otimes \widetilde{D}_{N^x_j}) = \widetilde{M}_{N^y_j} \otimes \widetilde{L}_{N^y_j}$$

and

$$(\widetilde{D}^T_{N^y_j} \otimes E_{N^x_j})(\widetilde{M}_{N^y_j} \otimes \widetilde{M}_{N^x_j})(\widetilde{D}_{N^y_j} \otimes E_{N^x_j}) = \widetilde{L}_{N^y_j} \otimes \widetilde{M}_{N^y_j}.$$

Therefore, using Lemma 5.1 one may see that

$$(\widetilde{L}_{N^2} U, U) \sim (\widetilde{L}^s_{N^2} U, U).$$

From Theorem 4.6,

$$(\widetilde{L}^s_{N^2} U, U) \sim (\widehat{B}^s_{h^2} U, U).$$

These arguments complete the proof. ∎

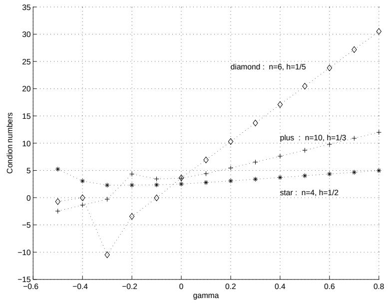

FIGURE 1. Condition numbers of $\widehat{B}_h^{-1}\widetilde{L}_N$ according to γ

6. Computational results. Because there are numerical results reported already in [8], we discuss a few numerical experiments of one dimensional problem. Consider

$$Lu = -(p(x)u')' + q(x)u, \quad x \in (0,1) \tag{6.1}$$

with boundary conditions $u(0) = u(1) = 0$. Let us take the preconditioner operator B as

$$Bu = -u'' + \gamma u, \quad x \in (0,1) \tag{6.2}$$

with same homogeneous Dirichlet boundary conditions where γ will be taken later. In the following numeric tests, the uniform mesh size $h = h_j^x$ and uniform degree $N = N_j^x$ for $j = 1, \cdots, M^x$ are used.

Example 1. With a chosen $p(x) = 1 + x^2$ and $q(x) = e^x$ in (6.1), we discuss the optimal preconditioning operator (6.2) by considering several γ. We will take $\gamma = k \times 0.1$ where k is an integer between -5 and 8. The Figure 1 shows that the condition number of the preconditioned matrix $\widehat{B}_h^{-1}\widetilde{L}_N$ becomes small relatively if we choose γ near at 0 among other γ's. These phenomena were pointed out in [9] if (6.1) is discretized by a finite difference scheme, that is to say, γ may be chosen by considering the advection coefficient in (6.1). Hence one may choose $\gamma = 0$ in this case.

Example 2. Stimulated the numerical results in Example 1, we will take $\gamma = 0$ for the preconditioner operator B while $p(x) = q(x) = 1$ in (6.1) are taken. The condition numbers of the matrix $\hat{L}_N$ are shown in Figure 2 for various mesh sizes and degrees of polynomials. Then we enumerate condition numbers of the preconditioned matrix $\widehat{B}_h^{-1}\widetilde{L}_N$ in Figure 2 also, which shows that the condition numbers can be fixed for mesh sizes and degrees of polynomials. We point out that the condition numbers glow as h^{-2} for a fixed degree of polynomial and $O(N^3)$ for a fixed mesh size. These also can be proven in a standard finite element theory and in spectral methods(see [1]). These phenomena are depicted in Figure 3.

7. Conclusion. As we have shown that the condition numbers of the preconditioned systems are well bounded, it is possible to find multigrid algorithms for solving elliptic boundary value problems with high-order discretizations based on LGL quadrature nodes. As an immediate application, one may apply the techniques here to a first-order system of least-squares for an elliptic problem whose systems are found

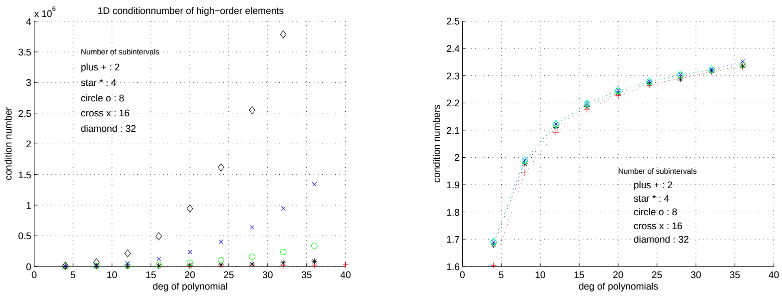

FIGURE 2. Condition numbers of $\widetilde{L}_N$ (left) and $\widehat{B}_h^{-1}\widetilde{L}_N$ (right).

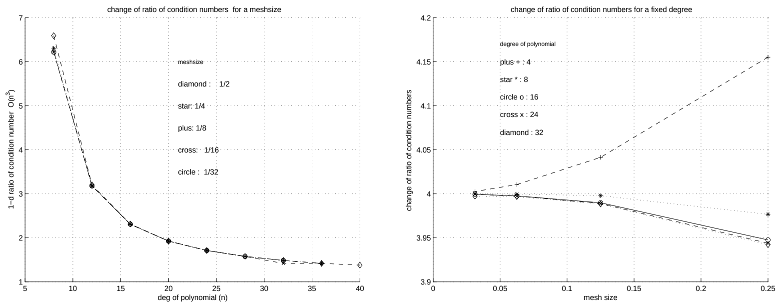

FIGURE 3. Increasing rates of condition numbers of $\widetilde{L}_N$.

in [2] combining the results here and [11] (for example, one may refer to [10]). In order to avoid the irregular nodes of LGL, one may use Chebyshev-Gauss-Lobatto nodes for discretizations which has a nested property. This case will be dealt in a coming paper.

Acknowledgement. The author deeply thank to Prof. Manteuffel for his kind guidance on preparing for the present topic.

REFERENCES

[1] C. Bernardi and Y. Maday, *Approximation Spectrales de Problémes aux Limites Elliptiques*, Springer-Verlag, Paris (1992).

[2] Z. Cai, T. Manteuffel, and S. McCormick, *First-order system least squares for second-order partial differential equations: Part II*, SIAM J. Numer. Anal., 34(1997), pp. 425–454.

[3] M. O. Deville and E. H. Mund, *Finite element preconditioning for pseudospectral solutions of elliptic problems*, SIAM J. Sci. Stat. Comput., 11(1990), pp. 311–342.

[4] M. O. Deville, P. F. Fischer E. H. Mund, *High-order Methods for Incompressible Fluid Flow*, Cambridge Monographs on Applied and Computational Mathematics, Cambridge Univ. Press, Cambridge, UK, 2002.

[5] M. Dryja, B. Smith, O. Widlund, *Schwarz analysis of iterative substructuring algorithms for elliptic problems in three dimensions*, SIAM J. Numer. Anal. 31(1994), pp 1662-1694.

[6] P. Fischer, *An overlapping Schwarz method for spectral element solution of the incompreesible Navier-Stokes equations*, J. Comput. Phys. 133 (1997), pp 84-101.

[7] P. Fisher and E. Ronquist Spectral element methods for large-scale parallel Navier-Stokes calculations, Comput. Mehtod Appl. M. 116 (1994) pp 69-76.

[8] J. Heys, T. Manteuffel, S. McCormick and L. Olson, *Algebraic Multigrid(AMG) for High-Order Finite Elements*, J. Comput. Phys. 204(2005) pp 520-532.

[9] T. Manteuffel and J. Otto, *Optimal equivalent preconditioners*, SIAM J. Numer. Anal., 30(1993), pp. 790–812.

[10] S. D. Kim, H.-C. Lee and B. C. Shin, *Pseudospectral least-squares method for the second-order elliptic boundary value problem*, SIAM J. Numer. Anal., 41(2003), pp. 1370-1387.

[11] S. Parter and E. Rothman, *Preconditioning Legendre spectral collocation approximations to elliptic problems*, SIAM J. Numer. Anal., 32(1995), pp. 333–385.

[12] S. Parter, *Preconditioning Legendre spectral collocation methods for elliptic problems I: Finte element operators*, SIAM J. Numer. Anal., 39, No. 1, (2001) pp. 348-362

[13] L. Pavarino and O. Widlund, *A polylogorithmic bound for an iterative substituting method for spectral element in three dimensions*, SIAM J. Numer. Anal. 33 (1996), pp. 1303-1335

[14] A. Quarteroni and E. Zampieri, *Finite element preconditioning for Legendre spectral collocation approximations to elliptic equations and systems*, SIAM J. Numer. Anal., 29(1992), pp. 917–936.

[15] J. Ruge. AMG for high-order discretiztions of second-order elliptic problems in: Abstract of the Eleventh Copper Mountain Conference on Multigrid Methods(http://www.mgnet.org/mgnet/Conference /CopperMtn03/Talks/ruge.pdf), 2003.

[16] A. St-Cyr, M. Gander and S. Thomas, *On optimized Schwarz Preconditioning for High-order Spectral Methods*, Twelfth copper mountain conference on multigrid methods (2005)

[17] A. Toselli and O. Widlund, *Domain Decomposition Methods- Algorithms and Theory*, Springer Series in Computational Mathematics, Springer-Verlag, Berlin Heidelberg (2005).