Non-uniform Variational Subdivision

Outline:

I. Subdivision Literature and Preliminaries.
II. Trisection and Corner-Cutting Schemes.
III. Four-point interpolatory Schemes.
IV. Global Variational Subdivision.
V. Local Variational Subdivision.
VI. Smoothing Variational Subdivision Surfaces.
Non-uniform Variational Subdivision

Emphasis:

- Uniform and Non-uniform Schemes.
- Effect of Parametrization on Uniformity and Smoothness.
- Smoothness (Discrete Hölder Regularity).
- Approximation Order.
- Localizing Global Variational Schemes.
Part I

Literature and Preliminaries
## Corner-Cutting

<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Title</th>
<th>Year</th>
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<tbody>
<tr>
<td>de Rham, G.</td>
<td>Trisection Algorithms of 1947,’53,’56,’57,’58</td>
<td>1947–</td>
</tr>
<tr>
<td>Chaikin, G. M.</td>
<td>An algorithm for high speed curve generation</td>
<td>1974</td>
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<tr>
<td>Riesenfeld, R.</td>
<td>On Chaikin’s Algorithm</td>
<td>1975</td>
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<tr>
<td>Catmull &amp; Clark</td>
<td>Recursively generated B-Spline Surfaces on arbitrary topological meshes</td>
<td>1978</td>
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<td>Doo &amp; Sabin</td>
<td>Behavior of recursive division surfaces near extraordinary points</td>
<td>1978</td>
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<tr>
<td>Lane &amp; Riesenfeld</td>
<td>Uniform B-splines of arbitrary order</td>
<td>1980</td>
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<tr>
<td>de Boor</td>
<td>Corner-Cutting always works</td>
<td>1987</td>
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<td>Gregory &amp; Qu</td>
<td>Non-uniform corner-cutting</td>
<td>1988</td>
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<td>de Boor</td>
<td>Local corner cutting and smoothness of limit curve</td>
<td>1990</td>
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<td>Dyn, Gregory &amp; Levin</td>
<td>Analysis of uniform binary subdivision</td>
<td>1991</td>
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## Subdivision Literature

### Interpolatory Subdivision

<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Description</th>
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<tbody>
<tr>
<td>Dubuc</td>
<td>Interpolation through an iterative scheme.</td>
<td>1986</td>
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<tr>
<td>Dyn, Levin &amp; Gregory</td>
<td>A 4-point interpolatory subdivision scheme</td>
<td>1987</td>
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<td>Gregory &amp; Qu</td>
<td>Non-uniform corner-cutting</td>
<td>1988</td>
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<tr>
<td>Deslauries &amp; Dubuc</td>
<td>Symmetric Iterative Refinement</td>
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<td>Dyn, Gregory &amp; Levin</td>
<td>Butterflies subdivision for surfaces</td>
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<td>Kobbelt</td>
<td>A variational approach to subdivision</td>
<td>1996</td>
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<tr>
<td>Kobbelt</td>
<td>Discrete Fairing (for surfaces)</td>
<td>1997</td>
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<td>Kuijt &amp; van Damme</td>
<td>Convexity preserving interpolatory subdivision</td>
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<td>Levin</td>
<td>Analysis of non-uniform binary subdivision</td>
<td>1999</td>
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<tr>
<td>Marinov, Dyn &amp; Levin</td>
<td>Geometrically controlled 4-point schemes</td>
<td>2004</td>
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## My work

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<tr>
<th>Kersey</th>
<th>Constrained subdivision curves</th>
<th>’03</th>
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<td>Near-interpolation subdivision surfaces</td>
<td>’04</td>
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<tr>
<td>Kersey</td>
<td>Smoothness of Non-uniform variational subdivision</td>
<td>’04</td>
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<td>Kersey</td>
<td>An abstract approach to variational refinement</td>
<td>’04</td>
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<td>Kersey</td>
<td>Local Variational subdivision</td>
<td>’05</td>
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**Current Work:**

- Approximation order in non-uniform variational refinement.
- Convergence of local variational subdivision.
Non-Interpolatory

Example: Corner-Cutting (Convex Scheme)
Non-Interpolatory

Example: Corner-Cutting (Convex Scheme)
Non-Interpolatory

Example: Corner-Cutting (Convex Scheme)
Non-Interpolatory

Example: Corner-Cutting (Convex Scheme)
Non-Interpolatory

Non-interpolatory (corner-cutting) schemes

- Trisection: de Rham
- Corner-Cutting: Chaikin
- B-Spline Subdivision: Lane and Riesenfeld
- Corner-Cutting Always works: de Boor
- Non-uniform Corner-cutting: Gregory and Qu
Example: Four-Point and variational subdivision

- Non-convex
Example: Four-Point and variational subdivision
• Non-convex
Interpolatory

Examples: Four-Point and variational subdivision
Non-convex
Interpolatory

Non-convex
Interpolatory

Interpolatory Subdivision

- Local Four-point Subdivision: Dubuc
- Local Four-point Subdivision with Tension: Gregory, Dyn, Levin
- Local n-point Subdivision: Dubuc and Deslauries
- Local Non-uniform Four-point Subdivision: Levin
- Global Uniform Variational Subdivision: Kobbelt
- Global Non-uniform Variational Subdivision: Kersey
- Local Non-uniform Variational Subdivision: Kersey
Representation of PWL Curves

At each stage in the subdivision process the curves are parametric piecewise linear curves.

- Coefficients $f_i \in \mathbb{R}^2$.
- Parameter values (knots) $t_i \in [a, b]$.
- Linear B-splines $N_i(t)$ with $N_i(t_j) = \delta_{ij}$.
- Piecewise linear spline curve: $f(t) = \sum_i f_i N_i(t)$.
- Interpolation: $f(t_i) = f_i$. 
General Linear Subdivision

Given:

- Coefficients at “level 0”: $f_i = f_i^0 \in \mathbb{R}^2$.
- Parameters at level 0: $t_i = t_i^0 \in [a, b]$.

Find:

- Coefficients at level k: $f_i^k := \sum_j \alpha_{ij}^k f_i^{k-1}$.
- Parameters at level k: $t_i^k$.

Note that the coefficients are determined by a linear process: $f^k := A_k f_{k-1}$. (Very general nonlinear schemes have been studied by Kuijt, et al.)
Uniform Schemes

With Uniform schemes the knots are equally spaced. That is, \( h := h_i \) is constant where \( h_i := t_{i+1} - t_i \). In dyadic subdivision \( h^k = h^{k-1}/2 \).

Examples of uniform schemes:

- de Rham’s trisection
- Chaikin’s algorithm
- Dubuc’s Four-point subdivision
- Gregory’s Four-point subdivision
- Kobbelt’s variational subdivision

Note that any of these schemes can be made nonuniform under different parametrizations.
Questions:

- Do these schemes converge, i.e., $f^k(t) \rightarrow f(t)$?
- Under what norms?
- What is the smoothness of $f(t)$?
- Does the scheme approximate given functions?

Corner-cutting schemes do not approximate; interpolatory schemes do.

Smoothness and approximation order depend on parametrization (uniformity).
Part II

Trisection and Corner-Cutting
The first publication on ‘corner-cutting’ was de Rham’s 1:1:1 trisection algorithm of 1947. Under a uniform parametrization the scheme is $C^0$, not $C^1$. 
The first publication on ‘corner-cutting’ was de Rham’s 1:1:1 trisection algorithm of 1947. Under a uniform parametrization the scheme is $C^0$, but not $C^1$. 

![Diagram of trisection algorithm]
Trisection – de Rham (1947)

But de Rham’s curve (limit of trisection of the square) is geometrically smooth, even though not $C^1$!!
Trisection $w : 1 - 2w : w$ is $C^1$-scheme under a uniform parametrization iff $w = 1/4$ (i.e., 1:2:1). In modern-day the 1:2:1 trisection is known as Chaikins algorithm.
Trisection $w : 1 - 2w : w$ is $C^1$-scheme under a uniform parametrization iff $w = 1/4$ (i.e., 1:2:1). In modern-day the 1:2:1 trisection is known as Chaikin’s algorithm.
Chaikin’s Algorithm (1974)

\[ f_{2i}^{k+1} = \frac{3}{4} f_i^k + \frac{1}{4} f_{i+1}^k, \quad f_{2i+1}^{k+1} = \frac{1}{4} f_i^k + \frac{3}{4} f_{i+1}^k \]
Chaikin’s Algorithm (1974)

\[
f_{2i}^{k+1} = \frac{3}{4} f_i^k + \frac{1}{4} f_{i+1}^k, \quad f_{2i+1}^{k+1} = \frac{1}{4} f_i^k + \frac{3}{4} f_{i+1}^k
\]
Chaikin’s Algorithm (1974)

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\]
For any corner-cutting scheme (even non-stationary) the limit is always Lipschitz continuous. That is,

$$|f(x) - f(y)| \leq \alpha |x - y|$$

for some $\alpha$. Such curves are $C^1$ almost everywhere.
Corner-Cutting Always Works – de Boor (1987)

3:1:1 trisection:
Corner-Cutting Always Works – de Boor (1987)

3:1:1 trisection:
B-spline subdivision – Riesenfeld and Lane

- Showed that Chaikin’s algorithm produces $C^1$ quadratic uniform B-spline curves.
- Derived subdivision algorithms for uniform B-splines of arbitrary order.

Based on the observation that:

$$f(t) := \sum_i f_i^0 B_m(t - i) = \cdots = \sum_i f_i^k B_m(2^k t - i)$$

For some refined coefficients $f_i^k$. (B-spline control polygon approaches limit curve.)
Towards a non-uniform scheme – Gregory and Qu (1998)

Subdivision:

\[ f_{2i}^{k+1} = (1 - \alpha_i^k) f_i^k + \alpha_i^k f_{i+1}^k, \quad t_{2i}^{k+1} = (1 - \alpha_i^k) t_i^k + \alpha_i^k t_{i+1}^k \]

\[ f_{2i+1}^{k+1} = \beta_i^k f_i^k + (1 - \beta_i^k) f_{i+1}^k, \quad t_{2i+1}^{k+1} = \beta_i^k t_i^k + (1 - \beta_i^k) t_{i+1}^k \]

- Linear (non-stationary) corner cutting scheme.
- Non-uniform parametrization.
- Subdivision scheme applied to coefficients AND knot sequence.
- Preserves arc length parametrizations.
- Partly non-uniform: parametrization does not affect trace of curve.
$d^k(t) := \sum_i d_i^k N_i(t)$ with $d_i := \frac{f_{i+1} - f_i}{h_i}$.

**Lemma 0.0** If $d^k \xrightarrow{u} d$ for some $d \in C([a, b])$, then $f^k \xrightarrow{u} f \in C^1([a, b])$ where $f' = d$.

**Theorem 0.0** The corner-cutting scheme converges to a $C^1$ curve provided

$$\alpha > 0, \beta > 0, 2\alpha + \beta < 1, \alpha + 2\beta < 1.$$

Consequence: For $0 < w < 1/3$, de Rham’s $w:1-2w:w$ is $C^1$ under the above parametrization.
Question: Does their lemma generalize to second (or higher) order derivatives/differences with arbitrary parametrizations and arbitrary subdivision schemes (not necessarily corner-cutting)?

Answer: It does, however not directly because we need ‘locally defined’ second divided differences.

I.e.,

\[ s_i = \frac{d_{i+1} - d_i}{h_{i,2}} \neq \frac{d_{i+1} - d_i}{h_i}, \]

with \( h_i := t_{i+1} - t_i \), \( h_{i,2} := h_i + h_{i+1} = t_{i+2} - t_i \), and \( s_i \) second divided differences.
Generalization of GQ

Symmetric Differences:

\[ s^k(t) := \sum_i s^k_i N_i(t) \quad \text{with} \quad s_i := \frac{d_{i+1} - d_i}{h_{i,2}} \]

\[ d^k(t) := \sum_i d^k_i N_i(t) \quad \text{with} \quad d_i := \frac{h_{i-1}d_i + h_id_{i-1}}{h_{i-1,2}} \]

\[ s^k(t) := \sum_i s^k_i N_i(t) \quad \text{with} \quad s_i := \frac{d_{i+1} - d_i}{h_i} = s_i + s_{i-1}. \]

**Theorem 0.0** (Kersey, 2004) Assume that \( \limsup h^k_i \to 0. \) If \( s^k \xrightarrow{u} \tilde{s} \) for some \( \tilde{s} \in C([a, b]), \) then \( f^k \xrightarrow{u} f \in C^2([a, b]) \) where \( f'' = d' = \tilde{s} = 2s. \)
Part III

Four-Point Interpolatory Schemes
Idea: Add new point between $f_1$ and $f_2$ by evaluating on the interpolating cubic polynomial.
Uniform Four Point Scheme – Dubuc (1986)

\[
f_{2i}^{k+1} = f_i^k
\]

\[
f_{2i+1}^{k+1} = -\frac{1}{16} f_{i-1}^k + \frac{9}{16} f_i^k + \frac{9}{16} f_{i+1}^k - \frac{1}{16} f_{i+2}^k
\]
Uniform Four Point Scheme – Dubuc (1986)

Properties:
- Cubic polynomial precision.
- Almost $C^2$ under uniform parametrization – Holder exponent arbitrary close to 2.
Uniform Four Point Scheme with Tension – Dyn, Levin, Gregory (1986)

\( f_{2i} = f_i \)

\[ f_{2i+1} = -wf_{i-1} + \left( \frac{1}{2} + w \right) f_i + \left( \frac{1}{2} + w \right) f_{i+1} - wf_{i+2} \]

\( w = 1/32 \)
Uniform Four Point Scheme with Tension – Dyn, Levin, Gregory (1986)

\[ f_{2i} = f_i \]

\[ f_{2i+1} = -wf_{i-1} + \left( \frac{1}{2} + w \right)f_i + \left( \frac{1}{2} + w \right)f_{i+1} - wf_{i+2} \]

\[ w = 1/16 \]
Uniform Four Point Scheme with Tension – Dyn, Levin, Gregory (1986)

\[ f_{2i} = f_i \]

\[ f_{2i+1} = -wf_{i-1} + \left( \frac{1}{2} + w \right) f_i + \left( \frac{1}{2} + w \right) f_{i+1} - wf_{i+2} \]
Properties of Weighted Four Point Scheme

- Reduces to Dubuc scheme when $w = \frac{1}{16}$.
- Sufficient $C^0$ condition: $0 < w < \frac{1}{4}$.
- Sufficient $C^1$ condition: $0 < w < \frac{1}{8}$.
- Deslauries and Dubuc generalizes to $2n$ schemes (by higher degree polynomial interpolation).
- For $|w| < 1/4$ approximation order to $f \in C^2$ is $O(h^2)$, with uniform mesh-spacing $h$.
- For $w = \frac{1}{16}$ approximation order to $f \in C^4$ is $O(h^4)$.
- Note that there is no approximation order between $h^2$ and $h^4$. Reason: Quadratic reproduction requires $w = 1/16$, which reproduces cubic.
Non-Uniform Four Point Scheme

Levin (1999)

\[ f_{2i} = f_i \]

\[ f_{2i+1} = -w_i^k f_{i-1} + \left( \frac{1}{2} + w_i^k \right) f_i + \left( \frac{1}{2} + w_i^k \right) f_{i+1} - w_i^k f_{i+2} \]

- Non-uniform scheme over a uniform parametrization.
- Non-stationary.
- Sufficient \( C^1 \) condition: \( 0 < \epsilon < w_i^k < \frac{1}{8} \).
Non-Uniform Four Point Scheme


\[ w_i := f(g(i)) \text{ with } e_i := |f_{i+1} - f_i|, \]

\[ g(i) := \begin{cases} \frac{3|e_i|}{|e_{i-1}| + |e_i| + |e_{i+1}|} & \text{if } |e_{i-1}| + |e_i| + |e_{i+1}| > 0, \\ 0 & \text{otherwise} \end{cases} \]

otherwise \( g(i)=0 \), and with

\[ f(x) = \begin{cases} W x, & 0 \leq x \leq 1, \\ W \text{ or } W \frac{3-x}{2}, & 1 < x \leq 3, \end{cases} \]
Non-Uniform Four Point Scheme

Properties:
- A “Local Chordal Parametrization” for parametric curves.
- Non-Linear.
- “Artifact-Free”
- No “corners” (i.e., no edges of zero length)
- Produces $C^0$ curves

Modifications: A modification is given that will guarantee $C^1$ limit curves by bounding the weights away from zero, i.e., $w_i \in [\epsilon, \frac{1}{8} - \epsilon]$ for some $\epsilon > 0$. The authors also derive a convexity preserving scheme.
Non-Uniform Four Point Scheme

Uniform [Dubuc 86]:

Non-uniform [MDL04]:

(Example from [MDL04]).
Non-Uniform Four Point Scheme

Another example: by [MDL04] method.
Global Variational Subdivision

Solve:

\[
\min_{f_i} \{ E(f) : f_{2i} = f_i \} \quad \text{with} \quad E(f) := \sum_i \left| \sum_{j=0}^{k} \alpha_{ij} f_{i+j} \right|^2.
\]

General \( C^k \) sufficient condition.

**Theorem 0.1** If \( \sum_{k=0}^{\infty} 2^{mk} \Delta^{k+l} f_k(t) \|_{\infty} < \infty \) for some \( l \in \mathbb{N} \) then the \( f^k \xrightarrow{u} f \) for some \( f \in C^m \).
Forward-difference scheme (Second difference below):

\[ E(f) := \sum_i |\Delta^2_i f|^2. \]

Euler-Lagrange Equation:

\[ \Delta^4 f_{2l-1}^{k+1} = 0, \quad l=1:n. \]

Note: These are not ‘divided differences’. That is, the parameter spacing is not involved, it is assumed to be uniform.
Recall the uniform four point scheme of Dubuc with the uniform variational difference scheme of Kobbelt.
Why Non-Uniform?

Now compare the uniform variational subdivision with the non-uniform variational scheme to be described. We can control the shape by the parametrization.

Uniform \((t_i = 0, .25, .5, .75, 1)\) and non-uniform \((t_i = 0, .1416, .5, .8584, 1)\) parametrizations.
Let $f(t) = \sum_i f_i N_i(t)$ be a piecewise linear spline curve with coefficients $f_i$ and knots $t_1, t_2, \ldots$, such that $f(t_j) = f_j$. Let

$$h_i := t_{i+1} - t_i$$
$$h_{i,2} := t_{i+2} - t_i = h_i + h_{i+1}$$
$$h_{i,3} := t_{i+3} - t_i = h_i + h_{i+1} + h_{i+2}$$

Possible choices of $t_i$:

- **Uniform**: $h_i = \text{constant}$
- **Chordal**: $h_i = |f_{i+1} - f_i|$
- **Centripetal**: $h_i = |f_{i+1} - f_i|^e$
Consider the following discretization of the thin beam functional:

\[
E(f) := \sum_i |s_i|^2 h_{i-1,2} \approx \int_a^b |f''(t)|^2 \, dt
\]

Variational Problem:

\[
\text{minimize}\{E(f) : f_{2i}^{k+1} = f_i^k\}
\]

Optimality condition (jump in third divided difference):

\[
\text{jmp}_{t_i}(\Delta^3 f) := \frac{h_{i-1,3}}{h_i} \Delta_{i-1,3} f - \frac{h_{i-2,3}}{h_{i-1}} \Delta_{i-2,3} f = 0.
\]
Advantages:
- Parametrization is built in to the scheme.
- Standard parametrizations can be used to control the shape of the curves.

Disadvantages:
- Extra computation needed to compute the ‘masks’.
- The scheme is still global.
Assumptions:

- A1: \((f^k)\) is sequence PWL curves minimizing \(E(f)\)
- A2: \(\max\{h^{k+1}_{2i-1}, h^{k+1}_{2i}\} \leq \alpha h^k_i\) for some \(1/2 \leq \alpha < 1\).
- A3: \(E(f^k)\) is uniformly bounded as \(k \to \infty\).
- A4: \(|s^k_i|^2 h^k_{i,2}\) are uniformly bounded for all \(i\) as \(k \to \infty\).
- A5: \(|s^k_i|\) are uniformly bounded for all \(i\) as \(k \to \infty\).
- A6: \(|v^k_i|\) are uniformly bounded for all \(i\) as \(k \to \infty\).
- A7: \(|\text{jmp}_{t_i}(\Delta^3 f^k)|\) are uniformly bounded for all \(i\) as \(k \to \infty\), with \(\text{jmp}_{t_i}(\Delta^3 f) := h_{i-1,3} v_{i-1} / h_i - h_{i-2,3} v_{i-2} / h_{i-1}\).
- A8: \(\frac{h^k_i h^k_{i+1}}{h^k_{i-2} h^k_{i-1}} \to 1\) as \(k \to \infty\).
Corollary 0.1 Assume $A1$, $A2$ and $A3$. Then
$f^k \longrightarrow u f \in C^0([a, b])$.

Corollary 0.1 Assume $A1$, $A2$ and $A3$. Then $f^k \longrightarrow u f$
for some $f \in C^1([a, b])$ with $f' = d$.

Corollary 0.1 Assume either $A1$, $A2$ and $A6$, or $A1$, $A2$, $A7$ and $A8$. Then, $f^k \longrightarrow u f$ for some $f \in C^2$.

- Proofs depend on extension of Gregory and Qu Lemma given earlier.
- Uniform boundedness of $s_i$ causes contraction of $d(t)$ to get $C^1$. 
(Current work)

Approximation Order: Suppose that variational scheme reproduces $C^k$ curves for $k \leq 3$. Then the approximation order is $O(h^k)$.

Proof based on linearity of the scheme and reproduction of polynomials.

\[ E(f) := \sum_{i} |\alpha_i s_i|^2 h_{i-1,2} \approx \int_{a}^{b} |\alpha(t) f''(t)|^2 \, dt \]
Local Variational Subdivision
Why Local?

Dubuc’s four point scheme and Kobbelt’s variational difference scheme.

$C^1$ non-regular curves, with cusps.
Why Local?

- For both schemes the number of points grows exponentially. That is \( n = 2^{\text{level}} n_0 \).

- Both schemes are \( O(n) = O(2^{\text{level}} n_0) \). (The variational solves a 5 banded matrix.)

- The variational scheme is global – all vertices are needed to compute the value at one point. The four point scheme is local.

- "Big Advantage": The 4-pt scheme can evaluate at any non-dyadic point to machine tolerance (only a support of 6 points is needed to do this).

Idea: Local variational subdivision (current work).
Ideas (current work):

1. Insert each new point $f_{2i+1}$ in the sequence, one at a time, and minimize energy functional $E_i(f)$. This generates a four-point scheme.

2. Insert three new points $f_{2i-1}, f_{2i+1}, f_{2i+3}$ to determine $f_{2i+1}$. This generates a six-point scheme.

3. To determine $f_i$ minimize the local energy functional

$$E_i(f) := \sum_{i=j-k}^{j+k} |s_i|^2 h_{i-1,2}.$$ 

As $k \to \infty$ this approximates the global functional.
1. Insert each new point $f_{2i+1}$, one point at a time:

2. For each: minimize $E_i(f) := \sum_i |s_i|^2 h_{i-1,2}$

3. Optimality:

$$-\frac{h_{i-1,2}}{h_{2i}} f[t_{i-1}, t_i, t_{2i+1}, t_{i+1}] + \frac{h_{i,2}}{h_{2i+1}} f[t_i, t_{2i+1}, t_{i+1}, t_{i+2}] = \ldots$$
Variational four-point scheme (K05)

**Theorem 0.1** Suppose that \( \frac{h_{i-1,2}}{h_{i,2}} = \frac{h_{2i+1}}{h_{2i}} \) for all \( i \). If \( \partial f_{2i+1} E_i(f) = 0 \), then

\[
\begin{align*}
  f_{2i+1} &= -w_{i-1} f_{i-1}(w_{i-1} + \frac{h_{2i+1}}{h_i}) f_i + (w_i + \frac{h_{2i}}{h_i}) f_{i+1} - w_i f_i \\
\text{with} \\
  w_{i-1} &= \frac{h_{2i} h_{2i+1}^2}{h_{i-1} h_{i-1,2} h_i} \quad \text{and} \quad w_i = \frac{h_{2i} h_{2i+1}}{h_{i,2} h_i h_{i+1}}
\end{align*}
\]
Variational four-point scheme (K05)

Stretching (Interval Tension):
\[ h_{i-1} := \alpha_i h_{i-1}, \quad h_{i+1} := \alpha_i h_{i+1} \]

Centripetal Parametrization: \[ h_i := |f_{i+1} - f_i|^e \]
Tension scaled by length of segment.
Variational four-point scheme (K05)
Variational four-point scheme (K05)

e=0.2, alpha==1
Variational four-point scheme (K05)

e=0.2, alpha 1.1*Centripetal
Variational six-point scheme (K05)

Global, 4-pt and 6-pt Subdivision:

Idea: Insert Three points at a time to determine the middle point (Solve 3x3 systems).
Variational six-point scheme (K05)

e=.2, alpha = uniform
Variational six-point scheme (K05)

$e=.2, \alpha = 1.1 \times \text{Centripetal}$
Discretized Hölder Regularity

Lipschitz Continuous function:

\[ |f(x) - f(y)| \leq C|x - y|, \quad \omega_f(h) \leq C \cdot h \]

Hölder Regularity \( H_{l+\alpha} \):

\[ |f^{(l)}(x) - f^{(l)}(y)| \leq C|x - y|^\alpha, \quad \omega_f^{(l)}(h) \leq C \cdot h^\alpha \]

Discretization:

\[ |l! \Delta^l f_{i+1}^k - l! \Delta^l f_i^k| \leq C \cdot |h_i^k|^\alpha \]
Discretized Hölder Regularity

As in Kuijt et al., let

$$
\rho_i^k := l! |\Delta^l f_{i+1}^k - \Delta^l f_i^k|.
$$

Then

$$
\frac{\rho_i^{k+1}}{\rho_i^k} \approx \frac{C(h_i^{k+1})^{\alpha_i}}{C(h_i^k)^{\alpha_i}} = \left( \frac{h_i^{k+1}}{h_i^k} \right)^{\alpha_i},
$$

and so

$$
\alpha_i \approx \log \left( \frac{\rho_i^{k+1}}{\rho_i^k} \right) / \log \left( \frac{h_i^{k+1}}{h_i^k} \right).
$$

Use arc length parametrization (a regular parametrization).
Discretized Hölder Regularity

Smoothness of Non-Uniform Local Variational Schemes.
Based on ‘Florida’ example shown earlier.

<table>
<thead>
<tr>
<th>Method</th>
<th>Tension $\alpha$</th>
<th>Parametrization</th>
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<tr>
<td>MDL04</td>
<td>DNA</td>
<td>Chordal $e=1$</td>
<td>1.9</td>
</tr>
<tr>
<td>Four-Point</td>
<td>Uniform</td>
<td>Initially Chordal $e=1$</td>
<td>1.9</td>
</tr>
<tr>
<td>Four-Point</td>
<td>1.1*Centripetal</td>
<td>Centripetal $e=.2$</td>
<td>1.3</td>
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<td>Six-Point</td>
<td>Uniform</td>
<td>Initially Centripetal $e=.2$</td>
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<tr>
<td>Six-Point</td>
<td>1.1*Centripetal</td>
<td>Centripetal $e=.2$</td>
<td>1.3</td>
</tr>
</tbody>
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Part VI

Variational Subdivision Surfaces
The *thin-plate spline* minimizes the functional:

\[
\int \int f_{uu}^2 + 2 f_{uv}^2 + f_{vv}^2 \, dA,
\]

Discretization functional over surfaces:

\[
E(p) := \sum_i \frac{1}{m_i} \sum_{j=1}^{m_i} \frac{1}{m_{ij}} \sum_{k=1}^{m_{ij}} |p_{ijk} - 2p_{ij} + p_i|^2.
\]

Variational problem:

\[
\text{minimize}_p \left\{ E(p) : p_i = q_i \text{ for } i \in I_1 \right\}.
\]
Optimality (Laplacian smoothing): For all $i \notin I_1$,

$$p_i = p_i - \frac{1}{\nu_i} U^2(p_i)$$

with

$$U(p_i) := \frac{1}{m_i} \sum_{j=1}^{m_i} p_{ij} - p_i,$$

$$U^2(p_i) := \frac{1}{m_i} \sum_{j=1}^{m_i} U(p_{ij}) - U(p_i) = 0,$$

$$\nu_i := 1 + \frac{1}{m_i} \sum_{j} \frac{1}{m_{ij}}.$$
Variational Surfaces – Kobbelt

Figure 0.0: Interpolated Mannequin Head Data
Let \( \{I_1, I_2, I_3\} \) be a partition of the index set \( i=1:n \), corresponding to those vertices that are interpolated, smoothed/near-interpolated, or unconstrained, respectively.

**Mixed smoothing and interpolation:**

\[
\text{minimize}_p \ \{ E(p) + \sum_{i \in I_2} w_i |p_i - q_i|^2 : p_i = q_i \text{ for } i \in I_1 \} ,
\]

**Near-interpolation:**

\[
\text{minimize}_p \ \{ E(p) : |p_i - q_i| \leq e_i \text{ for } i \in I_2, \ p_i = q_i \text{ for } i \in I_1 \} .
\]

Mixed smoothing and interpolation:

\[ \text{minimize}_p \left\{ E(p) + \sum_{i \in I_2} w_i |p_i - q_i|^2 : p_i = q_i \text{ for } i \in I_1 \right\}. \]

Optimality:

\[
\begin{align*}
  p_i &= q_i, \text{ for } i \in I_1 \\
  U^2(p_i) + w_i (p_i - q_i) &= 0 \text{ for } i \in I_2 \\
  U^2(p_i) &= 0 \text{ for } i \in I_3
\end{align*}
\]
Mixed smoothing and interpolation:

\[
\min_p \{ E(p) + \sum_{i \in I_2} w_i |p_i - q_i|^2 : p_i = q_i \text{ for } i \in I_1 \}.
\]

Iteration:

\[
p_i = q_i, \text{ for } i \in I_1
\]

\[
p_i \leftarrow \frac{1}{\nu_i + w_i} \left( w_i q_i + \nu_i p_i - U^2(p_i) \right), \text{ for } i \in I_2
\]

\[
p_i \leftarrow p_i - \frac{1}{\nu_i} U^2(p_i), \text{ for } i \in I_3.
\]
Near-interpolation:

\[
\text{minimize}_p \{E(p) : |p_i - q_i| \leq e_i \text{ for } i \in I_2, p_i = q_i \text{ for } i \in I_1\}
\]

Iteration:

\[
p_i = q_i, \text{ for } i \in I_1
\]

\[
p_i \leftarrow \frac{1}{\nu_i + w_i} (w_i q_i + \nu_i p_i - U^2(p_i)), \text{ for } i \in I_2
\]

\[
p_i \leftarrow p_i - \frac{1}{\nu_i} U^2(p_i), \text{ for } i \in I_3
\]

\[
w_i \leftarrow \frac{|p_i - q_i|}{e_i} w_i, \text{ for } i \in I_2.
\]
Figure 0.1: Interpolated Noisy Torus
Examples

Figure 0.1: Near-Interpolated Noisy Torus
Examples

Figure 0.1: Interpolated Noisy Mannequin Head Data
Examples

Figure 0.1: Near-Interpolated Noisy Mannequin Head Data
The End