Shortest Path for Disc Obstacles with Rational Radii

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Joint work with:

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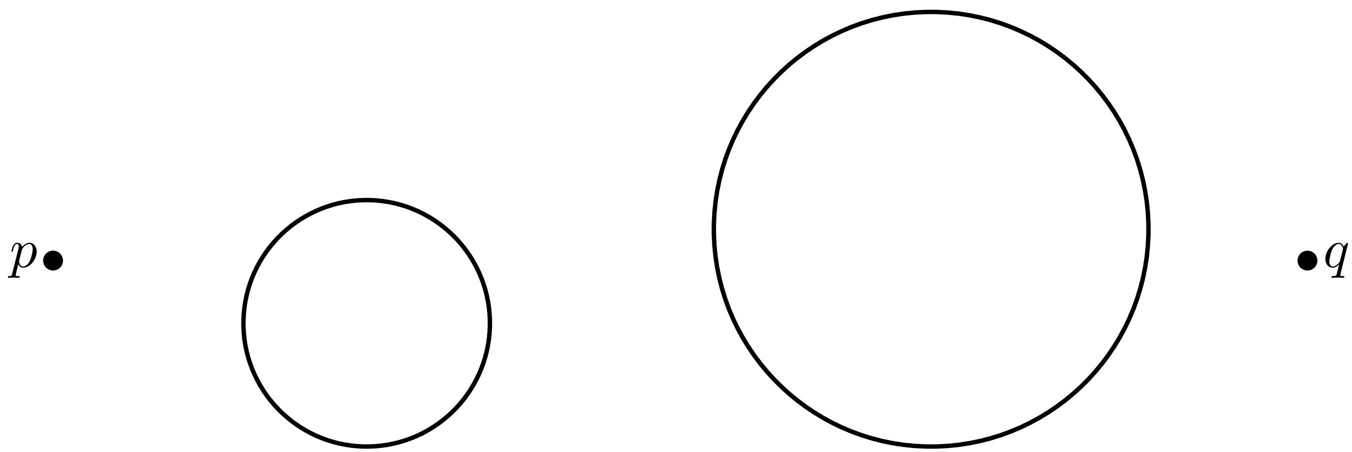
Given $p, q \in \mathbb{R}^2$ & discs $C_1, \cdots, C_n$:

Determine exactly the shortest path from p to q avoiding C_i's.



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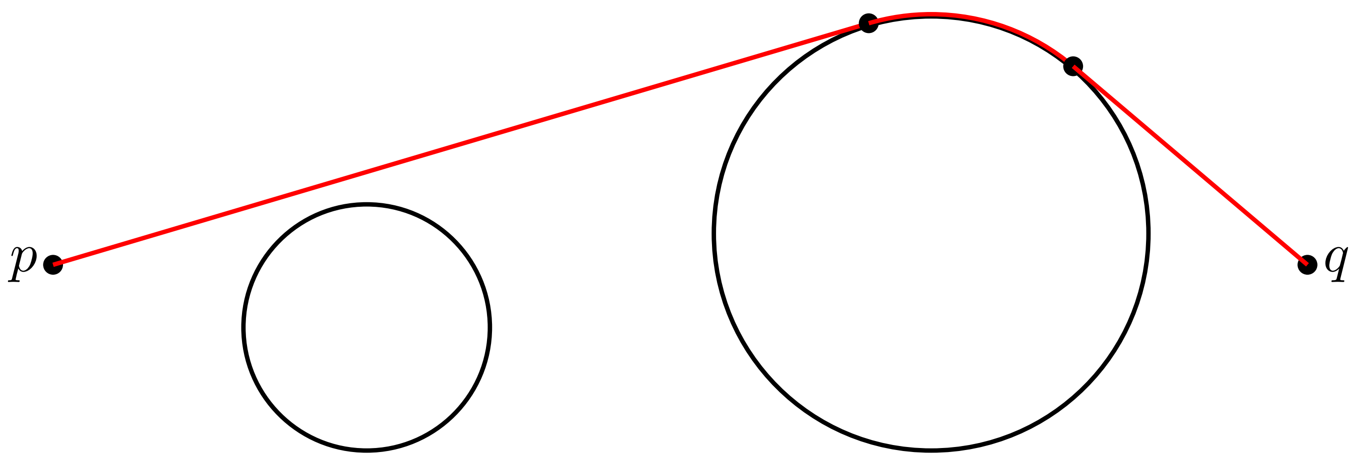

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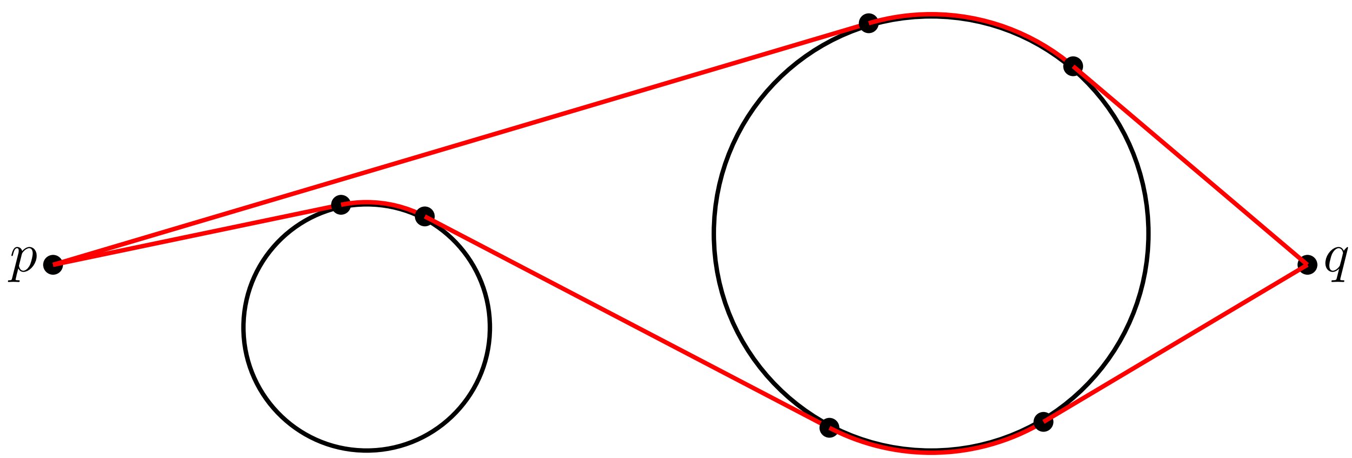

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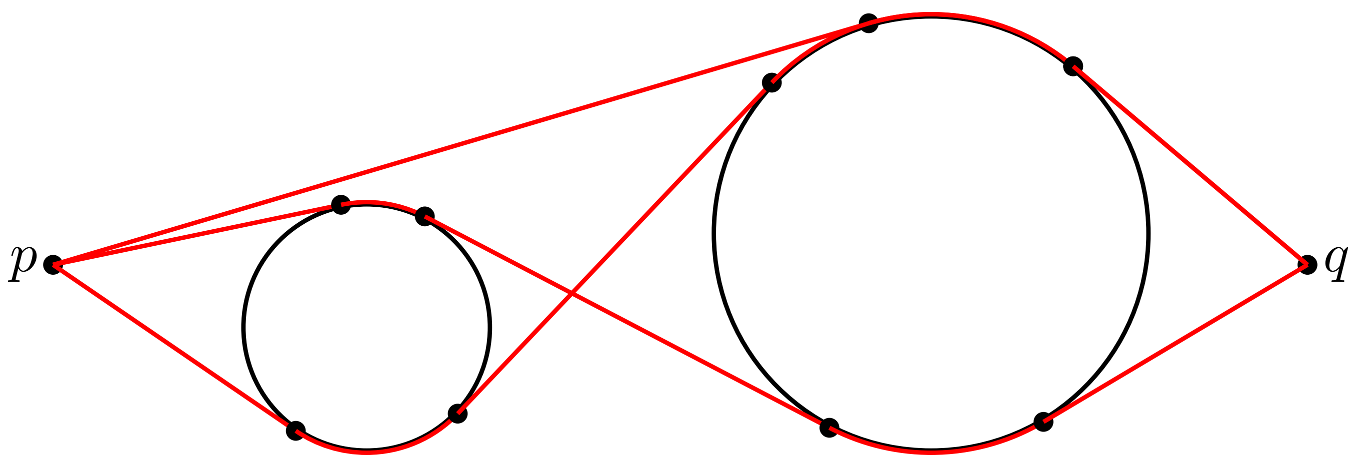

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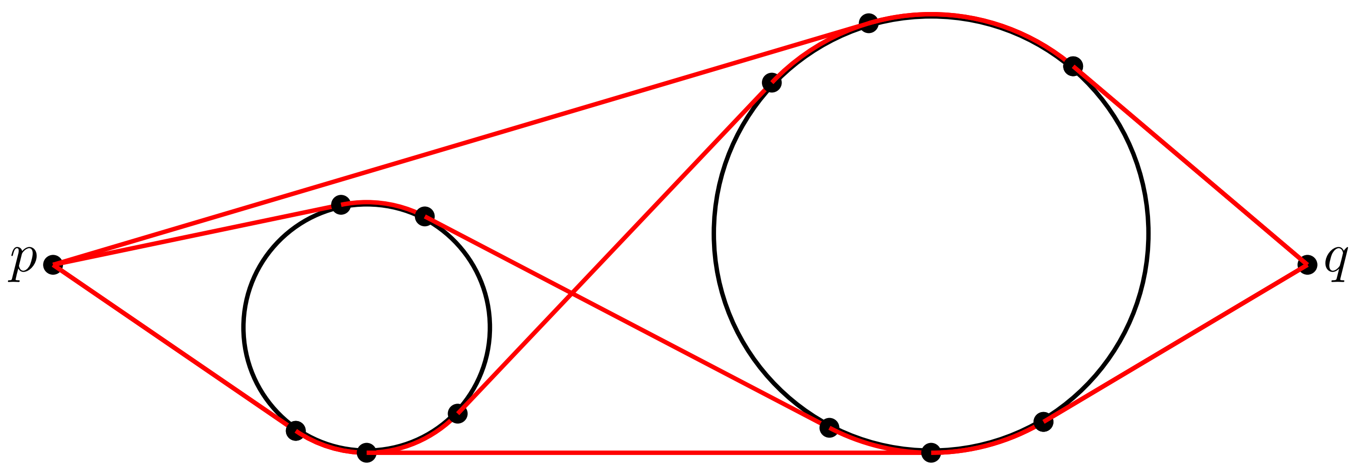

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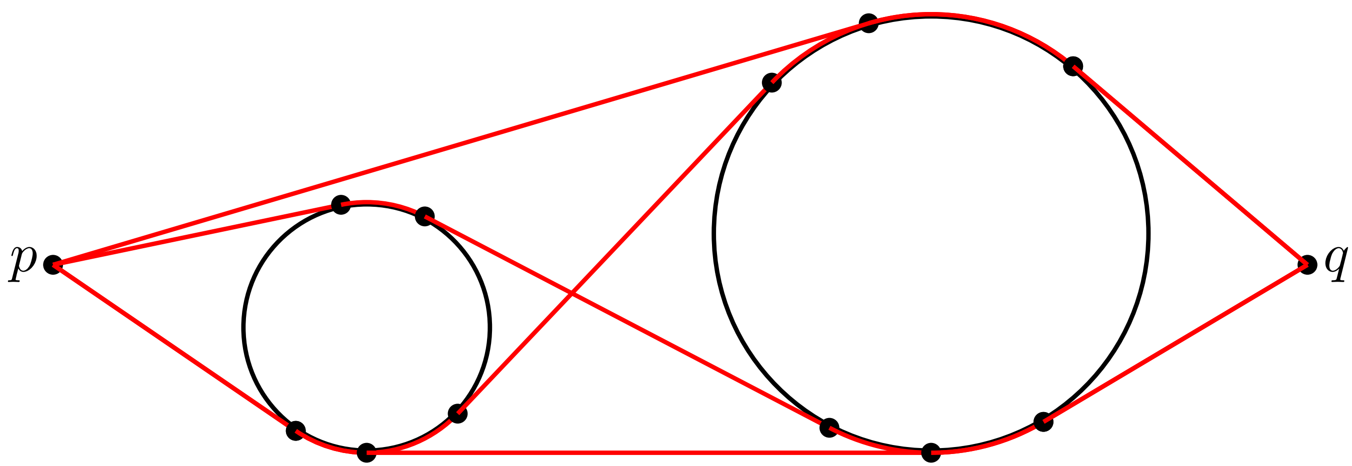

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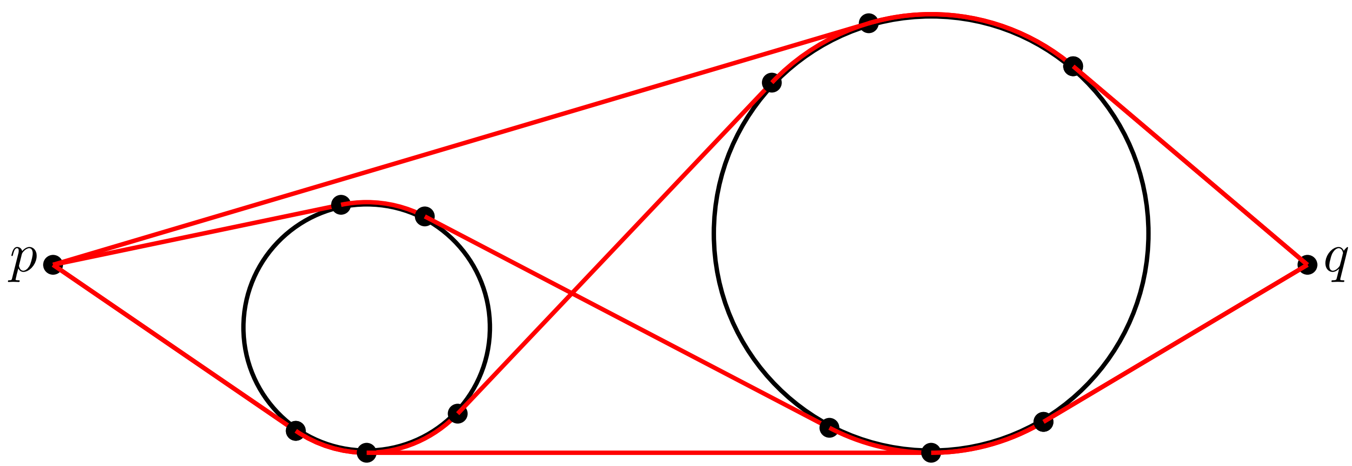

- Seemingly a typical problem in computational geometry – *feasible paths*.
- Non-algebraic, but turns out to be decidable.

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Constructive Root Bounds:

- General Form: For $0 \neq \alpha \in \mathbb{C}$, $p(x) \in \mathbb{Z}[x]$,
  $p(\alpha) = 0 \Rightarrow |\alpha| > F(p)$, F: *effective*.
- Possible to determine whether a given *algebraic number* is zero or not, from *finite number* of digits. (# digits can be determined a priori.) $\Rightarrow$ Bit Complexity
- Not known for general transcendental numbers.

Algebraic Problems

Definition

- $\alpha \in \mathbb{C}$ is *algebraic*, if $p(\alpha) = 0$ for some nontrivial $p \in \mathbb{Z}[x]$.
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Algebraic Problems: Decidable through the zero problem of an *algebraic* number, given *algebraic inputs*.

E.g. Given a line $l : ax + by + c = 0$ and a circle $C : (x - d)^2 + (y - e)^2 = r^2$ with algebraic inputs a, b, c, d, e, r, determine the relation between them.
$\Rightarrow$ Compute the discriminant which is *algebraic*.
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Our problem turns out to be one of the first (nontrivial) nonalgebraic example amenable to EGC approach.

Exact Geometric Computation

- Most successful approach to numerical non-robustness
- Tools: constructive root bounds, digraph representation, etc.
- Combines symbolic manipulation and numerical computation: fast and exact
- Softwares: LEDA, CGAL, Core Library
- Challenge: non-algebraic problems

Overall Approach

- Find Feasible Paths: $\mu = \mu_1; \mu_2; \cdots; \mu_k$
  - Alternating between line segment and circular arc
  - Boundary points are *algebraic*.
  - Sum up the lengths of $\begin{cases} \text{line segments:} & \sqrt{(\cdot - \cdot)^2 + (\cdot - \cdot)^2} \\ \text{circular arcs:} & r \cdot \theta \end{cases}$

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  - Compute a combinatorial (weighted) graph $G = (V, E)$, where vertices V: discs & edges E: joining two discs.
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Input of the Problem

Input:

- Two end points: $p = (p^x, p^y)$, $q = (q^x, q^y)$
- Centers: $c_n = (c_n^x, c_n^y)$ for $1 \leq n \leq N$
- Radii: r_n for $1 \leq n \leq N$

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Type:

- In general: algebraic inputs
- In our case: assume r_n's are rationally related, i.e., $r_n/r_m \in \mathbb{Q}, \forall m, n$.
  - Relatively easy solution
  - Includes important special case: all inputs are rational
- L-bit rational number: P/Q, where P, Q are L-bit integers. ($|P|, |Q| < 2^L$)

Length of Feasible Path

$$d(\mu) = \sum_i d(\mu_i) = \sum_i \alpha_i + \sum_j \theta_j r_j$$

- $\sum \alpha_i$: length of line segments $\Rightarrow$ algebraic
- $\sum \theta_j r_j$: length of circular arcs
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Commensurable Radii

Lemma. If the radii r_i are *rationally related* (or *commensurable*), then the difference of two feasible paths can be *systematically* transformed into the form:

$$\alpha + r\theta,$$

where α, r, $\cos\theta$ are algebraic and *computable* from the input.

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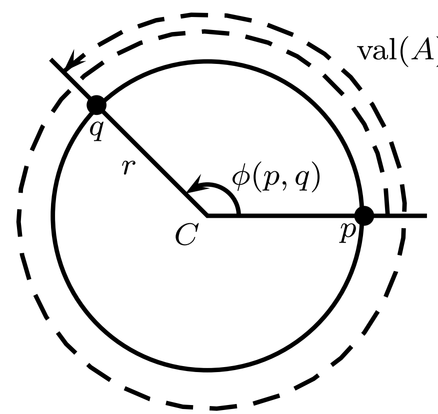
But How? $\Rightarrow$ Sum up the arclengths into an arclength on *one* disc.

Directed Arc: $A = [C, p, q, n]$

- C: disc (radius r & center)
- p, q: the start and the end point
- $\mathrm{val}(A) = r\theta$ and $\theta = 2n\pi + \phi(p, q)$, where $-\pi < \phi(p, q) < \pi$ is the directed angle of the arc from p to q.



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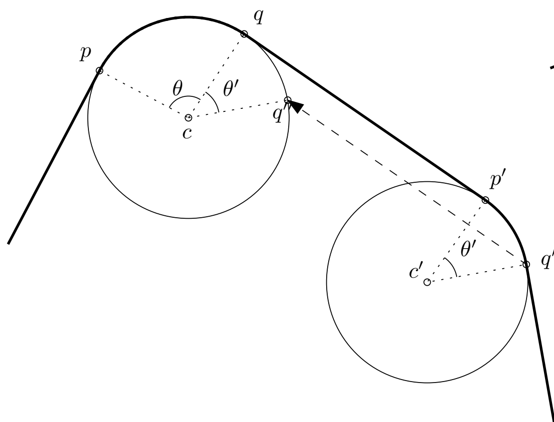
Operations:

- Normalization
- Negation
- Addition and Subtraction
- Scalar multiplication



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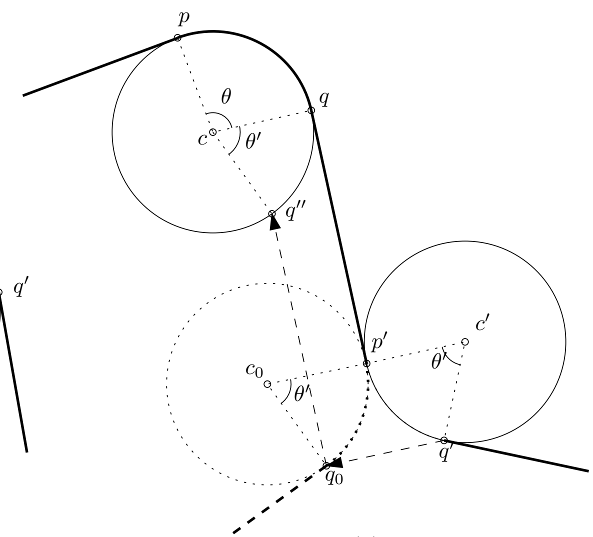

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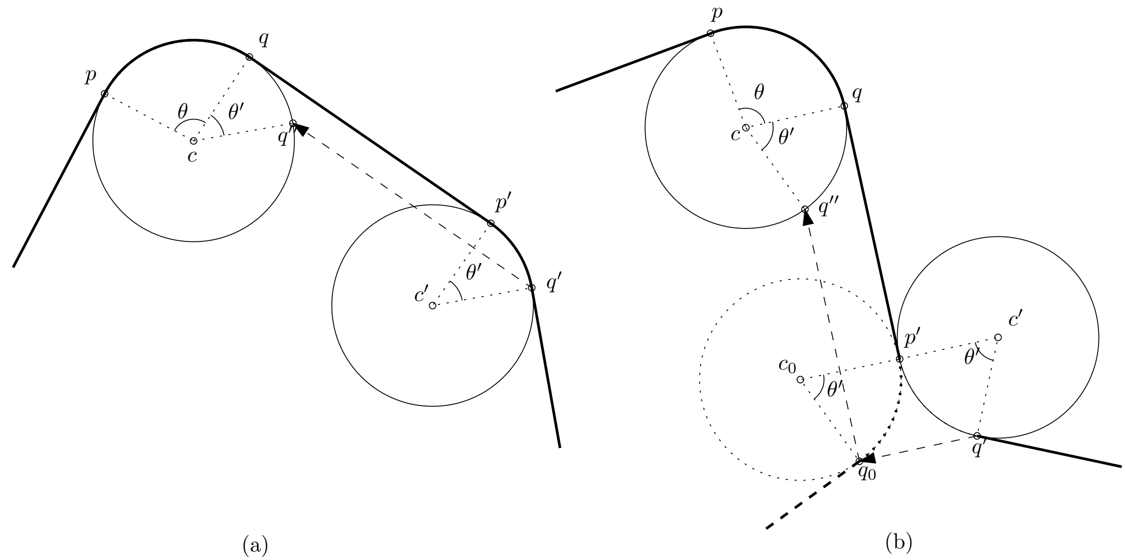

We have:

- The results of normalization, negation, addition and subtraction is algebraic for any algebraic input.
- The result of scalar multiplication is algebraic, if the the radii are commensurable.
- Can be shown using *Chebyshev polynomial* of the first kind $T_n(x)$:
  - $T_0(x) = 1$, $T_1(x) = x$, and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$, $n \geq 1$
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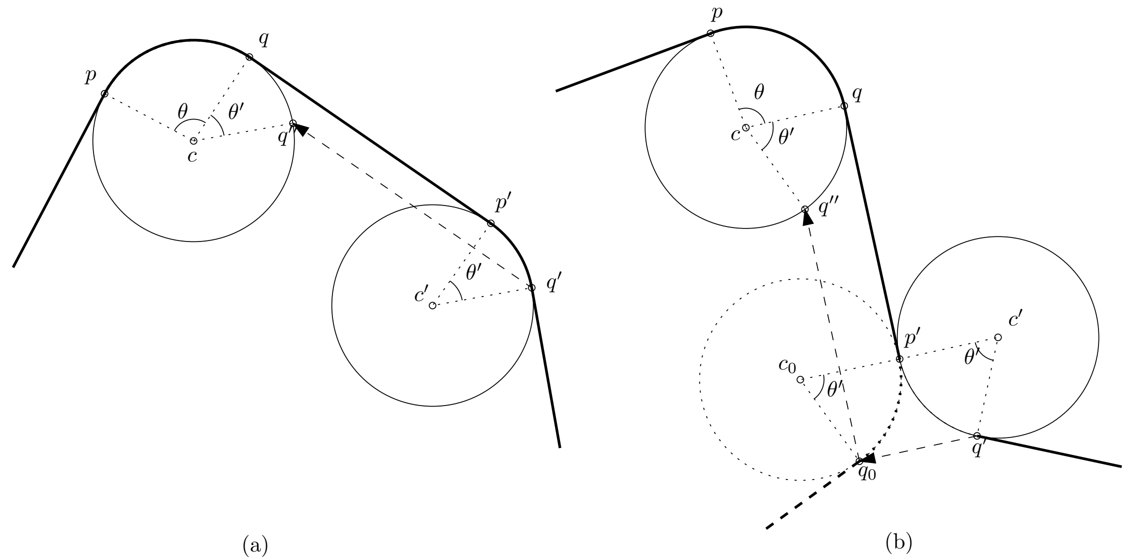

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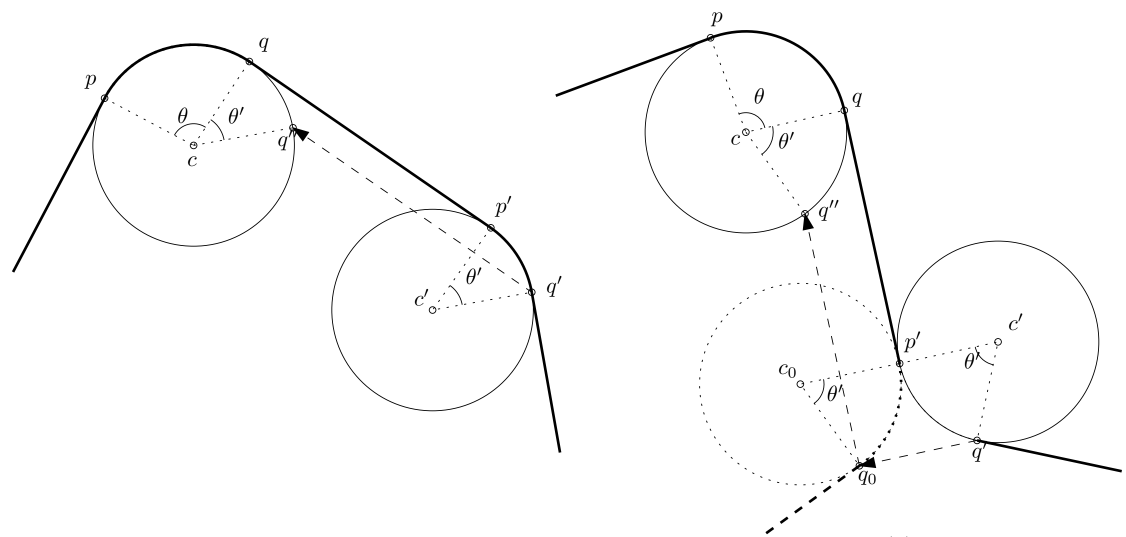

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How about the *bit complexity*? – How many digits are needed to compare the lengths of two feasible paths? $\Rightarrow$ Need a lower bound for $|\alpha + r\theta|$ for algebraic $\alpha, r, \cos\theta$.

Effective Bound from Transcendental Number Theory

Theorem. (Waldschmidt) Let $\alpha, \beta \in \mathbb{C}$ be nonzero algebraic numbers, and let $\log\beta$ be any determination of the logarithm of β. Assume

$$D \geq [\mathbb{Q}(\alpha,\beta):\mathbb{Q}], \qquad V \geq \max\{h(\beta), |\log\beta|/D, 1/D\},$$
$$1 < E \leq \min\{e^{DV}, 4DV/|\log\beta|\}, \qquad V^+ = \max\{V, 1\}.$$

Then we have

$$|\alpha + \log\beta| > \exp\{-2^{35}D^3V(h(\alpha)+\log(EDV^+))(\log(ED))(\log E)^{-2}\}.$$

Definition. $\alpha \in \mathbb{C}$: algebraic & $p(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$: its *minimal polynomial*

- Degree: $\deg(\alpha) := \deg(p) = n$
- Absolute logarithmic height: $h(\alpha) := \frac{1}{\deg(\alpha)} \log M(\alpha)$
- Mahler measure: $M(\alpha) := |a_n| \prod_{i=1}^{n} \max\{1, |\alpha_i|\}$, where $\alpha_1, \cdots, \alpha_n$ are all the conjugates of α.

Bit Complexity

- Assume the input is L-bit rational numbers, and N is the number of discs.
- View $\beta = e^{i\theta}$ and $\alpha \to i\alpha \Rightarrow |\alpha + \log\beta| \to |i\alpha' + i\theta| = |\alpha + r\theta|$ (can assume $r = 1$).

Corollary. Let α, $\theta \in \mathbb{C}$ be such that $\alpha, \cos\theta$ are nonzero algebraic numbers. Then

$$|\alpha + \theta| > \exp\{-2^{35} D^3 V (h(\alpha) + \log(EDV^+))(\log(ED))(\log E)^{-2}\},$$

where

$$\begin{aligned} D &\geq [\mathbb{Q}(i\alpha, e^{i\theta}) : \mathbb{Q}], & V &\geq \max\{h(e^{i\theta}), |\theta|/D, 1/D\}, \\ 1 < E &\leq \min\{e^{DV}, 4DV/|\theta|\} & V^+ &= \max\{V, 1\}. \end{aligned}$$

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$\Rightarrow$ Single Exponential in L and N!

Finally...

Remarks:

- Rare example of transcendental number theory giving positive computational result
  - c.f. Lyapunov's work on theoretical mechanics
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Thanks!