First-Order Least-Squares for the Distributed Control by the Stokes Equation

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The structure of flow control problem

1.Objective functional

- A reason why one wants to control the flow

ex) flow matching, drag minimization, preventing separation, preventing transition to turbulence, deterring temperature variations.

2. controls or design parameters

- Boundary value controls
- Distributed controls
- Shape controls

3. Constraints

§1. Optimization problem for the Stokes system

$\Omega \subset \mathbb{R}^n (n = 2 \text{ or } 3)$: Consider

$$\text{(1)} \qquad \min \mathcal{J}(\mathbf{u}, \mathbf{f}) = \frac{\sigma_1}{2} \int_\Omega |\mathbf{u} - \hat{\mathbf{u}}|^2 \, dx + \frac{\sigma_2}{2} \int_\Omega |\mathbf{f}|^2 \, dx$$

subject to the Stokes equation

$$\text{(2)} \qquad \begin{cases} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in} \quad \Omega \\ \nabla \cdot \mathbf{u} = 0 & \text{in} \quad \Omega \\ \mathbf{u} = \mathbf{0} & \text{on} \quad \partial\Omega \\ \int_\Omega p \, dx = 0, & \end{cases}$$

Lagrangian functional:

$$\text{(3)} \quad \mathcal{L}(\mathbf{u}, \mathbf{v}, \mathbf{f}, p, q) = \mathcal{J}(\mathbf{u}, \mathbf{f}) + \int_\Omega \nu (\Delta \mathbf{u}) \cdot \mathbf{v} - \nabla p \cdot \mathbf{v} + \mathbf{f} \cdot \mathbf{v} \, dx - \int_\Omega \nabla \cdot \mathbf{u} q \, dx$$

The optimality system:

$$
(4) \qquad \begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in} \quad \Omega \\ \nu\Delta\mathbf{v} + \nabla q = \sigma_1(\hat{\mathbf{u}} - \mathbf{u}) & \text{in} \quad \Omega \\ \nabla \cdot \mathbf{u} = 0 & \text{in} \quad \Omega \\ \nabla \cdot \mathbf{v} = 0 & \text{in} \quad \Omega \\ -\sigma_2\mathbf{f} = \mathbf{v} & \text{in} \quad \Omega \end{cases}
$$

and the problem is : Find $(\mathbf{u}, \mathbf{v}, \mathbf{f}, p, q) \in \mathbf{H}_0^1(\Omega)^3 \times L_0^2(\Omega)^2$ satisfies (4), where

$$
L_0^2(\Omega) := \left\{ q \in L^2(\Omega) : \int_\Omega q \; dx \equiv 0 \right\} \text{ and}
$$

$$
H_0^1(\Omega)^n := \{\mathbf{v} = (v_1, \ldots, v_n) : v_i \in H_0^1(\Omega), i = 1, \ldots, n.\}.
$$

Let $\underline{\mathbf{U}} = \nabla \mathbf{u}^t = (\nabla u_1, \nabla u_2, \ldots, \nabla u_n)$, and $\underline{\mathbf{V}} = \nabla \mathbf{v}^t = (\nabla v_1, \nabla v_2, \ldots, \nabla v_n)$, the system (4) may be

(5)
$$\begin{cases} -\nu(\nabla \cdot \underline{\mathbf{U}})^t + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \underline{\mathbf{U}} - \nabla \mathbf{u}^t = \underline{\mathbf{0}} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma, \end{cases}$$

(6)
$$\begin{cases} \nu(\nabla \cdot \underline{\mathbf{V}})^t + \nabla q = \sigma_1(\hat{\mathbf{u}} - \mathbf{u}) & \text{in } \Omega, \\ \underline{\mathbf{V}} - \nabla \mathbf{v}^t = \underline{\mathbf{0}} & \text{in } \Omega, \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = \mathbf{0} & \text{on } \Gamma, \end{cases}$$

(7)
$$\mathbf{f} = -\frac{1}{\sigma_2}\mathbf{v}.$$

From definition of $\mathbf{U}$, the continuity condition $\nabla \cdot \mathbf{u}$, and the Dirichlet boundary condition $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$, we obtain the equivalent extended optimal system for (4):

$$
\left\{
\begin{aligned}
-\nu(\nabla \cdot \underline{\mathbf{U}})^t + \nabla p &= -\frac{\mathbf{v}}{\sigma_2} && \text{in } \Omega \\
\underline{\mathbf{U}} - \nabla \mathbf{u}^t &= \underline{\mathbf{0}} && \text{in } \Omega \\
\nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega \\
\nabla \times \underline{\mathbf{U}} &= \mathbf{0} && \text{in } \Omega \\
\nabla(\mathrm{tr}\underline{\mathbf{U}}) &= \mathbf{0} && \text{in } \Omega \\
\mathbf{u} &= \mathbf{0} && \text{on } \Gamma \\
\mathbf{n} \times \underline{\mathbf{U}} &= \mathbf{0} && \text{on } \Gamma,
\end{aligned}
\right. \tag{8}
$$

$$
(9) \qquad \begin{cases} \nu(\nabla \cdot \underline{\mathbf{V}})^t + \nabla q = \sigma_1(\hat{\mathbf{u}} - \mathbf{u}) & \text{in } \Omega \\ \underline{\mathbf{V}} - \nabla \mathbf{v}^t = \underline{\mathbf{0}} & \text{in } \Omega \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega \\ \nabla \times \underline{\mathbf{V}} = \mathbf{0} & \text{in } \Omega \\ \nabla(\mathrm{tr}\underline{\mathbf{V}}) = \mathbf{0} & \text{in } \Omega \\ \mathbf{v} = \mathbf{0} & \text{on } \Gamma \\ \mathbf{n} \times \underline{\mathbf{V}} = \mathbf{0} & \text{on } \Gamma, \end{cases}
$$

§2. First order system least squares

(10)

$$G_1(\underline{\mathbf{U}}, \mathbf{u}, p, \underline{\mathbf{V}}, \mathbf{v}, q; \hat{\mathbf{u}}) = \|\nu(\nabla \cdot \underline{\mathbf{U}})^t - \nabla p - \frac{\mathbf{v}}{\sigma_2}\|_{-1}^2 + \|\underline{\mathbf{U}} - \nabla \mathbf{u}^t\|^2 + \|\nabla \cdot \mathbf{u}\|^2 + \|\nu(\nabla \cdot \underline{\mathbf{V}})^t + \nabla q + \sigma_1(\mathbf{u} - \hat{\mathbf{u}})\|_{-1}^2 + \|\underline{\mathbf{V}} - \nabla \mathbf{v}^t\|^2 + \|\nabla \cdot \mathbf{v}\|^2.$$

(11)

$$G_2(\underline{\mathbf{U}}, \mathbf{u}, p, \underline{\mathbf{V}}, \mathbf{v}, q; \hat{\mathbf{u}}) = \|\nu(\nabla \cdot \underline{\mathbf{U}})^t - \nabla p - \frac{\mathbf{v}}{\sigma_2}\|^2 + \|\underline{\mathbf{U}} - \nabla \mathbf{u}^t\|^2 + \|\nabla \cdot \mathbf{u}\|^2 + \|\nabla \times \underline{\mathbf{U}}\|^2 + \|\nabla \mathrm{tr}(\underline{\mathbf{U}})\|^2 + \|\nu(\nabla \cdot \underline{\mathbf{V}})^t + \nabla q + \sigma_1(\mathbf{u} - \hat{\mathbf{u}})\|^2 + \|\underline{\mathbf{V}} - \nabla \mathbf{v}^t\|^2 + \|\nabla \cdot \mathbf{v}\|^2 + \|\nabla \times \underline{\mathbf{V}}\|^2 + \|\nabla \mathrm{tr}(\underline{\mathbf{V}})\|^2.$$

Theorem 1. *There are two constant* C_1 *and* C_2 *dependent on* $\sigma_1,\ \sigma_2$ *and* ν *such that for any* $(\underline{\mathbf{U}}, \mathbf{u}, p, \underline{\mathbf{V}}, \mathbf{v}, q) \in \mathcal{W}_1 \times \mathcal{W}_1,$ *we have*

$$C_1\ M_1(\underline{\mathbf{U}}, \mathbf{u}, p, \underline{\mathbf{V}}, \mathbf{v}, q) \leq G_1(\underline{\mathbf{U}}, \mathbf{u}, p, \underline{\mathbf{V}}, \mathbf{v}, q; \mathbf{0}) \tag{12}$$

and

$$G_1(\underline{\mathbf{U}}, \mathbf{u}, p, \underline{\mathbf{V}}, \mathbf{v}, q; \mathbf{0}) \leq C_2\ M_1(\underline{\mathbf{U}}, \mathbf{u}, p, \underline{\mathbf{V}}, \mathbf{v}, q), \tag{13}$$

where

$$M_1(\underline{\mathbf{U}}, \mathbf{u}, p, \underline{\mathbf{V}}, \mathbf{v}, q) = \|\underline{\mathbf{U}}\|^2 + \|\mathbf{u}\|_1^2 + \|p\|^2 + \|\underline{\mathbf{V}}\|^2 + \|\mathbf{v}\|_1^2 + \|q\|^2.$$

§Appendix A. Regularity estimates.

(14)

$$\begin{cases} -\Delta\mathbf{u} + \nabla p/\nu + \dfrac{\mathbf{v}}{\sigma_2\nu} = \mathbf{0} & \text{in} \quad \Omega \\ \Delta\mathbf{v} + \nabla q/\nu + \dfrac{\sigma_1}{\nu}\mathbf{u} = \dfrac{\sigma_1}{\nu}\hat{\mathbf{u}} & \text{in} \quad \Omega \\ \nabla\cdot\mathbf{u} = 0 & \text{in} \quad \Omega \\ \nabla\cdot\mathbf{v} = 0 & \text{in} \quad \Omega \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{u} = 0 & \text{on} \quad \partial\Omega \\ \mathbf{v} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We use ADN theory to prove that (14) applies when Ω has $C^{1,1}$ boundary. Let $l = \{l_{ij}\}$, for $1 \le i, j \le 2n+2$ and $B = \{B_{\mu j}\}$, for $1 \le \mu \le 2n, 1 \le j \le 2n+2$ denote the differential operator and boundary operator corresponding to (14). For example, $n = 3$,

(15)

$$l = \begin{bmatrix} -\Delta & 0 & 0 & \frac{1}{\sigma_2\mu} & 0 & 0 & \partial_1 & 0 \\ 0 & -\Delta & 0 & 0 & \frac{1}{\sigma_2\mu} & 0 & \partial_2 & 0 \\ 0 & 0 & -\Delta & 0 & 0 & \frac{1}{\sigma_2\mu} & \partial_3 & 0 \\ \frac{\sigma_1}{\nu} & 0 & 0 & \Delta & 0 & 0 & 0 & \partial_1 \\ 0 & \frac{\sigma_1}{\nu} & 0 & 0 & \Delta & 0 & 0 & \partial_2 \\ 0 & 0 & \frac{\sigma_1}{\nu} & 0 & 0 & \Delta & 0 & \partial_3 \\ \partial_1 & \partial_2 & \partial_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_1 & \partial_2 & \partial_3 & 0 & 0 \end{bmatrix}, U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \\ \frac{p}{\nu} \\ \frac{q}{\nu} \end{bmatrix}, F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\sigma_1}{\nu}\hat{u_1} \\ \frac{\sigma_1}{\nu}\hat{u_2} \\ \frac{\sigma_1}{\nu}\hat{u_3} \\ 0 \\ 0 \end{bmatrix} \text{ and}$$

(16)

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3)$ and $\hat{\mathbf{u}} = (\hat{u_1}, \hat{u_2}, \hat{u_3})$.

Theorem 2. *Assume that the system (14) is uniformly elliptic(and in 2D satisfies the Supplementary Condition) and assume that the boundary conditions satisfy the Complementing Condition. Furthermore, assume that for some* $k \geq 0$, $U \in H^{k+t_j}(\Omega)^{2n+2}$, $F \in H^{k-s_i}(\Omega)^{2n+2}$, *and* $G \in H^{k-r_\mu-1/2}(\Gamma)^{2n}$. *Then, there exist a constant* $C(\nu, \sigma_1, \sigma_2, k, \Omega) > 0$ *such that*

(17)

$$\sum_{j=1}^{2n+2} \|U_j\|_{k+t_j,\Omega} \leq C\Big(\sum_{i=1}^{2n+2} \|F_i\|_{k-s_i,\Omega} + \sum_{\mu=1}^{2n} \|G_h\|_{k-r_\mu-1/2,\Gamma} + \sum_{j=1}^{2n+2} \|U_j\|_{0,\Omega}\Big),$$

where $s_i = 0$ $(1 \leq i \leq 2n)$, $s_{2n+1} = s_{2n+2} = -1$, $t_j = 2$ $(1 \leq j \leq 2n)$, $t_{2n+1} = t_{2n+2} = 1$ *and* $r_\mu = -2$ $(1 \leq \mu \leq 2n)$. *Moreover, if the problem* (14) *has a unique solution, then the* L^2 *norm on the right-hand side of* (17) *can be omitted.*

Proposition 3. *Suppose that* Ω *has* $C^{1,1}$ *boundary. Then, for* $\mathbf{u}, \mathbf{v} \in H^2(\Omega)^n$ *and* $p, q \in H^1(\Omega)$*, we have the* H^2 *regularity estimate*

$$
\begin{aligned}
&\|\mathbf{u}\|_2 + \|\mathbf{v}\|_2 + \|p\|_1 + \|q\|_1 \\
&\qquad \leq C(\nu, \sigma_1, \sigma_2, \Omega)\Big\{\|\nu\Delta\mathbf{u} - \nabla p - \frac{\mathbf{v}}{\sigma_2}\| + \|\nu\Delta\mathbf{v} + \nabla q + \sigma_1\mathbf{u}\| \\
&\qquad\qquad + \|\nabla\cdot\mathbf{u}\|_1 + \|\nabla\cdot\mathbf{v}\|_1\Big\}.
\end{aligned} \tag{18}
$$

Theorem 4. *Assume that the domain* Ω *is a bounded convex polyhedron or has* $C^{1,1}$ *boundary, the regularity bound* (18) *holds. Then there are two constants* C_1 *and* C_2 *dependent on* σ_1, σ_2 *and* ν *such that for any* $(\underline{\mathbf{U}}, \mathbf{u}, p, \underline{\mathbf{V}}, \mathbf{v}, q) \in \mathcal{W}_2 \times \mathcal{W}_2$*, we have*

$$C_1 \; M_2(\underline{\mathbf{U}}, \mathbf{u}, p, \underline{\mathbf{V}}, \mathbf{v}, q) \leq G_2(\underline{\mathbf{U}}, \mathbf{u}, p, \underline{\mathbf{V}}, \mathbf{v}, q; \mathbf{0}), \tag{19}$$

and

$$G_2(\underline{\mathbf{U}}, \mathbf{u}, p, \underline{\mathbf{V}}, \mathbf{v}, q; \mathbf{0}) \leq C_2 \; M_2(\underline{\mathbf{U}}, \mathbf{u}, p, \underline{\mathbf{V}}, \mathbf{v}, q), \tag{20}$$

where

$$M_2(\underline{\mathbf{U}}, \mathbf{u}, p, \underline{\mathbf{V}}, \mathbf{v}, q) = \|\underline{\mathbf{U}}\|_1^2 + \|\mathbf{u}\|_1^2 + \|p\|_1^2 + \|\underline{\mathbf{V}}\|_1^2 + \|\mathbf{v}\|_1^2 + \|q\|_1^2$$