Exponential Splines

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Cardianl Exponential Splines

$L_{\vec{\alpha}} = \prod_{n=1}^{N}(D - \alpha_n I)$: differential operator ($\vec{\alpha} = (\alpha_1, \cdots, \alpha_N)$)

$\delta(t)$: Dirac function

$j = \sqrt{-1}$

Definition. An exponential spline with parameter $\vec{\alpha}$ is a function $s(t)$ such that

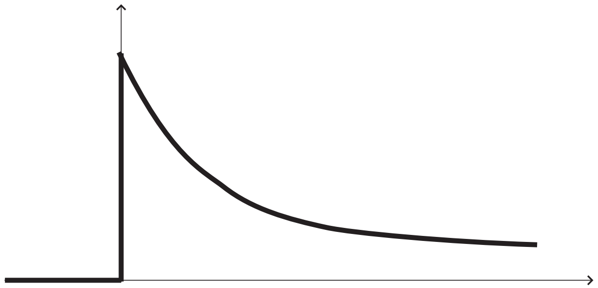
$$L_{\vec{\alpha}}\{s(t)\} = \sum_{k\in\mathbb{Z}} a[k]\delta(t-k).$$

$$\rho_{\vec{\alpha}}(t)\text{: Green function of } L_{\vec{\alpha}} \Leftrightarrow L_{\vec{\alpha}}\{\rho_{\vec{\alpha}}(t)\} = \delta(t)$$

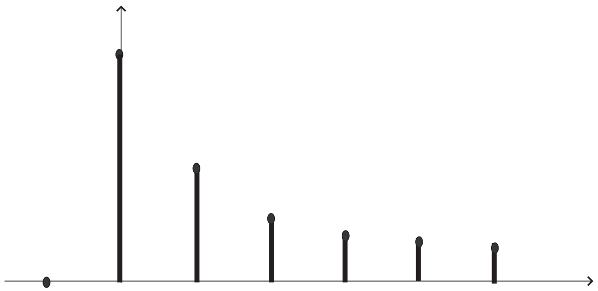
Remark. $\rho_{\vec{\alpha}}(t) = \rho_{\alpha_1} * \cdots * \rho_{\alpha_N}(t)$

Example of Exponential Spline

$$\rho_\alpha(t) = 1_+(t)e^{\alpha t} \rightarrow \ \mathcal{F} \ \rightarrow \hat{\rho}_\alpha(w) = \tfrac{1}{jw-\alpha}$$

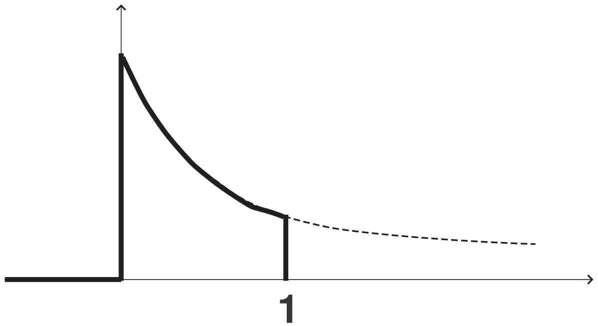


$$p_\alpha(k) = \begin{cases} e^{\alpha k}, & k > 0 \\ 0, & k < 0. \end{cases} \quad \rightarrow \ z \ \rightarrow P_\alpha(z) = \tfrac{1}{1-e^\alpha z^{-1}} = 1 + e^\alpha z^{-1} + \cdots$$



Example of Exponential Spline (Cont'd)

$$\hat{\beta}_\alpha(w) = \frac{\hat{\rho}_\alpha(w)}{p_\alpha(e^{jw})} = \frac{1 - e^{\alpha}e^{-jw}}{jw - \alpha} \longrightarrow \mathcal{F}^{-1} \longrightarrow \beta_\alpha(t) = \rho_\alpha(t) - e^{\alpha} \cdot \rho_\alpha(t-1)$$



- Reproduction formula

$$\rho_\alpha(t) = \sum_{k=0}^{\infty} e^{\alpha k}\beta_\alpha(t-k) = \sum_{k=0}^{\infty} p_\alpha[k]\beta_\alpha(t-k)$$

Exponential B-splines

- $\beta_{\vec{\alpha}}(t) = (\beta_{\alpha_1} * \cdots * \beta_{\alpha_N})(t)$
- $\triangle_{\vec{\alpha}}\{f(t)\} = \sum_{k=0}^{N} d_{\vec{\alpha}}[k] f(t-k)$ where $d_{\vec{\alpha}}[k]$ is characterized by its z-transform
$$\triangle_{\vec{\alpha}}(z) = \prod_{n=1}^{N}(1 - e^{\alpha_n} z^{-1})$$
- $\beta_{\vec{\alpha}}(t) = \triangle_{\vec{\alpha}}\{\rho_{\vec{\alpha}}(t)\} = \sum_{k=0}^{N} d_{\vec{\alpha}}[k]\rho_{\vec{\alpha}}(t-k)$
- $\hat{\beta}_{\vec{\alpha}}(w) = \frac{\triangle_{\vec{\alpha}}(e^{jw})}{L_{\vec{\alpha}}(jw)}$
- Properties
  - Piecewise exponential
  - Compact support $[0, N]$
  - Smoothness C^{N-2}

Reproduction Formula

$$\rho_{\vec{\alpha}}(t) = \sum_{k \in \mathbb{Z}} p_{\vec{\alpha}}[k] \beta_{\vec{\alpha}}(t-k) (= \rho_{\alpha_1} * \cdots * \rho_{\alpha_N}(t))$$

with

$$P_{\vec{\alpha}}(z) = \triangle_{\vec{\alpha}}^{-1}(z) = \prod_{n=1}^{N} \frac{1}{(1 - e^{\alpha_n} z^{-1})}$$

Theorem

$\{\beta_{\vec{\alpha}}(t-k)\}_{k \in \mathbb{Z}}$ is a Riesz basis for $V_{\vec{\alpha}} = \{f :$ cardinal e-spline$\} \cap L^2$ iff $\alpha_n - \alpha_m \neq j2\pi k$, $k \in \mathbb{Z}$ for all pair of distinct, purely imaginary roots.

B-spline property

- $\beta_{\vec{\alpha_1}} * \beta_{\vec{\alpha_2}}(t) = \beta_{(\vec{\alpha_1}:\vec{\alpha_2})}(t)$
- $\beta_{-\vec{\alpha}}(t) = \left(\prod_{n=1}^{N} e^{-\alpha_n} \right) \beta_{\vec{\alpha}}(-t+N)$
  - Example

    $\beta_{(0,\alpha,-\alpha)}(t+t_0) = \beta_{(0,\alpha,-\alpha)}(-t+t_0)$ with $t_0 = 3/2$

Interpolation

- Given discrete signal $x[k]$
  choose spline $s(t) = \sum_{k\in\mathbb{Z}} c[k]\beta_{\vec{\alpha}}(t-k)$ s.t. $s[k] = x[k]$
- Let $b_{\vec{\alpha}}(k) = \beta_{\vec{\alpha}}(k)$ and $B_{\vec{\alpha}}(z) = \sum_{k=0}^{N} b_{\vec{\alpha}}(k)z^{-k}$.
  Since $x[k] = (b_{\vec{\alpha}} * c)[k]$, $X(z) = B_{\vec{\alpha}}(z)C(z)$.
  Thus $C(z) = \frac{1}{B_{\vec{\alpha}}(z)}X(z)$

$$x[k] \longrightarrow \frac{1}{B_{\vec{\alpha}}(z)} \longrightarrow c[k]$$

Recursive IIR filter

Convolution

- Input
  $s_1(t) = \sum_{k \in \mathbb{Z}} c_1[k] \beta_{\vec{\alpha}_1}(t-k)$, $s_2(t) = \sum_{k \in \mathbb{Z}} c_2[k] \beta_{\vec{\alpha}_2}(t-k)$
- B-spline convolution property
  $\beta_{\vec{\alpha}_1} * \beta_{\vec{\alpha}_2}(t) = \beta_{(\vec{\alpha}_1 : \vec{\alpha}_2)}(t)$
- Convolution
  $(s_1 * s_2)(t) = \sum_{k \in \mathbb{Z}} (c_1 * c_2)[k] \beta_{(\vec{\alpha}_1 : \vec{\alpha}_2)}(t-k)$

Differentiation

- B-spline differentials

$$L_{\vec{\alpha}_1}\{\beta_{(\vec{\alpha}_1:\vec{\alpha}_2)}(t)\} = \triangle_{\vec{\alpha}_1}\{\beta_{\vec{\alpha}_2}(t)\}$$

$$\begin{aligned} L_{\vec{\alpha}_1}\{s_{(\vec{\alpha}_1:\vec{\alpha}_2)}(t)\} &= L_{\vec{\alpha}_1}\left\{\sum_k c[k]\beta_{(\vec{\alpha}_1:\vec{\alpha}_2)}(t-k)\right\} \\ &= \sum_k c[k]\triangle_{\vec{\alpha}_1}\{\beta_{\vec{\alpha}_2}(t-k)\} \\ &= \sum_k (c * d_{\vec{\alpha}_1})[k]\beta_{\vec{\alpha}_2}(t-k) \end{aligned}$$

$$s[k] \longrightarrow \frac{1}{B_{\vec{\alpha}}(z)} \longrightarrow \triangle_{\vec{\alpha}_1}(z) \longrightarrow B_{\vec{\alpha}_2}(z)$$