Riesz Wavelet Bases

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Contents

Fourier Transform

$$\hat{f}(\xi) := \int f(t)e^{-i\xi t}\, dt$$

- frequency : perfect
- time-localization : N/A

In many application, we are interested in time-frequency localization.

WFT

- WFT : Windowed Fourier Transform

$$G_b f(\xi) := \int f(t) e^{-i\xi t} g(t-b) dt$$

- $G_b f(\xi)$ can be interpreted loosely as the content of f near time b and near frequency ξ
- drawback : $e^{-i\xi t} g(t-b)$, regardless of the value ξ, have the same length

Notice : High frequency is very narrow, while low frequency is much broader.

Wavelet Transform

As a result, the wavelet transform is better able than the windowed Fourier Transform to "zoom in" on very short-lived high frequency phenomena.

Discrete signal processing and compression

- Decomposition

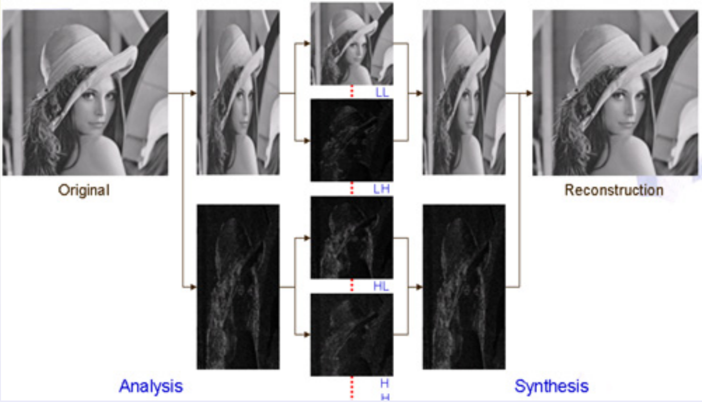
$$[S_0 := \{s_k\}] \rightarrow [S_{-1} : D_{-1}] \rightarrow \cdots \rightarrow [S_{-n} : D_{-n} : \cdots : D_{-1}]$$

Algorithm
$$S_{-k} = (\downarrow 2)h(-) * S_{-k+1}$$
$$D_{-k} = (\downarrow 2)g(-) * S_{-k+1}$$

- Reconstruction

$$[S_0 := \{s_k\}] \leftarrow [S_{-1} : D_{-1}] \leftarrow \cdots \leftarrow [S_{-n} : D_{-n} : \cdots : D_{-1}]$$

Algorithm $S_{-k+1} = \tilde{h} * (\uparrow 2)S_{-k} + \tilde{g} * (\uparrow 2)d_{-k}$


Original
LL
LH
HL
H
Reconstruction
Analysis
Synthesis

Perfect Reconstruction

Let

$$h(\xi) = \frac{1}{2} h_k e^{-ik\xi}.$$

To be a perfect reconstruction,

$$h(\xi)\overline{\tilde{h}(\xi)} + h(\xi+\pi)\overline{\tilde{h}(\xi+\pi)} = 1$$
$$h(\xi)\overline{\tilde{g}(\xi)} + h(\xi+\pi)\overline{\tilde{g}(\xi+\pi)} = 0$$
$$g(\xi)\overline{\tilde{g}(\xi)} + g(\xi+\pi)\overline{\tilde{g}(\xi+\pi)} = 1$$

Question How to design such a beautiful four sequence $h, g, \tilde{h}, \tilde{g}$?

Subdivision Scheme

$\gamma := \{\gamma(j)\}_{j\in\mathbb{Z}}$: a control point.
How to create nice curve from the control point γ ?

Subdivision Scheme

$$[S_a\gamma](i) = 2\sum_{j\in\mathbb{Z}} a(i-2j)\gamma(j)$$

$$\begin{aligned} \text{even stencil} \quad &: \quad [S_a\gamma](2i) = 2\textstyle\sum_{j\in\mathbb{Z}} a(2j)\gamma(i-j) \\ \text{odd stencel} \quad &: \quad [S_a\gamma](2i+1) = 2\textstyle\sum_{j\in\mathbb{Z}} a(2j+1)\gamma(i-j) \end{aligned}$$

Subdivision Scheme

Creating curve

- Construct $S\gamma \to S^2\gamma \to S^3\gamma \cdots$
- Create function $Q^n h$ corresponding to $S^n\gamma$

$$
\begin{array}{ccl}
S_a^2\gamma & \to & Q^2h(x) := \sum_{i\in\mathbb{Z}} S_a^2\gamma(i)h(2x-i) \\
S_a^3\gamma & \to & Q^3h(x) := \sum_{i\in\mathbb{Z}} S_a^3\gamma(i)h(2^2x-i) \\
\vdots & & \vdots \\
S_a^n\gamma & \to & Q^nh(x) := \sum_{i\in\mathbb{Z}} S_a^n\gamma(i)h(2^nx-i)
\end{array}
$$

where

$$Qf(x) = 2\sum a(i)f(2x-i).$$

Define

$$S_a^\infty\gamma := \lim_{n\to\infty} Q^n h \text{ in some sense.}$$

Subdivision Scheme

Assume that

$$\phi = S_a^{\infty}\delta = \lim_{n\to\infty} Q^n h \text{ in some sense.}$$

Actually, it can be easily verified that

- $\phi(x) = 2\sum_{i\in\mathbb{Z}} a(i)\phi(2x-i)$
- $S_a^{\infty}\gamma = \sum_{i\in\mathbb{Z}} \gamma(i)\phi(x-i)$

What is a Wavelet?

We call ψ is a **(Riesz) wavelet** in $L_2(\mathbb{R})$ if

- Span$\{\psi_{j,k} := 2^{k/2}\psi(2^k \cdot -j) \mid j,k \in \mathbb{Z}\}$: dense in $L_2(\mathbb{R})$
- $\exists$ $\mathbf{A}$, $\mathbf{B} > 0$ such that

$$\mathbf{A}\|c\|_{\ell_2} \leq \left\| \sum_{j,k} c_{j,k}\psi_{j,k} \right\|_{L_2(\mathbb{R})} \leq \mathbf{B}\|c\|_{\ell_2} \quad \forall\, c \in \ell_2.$$

Then, we have

$$f = \sum_{j,k} \underbrace{< f, \psi_{j,k} >}_{digitalization} \psi_{j,k}, \quad \forall f \in L^2(\mathbb{R}),$$

Definition

The shifts of ϕ in $L^p(\mathbb{R})$ are said to be *stable* if

$$C_1\|\alpha\|_p \leq \|\sum_k \alpha(k)\phi(\cdot - k)\|_p \leq C_2\|\alpha\|_p, \quad \forall\alpha \in l_0(\mathbb{Z})$$

Definition

ϕ is said to have o.n.shifts if

$$< \phi, \phi(\cdot - i) >= \delta_{i0}$$

Desirable propeties

- compactness
- orthonormality
- symmetry(or antisymmetry)
- smoothness(or high vanishing moment)

MRA

Definition

$\{V_j : j \in \mathbb{Z}\}$:= an **MRA** with a scaling function ϕ if

1. $0 \swarrow \cdots \subset V_j \subset V_{j+1} \cdots \nearrow L^2(\mathbb{R})$;
2. $D(V_j) = V_{j+1}$, where $D(V_j) := \{2^{1/2} f(2x) | f \in V_j\}$,;
3. $\{\phi(\cdot - k) | k \in \mathbb{Z}\}$ is an orthonormal(Riesz) basis for V_0,

Scaling function

Theorem

Assume that $\exists\ \phi$ *such that*

$$\phi = 2\sum_{i\in\mathbb{Z}} \phi(2\dot{-}i).$$

Define $V_j := \{2^{j/2}\phi(2^j - i) : i \in \mathbb{Z}\}$. *Then, with mild condition, we can construct MRA* $\{V_j\}_{j\in\mathbb{Z}}$.

O.N. MRA based wavelet function

Theorem

Assume ϕ generates o.n. MRA. Plus mild conditon
Then, we can obtain a o.n.wavelet ψ.

$\exists\ a$ satisfying

$$\phi(x) = \sum a(k)\phi(2x - k)$$

Define

$$\psi(x) := \sum (-1)^k a(1 - k)\phi(2x - k)$$

Then, such a ψ is a wavelet function.

I.Daubechies' O.N. wavelet family

- Two-scale equation

$$\phi(x) = \sum a(k)\phi(2x-k)$$

Let $\hat{a}(\xi) = \frac{1}{2}\sum a(k)e^{-i\xi}$

- $|\hat{a}(\xi)|^2 + |\hat{a}(\xi+\pi)|^2 = 1$
- $\hat{a}(\xi) = (\frac{1+e^{-i\xi}}{2})^m p(\xi)$

Example $m = 2$

$$a(0) = \frac{1+\sqrt{3}}{4}, a(1) = \frac{3+\sqrt{3}}{4}, a(2) = \frac{3-\sqrt{3}}{4}, a(3) = \frac{1-\sqrt{3}}{4}$$

$$\hat{a}(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^2 \left((1+\sqrt{3}) + (1-\sqrt{3})e^{-i\xi}\right)$$

Note : a is not symmetric $\Rightarrow$ ϕ is not symmetric

Biorthogoanl MRA

- Motivation : We cannot obtain a wavelet which is (anti)-symmetric and orthogonl exept Harr wavelet.

Definition

$\phi, \tilde{\phi}$: biorthogonal if

$$< \phi(\cdot), \tilde{\phi}(\cdot - k) >= \delta_{0,k}$$

Theorem

Let

$$\phi(x) = \sum a(k)\phi(2x-k)$$
$$\tilde{\phi}(x) = \sum b(k)\tilde{\phi}(2x-k)$$

Suppose both ϕ *and* $\tilde{\phi}$ *construct a MRA and biorthogonal. Define*

$$\psi(x) := \sum(-1)^k b(1-k)\phi(2x-k)$$
$$\tilde{\psi}(x) := \sum(-1)^k a(1-k)\tilde{\phi}(2x-k)$$

Then, $f = \sum < f, \hat{\psi}_{j,k} > \psi_{j,k}, \quad \forall f \in L^2$

MRA biorthogonal wavelet

(1) Find $a, \tilde{a}$ and $b, \tilde{b}$ satisfying

$$\begin{bmatrix} \hat{a}(\xi) & \hat{a}(\xi+\pi) \\ \hat{b}(\xi) & \hat{b}(\xi+\pi) \end{bmatrix} \begin{bmatrix} \hat{\tilde{a}}(\xi) & \hat{\tilde{a}}(\xi+\pi) \\ \hat{\tilde{b}}(\xi) & \hat{\tilde{b}}(\xi+\pi) \end{bmatrix}^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(2) some additional conditions on the filters

Then, we obtain $\phi, \psi, \tilde{\phi}, \tilde{\psi} \in L_2(\mathbb{R})$ such that

$$\begin{aligned} \hat{\phi}(2\xi) &= \hat{a}(\xi)\hat{\phi}(\xi) & \hat{\psi}(2\xi) &= \hat{b}(\xi)\hat{\phi}(\xi) \\ \hat{\tilde{\phi}}(2\xi) &= \hat{\tilde{a}}(\xi)\hat{\tilde{\psi}}(\xi) & \hat{\tilde{\psi}}(2\xi) &= \hat{\tilde{b}}(\xi)\hat{\tilde{\psi}}(\xi) \end{aligned}$$

and,

$$\begin{aligned} \left\langle \phi, \tilde{\phi}(\cdot - j) \right\rangle &= \delta_{0,j}, & \left\langle \phi, \tilde{\psi}(\cdot - j) \right\rangle &= 0, \\ \left\langle \tilde{\phi}, \psi(\cdot - j) \right\rangle &= 0, & \left\langle \psi, \tilde{\psi}(\cdot - j) \right\rangle &= \delta_{0,j}. \end{aligned}$$

CDF's Method

Assume that a is given.

(1) Choose $\tilde{a}$ satisfying

$$\hat{a}(\xi)\hat{\tilde{a}}(\xi)^* + \hat{a}(\xi+\pi)\hat{\tilde{a}}(\xi+\pi)^* = 1.$$

(2) Put

$$\hat{b}(\xi) = e^{-i\xi}\hat{\tilde{a}}(\xi+\pi)^*, \quad \hat{\tilde{b}}(\xi) = e^{-i\xi}\hat{\tilde{a}}(\xi+\pi)^*$$

Then, under mild conditions,

$$\exists \text{ Riesz wavelet } \psi, \tilde{\psi} \in L_2(\mathbb{R}).$$

Issues

1. Existence of Refinable function ϕ on L_p-spaces : p-Joint spectral radius
2. Multiwavelet
3. Non-stationary and Non-uniform subdivision scheme

Thanks for your attentions !!