Talk in Busan:

Bivariate Orthogonal Polynomials on Triangular Domains

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Abstract:

In this talk, Jacobi-weighted orthogonal polynomials

$$P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) \quad \alpha,\beta,\gamma > -1,$$

on the triangular domain T are constructed. We show that these polynomials $P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$ over the triangular domain T satisfy the following properties:

$$P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) \in \mathcal{L}_n, n \geq 1, r = 0,1,\ldots,n,$$

and

$$\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) \perp \mathcal{P}_{n,s}^{(\alpha,\beta,\gamma)}(u,v,w), r \neq s.$$

And hence,

$$\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w),\ n = 0,1,2,\ldots,\ r = 0,1,\ldots,n$$

form an orthogonal system over the triangular domain T with respect to the Jacobi weight function. These Jacobi-weighted orthogonal polynomials on triangular domains are given in Bernstein basis form and thus preserve many properties of the Bernstein polynomial basis.

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Introduction:

Recent years have seen a great deal in the field of orthogonal polynomials, a subject closely related to many important branches of analysis. Among these orthogonal polynomials, the Jacobi orthogonal polynomials are the most important. However, the cases of two or more variables of orthogonal polynomials on triangular domains have been studied by few researchers; although the main definitions and some simple properties were considered many years ago.

Previous Contributions:

Orthogonal polynomials with Jacobi weight function

$$w^{(\alpha,\beta,\gamma)}(u,v,w) = u^{\alpha}v^{\beta}(1-w)^{\gamma},\ \alpha,\beta,\gamma > -1$$

on triangular domain T are defined in [Sauer 1994]. These polynomials

$$P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$$

are orthogonal to each polynomial of degree $\leq n-1$, with respect to the defined weight function

$$w^{(\alpha,\beta,\gamma)}(u,v,w) \text{ on } T.$$

However, $P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$,$P_{n,s}^{(\alpha,\beta,\gamma)}(u,v,w)$, $r \neq s$, are not orthogonal with respect to the weight function $w^{(\alpha,\beta,\gamma)}(u,v,w)$ on T.

In [Farouki, Goodman, Sauer 2003], orthogonal polynomials with respect to the weight function

$$w(u, v, w) = 1$$

on a triangular domain T are defined. These polynomials $P_{n,r}(u, v, w)$ are orthogonal to each polynomial of degree $\leq n - 1$ and also orthogonal to each polynomial $P_{n,s}(u, v, w), \quad r \neq s$.

In this talk, we construct orthogonal polynomials $P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$ with respect to the Jacobi weight function

$$w^{(\alpha,\beta,\gamma)}(u,v,w) = u^{\alpha}v^{\beta}(1-w)^{\gamma}, \alpha, \beta, \gamma > -1$$

on triangular domain T. These Jacobi-weighted orthogonal polynomials on triangular domains are given in the Bernstein basis form, and thus preserving many geometric properties of the Bernstein polynomial basis. We show that these polynomials $P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$ over the triangular domain T satisfy the properties:

$$P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) \in \mathcal{L}_n, n \geq 1, r = 0, 1, \ldots, n,$$

and for $r \neq s$ we proved that

$$\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) \perp \mathcal{P}_{n,s}^{(\alpha,\beta,\gamma)}(u,v,w).$$

And hence, these bivariate polynomials for

$$\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w), r = 0, 1, \ldots, n,\ n = 0, 1, 2, \ldots$$

form an orthogonal system over the triangular domain T with respect to the weight function

$$w^{(\alpha,\beta,\gamma)}(u,v,w), \alpha, \beta, \gamma > -1.$$

Orthogonal Polynomials on Squares:

The construction of bivariate orthogonal polynomials on the square is straightforward. We consider the tensor product of the set of orthogonal polynomials over the square

$$G = \{(x, y) : -1 \leq x \leq 1, \quad -1 \leq y \leq 1\}.$$

Let

$$\{P_n^{(\alpha_1,\beta_1)}(x)\}$$

be the Jacobi polynomials over $[-1, 1]$ with respect to the weight function

$$w_1(x) = (1 - x)^{\alpha_1}(1 + x)^{\beta_1}.$$

And let

$$\{Q_m^{(\alpha_2,\beta_2)}(y)\}$$

be the Jacobi polynomials over $[-1, 1]$ with respect to the weight function

$$w_2(y) = (1 - y)^{\alpha_2}(1 + y)^{\beta_2}.$$

We define the bivariate polynomials $\{R_{nm}(x,y)\}$ on G formed by the tensor products of the Jacobi polynomials by

$$R_{nm}(x,y) := P_{n-m}^{(\alpha_1,\beta_1)}(x)Q_m^{(\alpha_2,\beta_2)}(y),$$

$$n = 0,1,2,\ldots \text{ and } m = 0,1,\ldots,n.$$

Then $\{R_{nm}(x,y)\}$ are orthogonal on the square G with respect to the weight function

$$w(x,y) = w_1^{(\alpha_1,\beta_1)}(x)w_2^{(\alpha_2,\beta_2)}(y).$$

However, the construction of orthogonal polynomials over a triangular domain is not straightforward like the tensor product over the square G.

Definitions:

For $m \geq 1$,,
Π_n: the space of all polynomials of degree n over the triangular domain T.
$\mathcal{L}_m$: The space of polynomials of degree m that are orthogonal to all polynomials of degree $< m$ over a triangular domain T, i.e.

$$\mathcal{L}_m = \{p \in \Pi_m : \quad p \perp \Pi_{m-1}\},$$

Univariate Bernstein Polynomials:

The Bernstein polynomials $b_i^n(u)$, $u \in [0,1]$, $i = 0, 1, \ldots, n$ are defined by

$$b_i^n(u) = \begin{cases} \begin{pmatrix} n \\ i \end{pmatrix} u^i (1-u)^{n-i}, & i = 0, 1, \ldots, n \\ 0, & else \end{cases}$$

where the binomial coefficients are given by

$$\begin{pmatrix} n \\ i \end{pmatrix} = \begin{cases} \frac{n!}{i!(n-i)!}, & \text{if } 0 \le i \le n \\ 0, & else \end{cases} .$$

Univariate Jacobi Polynomials:

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ of degree n are the orthogonal polynomials, except for a constant factor, on $[-1,1]$ with respect to the weight function

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha, \beta > -1.$$

It is appropriate to take $u \in [0,1]$ for both Bernstein and Jacobi polynomials. The following two lemmas will be needed in the construction of the orthogonal bivariate polynomials and the proof of the main results.

Relations between Bernstein and Jacobi:

Lemma. *(see Rababah 2004) The Jacobi polynomial* $P_r^{(\alpha,\beta)}(u)$ *of degree* $r = 0, 1, \ldots$ *has the following Bernstein representation*

$$P_r^{(\alpha,\beta)}(u) = \sum_{i=0}^{r} (-1)^{r-i} \frac{\begin{pmatrix} r+\alpha \\ i \end{pmatrix} \begin{pmatrix} r+\beta \\ r-i \end{pmatrix}}{\begin{pmatrix} r \\ i \end{pmatrix}} b_i^r(u) \, .$$

The Pochhammer symbol is more appropriate, but the combinatorial notation gives more compact and readable formulas, these have also been used in the book by Szegö.

Lemma. *(see Rababah 2004) The Jacobi polynomials*
$P_0^{(\alpha,\beta)}(u), ..., P_n^{(\alpha,\beta)}(u)$ *of degree* $\leq n$ *can be expressed in terms of the Bernstein basis of fixed degree* n *by the following formula*

$$P_r^{(\alpha,\beta)}(u) = \sum_{i=0}^{n} \mu_{i,r}^n b_i^n(u), \quad r = 0, 1, \ldots, n,$$

where, for $i = 0, \ldots, n,$

$$\mu_{i,r}^n = \binom{n}{i}^{-1} \sum_{k=\max(0,i+r-n)}^{\min(i,r)} (-1)^{r-k} \binom{n-r}{i-k} \binom{r+\alpha}{k} \binom{r+\beta}{r-k}.$$

Barycentric Coordinates:

Consider a base triangle in the plane with the vertices $\mathbf{p}_k = (x_k, y_k)$, $k = 1, 2, 3$. Then every point $\mathbf{p}$ inside the triangle T can be written using the barycentric coordinates (u, v, w), where $u + v + w = 1, \quad u, v, w \geq 0$ as $\mathbf{p}$=$u\mathbf{p}_1 + v\mathbf{p}_2 + w\mathbf{p}_3$. The barycentric coordinates are the ratio of areas of subtriangles of the base triangle as follows

$$u = \frac{area(\mathbf{p}, \mathbf{p}_2, \mathbf{p}_3)}{area(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)}, \quad v = \frac{area(\mathbf{p}_1, \mathbf{p}, \mathbf{p}_3)}{area(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)}, \quad w = \frac{area(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p})}{area(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)},$$

where $area(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \neq 0$, which means that $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are not collinear.

Generalized Bernstein polynomials:

Let T be a triangular domain defined by

$$T = \{(u, v, w) : u, v, w \geq 0, u + v + w = 1\}.$$

Let the notation $\alpha = (i, j, k)$ denotes triples of non-negative integers, where $|\alpha| = i + j + k\cdot$ The generalized Bernstein polynomials of degree n on the triangular domain T are defined by the formula

$$b_\alpha^n(u, v, w) = \begin{pmatrix} n \\ \alpha \end{pmatrix} u^i v^j w^k, \qquad |\alpha| = n,$$

where

$$\begin{pmatrix} n \\ \alpha \end{pmatrix} = \frac{n!}{i!j!k!}.$$

Note that the generalized Bernstein polynomials are non-negative over T, and form a partition of unity, i.e.

$$1 = (u + v + w)^n = \sum_{\substack{0 \leq i,j,k \leq n \\ i+j+k=n}} \frac{n!}{i!j!k!} u^i v^j w^k.$$

The sum involves a total of $\frac{1}{2}(n+1)(n+2)$ linearly independent polynomials. These polynomials define the Bernstein basis for the space Π_n over the triangular domain T.
The Bernstein polynomials $b^n_\alpha(u,v,w)$, $|\alpha| = n$, on T satisfy,

$$\int\int_T b^n_\alpha(u,v,w)dA = \frac{\Delta}{(n+1)(n+2)},$$

where Δ is double the area of T.

Triangular Bézier Surfaces:

Any polynomial P(u,v,w) of degree n can be written in the Bernstein form

$$P(u,v,w) = \sum_{|\alpha|=n} d_\alpha b_\alpha^n(u,v,w),$$

with Bézier coefficients d_α. We can also use the degree elevation algorithm for the Bernstein representation. This is obtained by multiplying both sides by $1 = u + v + w$, and writing

$$P(u,v,w) = \sum_{|\alpha|=n+1} d_\alpha^{(1)} b_\alpha^{n+1}(u,v,w).$$

The new coefficients $d_\alpha^{(1)}$ are defined by,

$$d_{ijk}^{(1)} = \frac{1}{n+1}(id_{i-1,j,k} + jd_{i,j-1,k} + kd_{i,j,k-1}),$$

where $i + j + k = n + 1$.

Inner Product:

Let $P(u, v, w)$ and $Q(u, v, w)$ be two bivariate polynomials over T, then we define their inner product over T by

$$\langle P, Q \rangle = \frac{1}{\Delta} \iint\limits_{T} PQ dA.$$

We say that P and Q are orthogonal if $\langle P, Q \rangle = 0$.

Degree-ordered System of OP:

A basis of linearly independent and mutually orthogonal polynomials in the barycentric coordinates (u, v, w) are constructed over T. These polynomials are represented in the following triangular table

$$P_{0,0}^{(\alpha,\beta,\gamma)}(U)$$

$$P_{1,0}^{(\alpha,\beta,\gamma)}(U), \quad P_{1,1}^{(\alpha,\beta,\gamma)}(U)$$

$$P_{2,0}^{(\alpha,\beta,\gamma)}(U), \quad P_{2,1}^{(\alpha,\beta,\gamma)}(U), P_{2,2}^{(\alpha,\beta,\gamma)}(U)$$

$$\vdots$$

$$P_{n,0}^{(\alpha,\beta,\gamma)}(U), P_{n,1}^{(\alpha,\beta,\gamma)}(U), P_{n,2}^{(\alpha,\beta,\gamma)}(U), \ldots, P_{n,n}^{(\alpha,\beta,\gamma)}(U)$$

The kth row of this triangle table contains $k+1$ polynomials. Thus, for a basis of linearly independent polynomials of total degree n, there are $\frac{1}{2}(n+1)(n+2)$ polynomials.

Orthogonal Polynomials on Triangular Domains:

A simple closed-form representation of degree-ordered system of orthogonal polynomials is constructed on a triangular domain T. Since the Bernstein polynomials are stable, it is convenient to write these polynomials in Bernstein form.

Let $f(u, v, w)$ be an integrable function over T, and consider the operator

$$S_n(f) = (n+1)(n+2) \sum_{|\alpha|=n} \langle f, b_\alpha^n \rangle \, b_\alpha^n.$$

For $n \geq m$,

$$\lambda_{m,n} = \frac{(n+2)!n!}{(n+m+2)!(n-m)!}$$

is an eigenvalue of the operator S_n, and $\mathcal{L}_m$ is the corresponding eigenspace, see Derriennic.

Preliminary Results:

The following Lemmas will be needed in the proof of the main results.

Lemma. *(See Farouki, Goodman, and Sauer 2003) Let* $P = \sum_{|\alpha|=n} c_\alpha b_\alpha^n \in \mathcal{L}_m$ *and* $Q = \sum_{|\alpha|=n} d_\alpha b_\alpha^n \in \Pi_n$ *with* $m \leq n\cdot$ *Then we have*

$$\langle P, Q \rangle = \frac{(n!)^2}{(n+m+2)!(n-m)!} \sum_{|\alpha|=n} c_\alpha d_\alpha.$$

Lemma. *(See Farouki, Goodman, and Sauer 2003, and Peters, and Reif 2000) Let* $P = \sum_{|\alpha|=n} c_\alpha b_\alpha^n \in \Pi_n$. *Then we have*

$$P \in \mathcal{L}_n \Leftrightarrow \sum_{|\alpha|=n} c_\alpha d_\alpha = 0 \text{ for all } Q = \sum_{|\alpha|=n} d_\alpha b_\alpha^n \in \Pi_{n-}$$

Consider the polynomials

$$q_{n,r}(w) = \sum_{j=o}^{n-r} (-1)^j \binom{n+r+1}{j} b_j^{n-r}(w).$$

The polynomial $q_{n,r}(w)$ is a scalar multiple of $P_{n-r}^{(0,2r+1)}(1-2w)$, and we have the following Lemma

Lemma. *(See Farouki et al 2003) For* $r = 0, \ldots, n$ *and* $i = 0, \ldots, n-r-1$, $q_{n,r}(w)$ *is orthogonal to* $(1-w)^{2r+i+1}$ *on* $[0,1]$, *and hence*

$$\int_0^1 q_{n,r}(w) P(w)(1-w)^{2r+1} dw = 0,$$

for every polynomial $P(w)$ *of degree* $\leq n-r-1$.

Let $\sigma = \alpha + \beta + \gamma$, then the following lemma will be used in the proof of theorem.

Lemma. *The following identity holds*

$$S = \sum_{j=0}^{n-r} (-1)^j \frac{\binom{n+r+\sigma+1}{j}\binom{n-r}{j}}{\binom{n+r+i+\sigma+1}{j}}$$

$$= \frac{\binom{i}{n-r}}{\binom{n+r+i+\sigma+1}{n-r}}.$$

Proof: Using equation (5.21) in Knuth, and negating the binomial term in the numerator, we get

$$\frac{\binom{n-r}{j}}{\binom{n+r+i+\sigma+1}{j}} =$$

$$(-1)^{n-r-j}\frac{\binom{-2r-i-\sigma-2}{n-r-j}}{\binom{n+r+i+\sigma+1}{n-r}}.$$

Substituting these simplifications in the summation, we get

$$S \;=\; \frac{(-1)^{n-r}}{\binom{n+r+i+\sigma+1}{n-r}}\sum_{j=0}^{n-r}\binom{n+r+\sigma+1}{j} \binom{-2r-i-\sigma-2}{n-r-j}.$$

Using equation (5.22) in Knuth, we have

$$S = \frac{(-1)^{n-r} \binom{n-r-i-1}{n-r}}{\binom{n+r+i+\sigma+1}{n-r}},$$

and by negating the numerator, the identity holds.

Jacobi-weighted Orthogonal Polynomials:

For $n = 0, 1, 2, \dots$ and $r = 0, 1, \dots, n$, we define the bivariate polynomials

$$\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) = \sum_{i=0}^{r} c(i,\alpha,\beta) b_i^r(u,v) \sum_{j=0}^{n-r} (-1)^j \binom{n+r+1}{j} b_j^{n-r}(w, u+v)$$

where $\alpha, \beta, \gamma > -1$,

$$c(i,\alpha,\beta) = (-1)^{r-i} \frac{\binom{r+\alpha}{i}\binom{r+\beta}{r-i}}{\binom{r}{i}}, \quad i = 0, 1, \dots$$

and

$$b_i^r(u,v) = \binom{r}{i} u^i v^{r-i}, \quad i = 0, 1, \dots, r.$$

Preliminaries for Proof:

We show that the polynomials $\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) \in \mathcal{L}_n$, $n \geq 1$, $r = 0,1,\ldots,n$, and $\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)} \perp \mathcal{P}_{n,s}^{(\alpha,\beta,\gamma)}$ for $r \neq s$. Thus, choosing $P_{0,0}^{(\alpha,\beta,\gamma)} = 1$, then the polynomials $\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$ for $0 \leq r \leq n$ and $n = 0,1,2,\ldots$ form a degree-ordered orthogonal sequence over T. We first rewrite these polynomials in the Jacobi polynomials form

$$\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(U) = \sum_{i=0}^{r} (-1)^{r-i} \frac{\begin{pmatrix} r+\alpha \\ i \end{pmatrix} \begin{pmatrix} r+\beta \\ r-i \end{pmatrix}}{\begin{pmatrix} r \\ i \end{pmatrix}} b_i^r(u,v)$$

$$\times \sum_{j=0}^{n-r} (-1)^j \begin{pmatrix} n+r+1 \\ j \end{pmatrix} b_j^{n-r}(w,u+v)$$

$$= \sum_{i=0}^{r} (-1)^{r-i} \frac{\binom{r+\alpha}{i}\binom{r+\beta}{r-i}}{\binom{r}{i}} \frac{b_i^r(u,v)}{(u+v)^r}(1-w)^r$$

$$\times \sum_{j=0}^{n-r} (-1)^j \binom{n+r+1}{j} b_j^{n-r}(w, 1-w).$$

Since

$$\frac{b_i^r(u,v)}{(u+v)^r} = b_i^r(\frac{u}{1-w}),$$

and using Lemma, we get

$$\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(U) = P_r^{(\alpha,\beta)}(\frac{u}{1-w})(1-w)^r q_{n,r}(w),\ r = 0, \ldots$$

where $P_r^{(\alpha,\beta)}(t)$ is the univariate Jacobi polynomial of degree r.

$\underline{\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) \in \mathcal{L}_n}$:

First, we show that the polynomials $\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$, $r = 0, ..., n$ are orthogonal to all polynomials of degree $< n$ over the triangular domain T.

Theorem. *For each* $n = 1, 2, \ldots,$ $r = 0, 1, \ldots, n,$ *and the weight function* $w^{(\alpha,\beta,\gamma)}(u,v,w) = u^\alpha v^\beta (1-w)^\gamma$ *such that* $\alpha, \beta, \gamma > -1$, *we have* $\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) \in \mathcal{L}_n$.

Proof of Theorem:

Proof: For each $m = 0, \ldots, n-1$, and $s = 0, \ldots, m$ we construct the set of bivariate polynomials

$$Q_{s,m}^{(\alpha,\beta)}(u,v,w) = P_s^{(\alpha,\beta)}(\frac{u}{1-w})(1-w)^m w^{n-m-1}.$$

The span of these polynomials includes the set of Bernstein polynomials

$$b_j^m(\frac{u}{1-w})(1-w)^m w^{n-m-1} = b_j^m(u,v)w^{n-m-1}.$$

which span the space Π_{n-1}. Thus, it is sufficient to show that for each $m = 0, \ldots, n-1$, $s = 0, \ldots, m$, we have

$$I := \int\int_T \mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(U) Q_{s,m}^{(\alpha,\beta)}(U) w^{(\alpha,\beta,\gamma)}(U) dA = 0.$$

This is simplified to

$$I = \Delta \int_0^1 \int_0^{1-w} P_r^{(\alpha,\beta)}(\frac{u}{1-w}) q_{n,r}(w) P_s^{(\alpha,\beta)}(\frac{u}{1-w})$$

$$w^{n-m-1} u^\alpha v^\beta (1-w)^{\gamma+r+m} dudw.$$

By making the substitution $t = \frac{u}{1-w}$, we get

$w^{(\alpha,\beta,\gamma)}(U) = u^\alpha v^\beta (1-w)^\gamma = t^\alpha (1-t)^\beta (1-w)^{\alpha+\beta+\gamma}$.

And thus, we have

$$I = \Delta \int_0^1 \int_0^1 P_r^{(\alpha,\beta)}(t) q_{n,r}(w) P_s^{(\alpha,\beta)}(t)$$

$$(1-w)^{\alpha+\beta+\gamma+r+m+1} w^{n-m-1} t^\alpha (1-t)^\beta \, dtdw$$

$$= \Delta \int_0^1 P_r^{(\alpha,\beta)}(t) P_s^{(\alpha,\beta)}(t) t^\alpha (1-t)^\beta \, dt \int_0^1 q_{n,r}(w)$$

$$(1-w)^{\alpha+\beta+\gamma+r+m+1}w^{n-m-1}dw.$$

If $m < r$, then we have $s < r$, and the first integral is zero by the orthogonality property of the Jacobi polynomials. If $r \leq m \leq n-1$, we have by Lemma, the second integral equals zero. And thus the theorem follows.

Note that taking $w^{(\alpha,\beta,\gamma)}(u,v,w) = u^{\alpha}v^{\beta}(1-w)^{\gamma}$ enables us to separate the integrand in the proof of theorem. Note that the case $\alpha+\beta+\gamma = 0$ is discussed in Rababah, and Al-Qudah.

$\underline{\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) \perp \mathcal{P}_{n,s}^{(\alpha,\beta,\gamma)}(u,v,w)}$:

In the following theorem, we show that $\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(U)$ is orthogonal to each polynomial of degree n. And thus the bivariate polynomials $\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$, $r = 0, 1, \ldots, n$, and $n = 0, 1, 2, \ldots$ form an orthogonal system over the triangular domain T with respect to the weight function $w^{(\alpha,\beta,\gamma)}(u,v,w)$, $\alpha, \beta, \gamma > -1$.

Theorem. *For* $r \neq s$, *we have* $\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) \perp \mathcal{P}_{n,s}^{(\alpha,\beta,\gamma)}(u,v,w)$ *with respect to the weight function* $w^{(\alpha,\beta,\gamma)}(u,v,w) = u^{\alpha}v^{\beta}(1-w)^{\gamma}$ *such that* $\alpha,\beta,\gamma > -1$.

Proof of Theorem:

Proof: For $r \neq s$, we have

$$I := \int\int_T \mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(U)\mathcal{P}_{n,s}^{(\alpha,\beta,\gamma)}(U)w^{(\alpha,\beta,\gamma)}(U)dA$$

$$= \Delta \int_0^1 \int_0^{1-w} P_r^{(\alpha,\beta)}\left(\frac{u}{1-w}\right) P_s^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^{r+s}$$

$$q_{n,r}(w)q_{n,s}(w)w^{(\alpha,\beta,\gamma)}(U)dudw$$

By making the substitution $t = \frac{u}{1-w}$, we get $w^{(\alpha,\beta,\gamma)}(u,v,w) = t^{\alpha}(1-t)^{\beta}(1-w)^{\alpha+\beta+\gamma}$. And thus, we have

$$I = \Delta \int\limits_{0}^{1} P_r^{(\alpha,\beta)}(t) P_s^{(\alpha,\beta)}(t) t^{\alpha} (1-t)^{\beta} \, dt$$

$$\int\limits_{0}^{1} q_{n,r}(w) q_{n,s}(w) (1-w)^{\alpha+\beta+\gamma+r+s+1} dw$$

the first integral equals zero by the orthogonality property of the Jacobi polynomials for $r \neq s$, and thus the theorem follows.

Orthogonal polynomials in Bernstein basis:

The Bernstein-Bézier form of curves and surfaces exhibits some interesting geometric properties. So, we write the orthogonal polynomials $\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w), \quad r = 0, 1, \ldots, n$ and $n = 0, 1, 2, \ldots$ in the following Bernstein-Bézier form

$$\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) = \sum_{|\alpha|=n} a_{\alpha}^{n,r} b_{\alpha}^{n}(u,v,w).$$

We are interested in finding a closed form for the computation of the Bernstein coefficients $a_{\alpha}^{n,r}$. These are given explicitly in the following theorem

Theorem. *The Bernstein coefficients* $a_\alpha^{n,r}$ *are given explicitly by:* $a_{ijk}^{n,r} =$

$$\begin{cases} (-1)^k \dfrac{\begin{pmatrix} n+r+1 \\ k \end{pmatrix}\begin{pmatrix} n-r \\ k \end{pmatrix}}{\begin{pmatrix} n \\ k \end{pmatrix}} \mu_{i,r}^{n-k}, & 0 \le k \le n-r \\ 0, & k > n-r \end{cases}$$

where $\mu_{i,r}^{n-k}$ *are given in Rababah 2004.*

Proof of Theorem:

Proof: It is clear that $\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$ has degree $\leq n-r$ in the variable w, and thus

$$a_{ijk}^{n,r} = 0 \quad for\ k > n-r.$$

For $0 \leq k \leq n-r$, the remaining coefficients are determined as follows

$$\sum_{i+j=n-k} a_{ijk}^{n,r} b_{ijk}^{n}(U) = (-1)^k \begin{pmatrix} n+r+1 \\ k \end{pmatrix} b_k^{n-r}(w, u+$$

$$\times \sum_{i=0}^{r} (-1)^{r-i} \frac{\begin{pmatrix} r+\alpha \\ i \end{pmatrix} \begin{pmatrix} r+\beta \\ r-i \end{pmatrix}}{\begin{pmatrix} r \\ i \end{pmatrix}} b_i^r(u,v).$$

Comparing powers of w on both sides, we have

$$\sum_{i=0}^{n-k} a_{ijk}^{n,r} \frac{n!}{i!j!k!} u^i v^j = (-1)^k \begin{pmatrix} n+r+1 \\ k \end{pmatrix} \begin{pmatrix} n-r \\ k \end{pmatrix}$$

$$(u+v)^{n-r-k}\sum_{i=0}^{r}(-1)^{r-i}\frac{\binom{r+\alpha}{i}\binom{r+\beta}{r-i}}{\binom{r}{i}}b_i^r(u,v).$$

The left hand side of the last equation can be written in the form

$$\sum_{i=0}^{n-k}a_{ijk}^{n,r}\frac{n!}{i!j!k!}u^iv^j=\sum_{i=0}^{n-k}a_{ijk}^{n,r}\frac{n!(n-k)!}{i!j!k!(n-k)!}u^iv^j$$

$$= \sum_{i=0}^{n-k} a_{ijk}^{n,r} \frac{n!(n-k)!}{i!(n-k-i)!k!(n-k)!} u^i v^j$$

$$= \sum_{i=0}^{n-k} a_{ijk}^{n,r} \left(\begin{array}{c} n \\ k \end{array} \right) b_i^{n-k}(u,v).$$

Now, we get

$$\sum_{i=0}^{n-k} a_{ijk}^{n,r} \left(\begin{array}{c} n \\ k \end{array} \right) b_i^{n-k}(u,v) = (-1)^k \left(\begin{array}{c} n+r+1 \\ k \end{array} \right) \left(\begin{array}{c} n- \\ k \end{array} \right.$$

$$(u+v)^{n-r-k} \sum_{i=0}^{r} (-1)^{r-i} \frac{\left(\begin{array}{c} r+\alpha \\ i \end{array} \right) \left(\begin{array}{c} r+\beta \\ r-i \end{array} \right)}{\left(\begin{array}{c} r \\ i \end{array} \right)} b_i^r(u,v).$$

With some binomial simplifications, and using Lemma, we get

$$\sum_{i=0}^{n-k} a_{ijk}^{n,r} \left(\begin{array}{c} n \\ k \end{array} \right) b_i^{n-k}(u,v) = (-1)^k \left(\begin{array}{c} n+r+1 \\ k \end{array} \right)$$

$$\left(\begin{array}{c} n-r \\ k \end{array} \right) \sum_{i=0}^{r} \mu_{i,r}^{n-k} b_i^{n-k}(u,v),$$

where $\mu_{i,r}^{n-k}$ are the coefficients resulting from writing Jacobi polynomial of degree r in the Bernstein basis of degree $n-k$,.

Thus, the required Bernstein-Bézier coefficients are given by $a_{ijk}^{n,r} =$

$$\begin{cases} (-1)^k \dfrac{\begin{pmatrix} n+r+1 \\ k \end{pmatrix}\begin{pmatrix} n-r \\ k \end{pmatrix}}{\begin{pmatrix} n \\ k \end{pmatrix}} \mu_{i,r}^{n-k}, & 0 \le k \le n-r \\ 0, & k > n-r \end{cases}$$

Which completes the proof of the theorem.

To derive a recurrence relation for the coefficients $a_{ijk}^{n,r}$ of $\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$, we consider the generalized Bernstein polynomial of degree $n-1$

$$b_{ijk}^{n-1}(u,v,w) = \frac{(n-1)!}{i!j!k!}u^i v^j w^k$$

$$= \frac{(n-1)!}{i!j!k!}u^i v^j w^k (u+v+w)$$

$$= \frac{(i+1)n!}{n(i+1)!j!k!}u^{i+1}v^j w^k + \frac{(j+1)n!}{n(i!)(j+1)!k!}u^i v^{j+1} w^k +$$

$$+ \frac{(k+1)n!}{n(i!)(j!)(k+1)!}u^i v^j w^{k+1}$$

$$= \frac{(i+1)}{n}b_{i+1,j,k}^{n}(u,v,w) + \frac{(j+1)}{n}b_{i,j+1,k}^{n}(u,v,w) +$$

$$\frac{(k+1)}{n} b^n_{i,j,k+1}(u,v,w).$$

By construction of $\mathcal{P}^{(\alpha,\beta,\gamma)}_{n,r}(u,v,w)$, we have

$$\langle b^{n-1}_{ijk}(u,v,w), \mathcal{P}^{(\alpha,\beta,\gamma)}_{n,r}(u,v,w)\rangle = 0, \quad i+j+k = n-1.$$

Thus by Lemma, we have

$$(i+1)a^{n,r}_{i+1,j,k} + (j+1)a^{n,r}_{i,j+1,k} + (k+1)a^{n,r}_{i,j,k+1} = 0$$

and since we know from theorem that

$$a^{n,r}_{i,n-i,0} = \mu^n_{i,r} \text{ for } i = 0,1,\ldots,n,$$

we can generate $a^{n,r}_{i,j,k}$ recursively on k.

Generalizations:

The d-dimensional unit simplex in barycentric coordinates is defined by

$$T_d = \left\{ u = (u_0, ..., u_d) \,:\, u_j \geq 0, \sum_{j=0}^{d} u_j = 1 \right\}.$$

The Bernstein basis for polynomials of degree n over T_d are defined by

$$b_\alpha^n(u) = \left(\begin{array}{c} n \\ \alpha \end{array} \right) u^\alpha = \frac{n!}{\alpha_0! .... \ \alpha_d!} u_0^{\alpha_0} ... u_d^{\alpha_d}, |\alpha| = n$$

where $\alpha = (\alpha_0, ..., \alpha_d) \in N_0^{d+1}$ and $|\alpha| = \alpha_0 + \cdots + \alpha_d = n$.

The method of construction in this talk can be generalized to the case of multivariate polynomials over a d-dimensional simplex in any number of variables d.

Closure:

We have constructed Jacobi-weighted orthogonal polynomials $P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$, $\alpha,\beta,\gamma \geq -1$, $\alpha+\beta+\gamma=0$ on the triangular domain T. Since the Bernstein polynomials are stable, we write these polynomials in Bernstein basis form. The polynomials $P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) \in \mathcal{L}_n, n \geq 1, r = 0,1,\ldots,n$, and $\mathcal{P}_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) \perp \mathcal{P}_{n,s}^{(\alpha,\beta,\gamma)}(u,v,w)$ for $r \neq s$. And hence, these bivariate polynomials form an orthogonal system over the triangular domain T with respect to the above weight function.

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