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Toric surface patches

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We define a toric surface patch associated with a convex polygon, which has vertices with integer coordinates. This rational surface patch naturally generalizes classical Bézier surfaces. Several features of toric patches are considered: affine invariance, convex hull property, boundary curves, implicit degree and singular points. The method of subdivision into tensor product surfaces is introduced. Fundamentals of a multidimensional variant of this theory are also developed.

Keywords: Bézier surfaces, toric surfaces, geometric modeling

1. Introduction

Toric varieties were introduced in the early 1970's in algebraic geometry. This remarkable theory appeared to be quite close to combinatorics of convex polytopes and therefore much more elementary than other parts of the sophisticated building of algebraic geometry. This simplification makes the theory of toric varieties very attractive for different kind of applications [2].

In Computer Aided Geometric Design Bézier surfaces play the central role. From the viewpoint of algebraic geometry tensor product Bézier surfaces and Bézier triangles are projections of Segre and Veronese surface patches from higher-dimensional space, which are just the two simplest cases of real projective toric surfaces.

Probably J. Warren [12] was the first who noticed that other real toric surfaces can be used in CAGD. In particular he considered a rational Bézier triangular surface with zero weights at appropriate control points located near its corners and obtained a hexagonal patch. J. Warren [12] also predicted that “Further work incorporating techniques from toric variety theory [...] may lead to practical methods for rendering, subdividing and meshing patches with seven or more sides”. Here we present the first results in this direction.

Several traditional definitions of toric varieties are not so satisfactory from the CAGD point of view: some of them are much too abstract, others involve numerically unstable limit procedures. We propose a definition based on the concept of global coordinates [1,2] and on recent ideas in the theory of multisided patches [8].

In section 2 we give a definition of a toric surface patch and show that Bézier surfaces are just particular cases corresponding to very special lattice triangles and rec-

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tangles. The main properties of toric patches and several examples are considered. Some technical proofs are postponed to section 5. Section 3 provides a method for subdivision of a toric patch into smaller tensor product patches. Section 4 is devoted to the definition of real projective toric varieties of arbitrary dimension via global coordinates. Also a detailed analysis of its non-negative part is presented. In section 5 we introduce a concept of Bézier polytope, which develops a multidimensional variant of the theory. Conclusions and future work are discussed in section 6.

2. Parametrization of a toric patch

2.1. *Definition*

Consider a *lattice* $\mathbb{Z}^2$ of points with integer coordinates in the real affine plane $\mathbb{R}^2$. We call a convex polygon $\Delta \subset \mathbb{R}^2$ a *lattice polygon* if its vertices are in the lattice $\mathbb{Z}^2$. Let edges ϕ_i of Δ define lines $h_i(t) = 0$, $i = 1, \dots, r$. Unique affine linear forms $h_i(t) = \langle \nu_i, t \rangle + a_i$ are defined provided that two additional conditions are satisfied: (i) the normal vectors ν_i are inward oriented; (ii) ν_i are primitive lattice vectors, i.e., they are the shortest vectors in this direction with integer coordinates. Denote by $\widehat{\Delta} = \Delta \cap \mathbb{Z}^2$ the set of lattice points of the polygon Δ. It is easy to see that $h_i(m)$ is non-negative integer for all $i = 1, \dots, r$ and $m \in \widehat{\Delta}$.

Definition 1. A *toric surface patch* associated with a lattice polygon Δ is a piece of an algebraic surface parametrized by the rational map $\mathcal{B}_\Delta : \Delta \to \mathbb{R}^n$

$$\mathcal{B}_\Delta(t) = \frac{\sum_{m\in\widehat{\Delta}} w_m p_m F_m(t)}{\sum_{m\in\widehat{\Delta}} w_m F_m(t)}, \quad F_m(t) = c_m h_1(t)^{h_1(m)} \cdots h_r(t)^{h_r(m)}, \tag{1}$$

with control points $p_m \in \mathbb{R}^3$ and weights $w_m > 0$ indexed by lattice points $m \in \widehat{\Delta}$. $F_m(t)$ are called *basis functions* and integers $c_m > 0$ are their *coefficients*.

At this moment we do not fix all the coefficients c_m of the basis functions $F_m(t)$, as they can vary from case to case. Bézier surfaces are particular cases of toric surface patches with the special coefficients c_m.

Example 2.

(i) Let $\Delta = \Delta_k$ be a triangle with vertices $(0,0)$, $(k,0)$ and $(0,k)$. Then $\widehat{\Delta}_k$ consists of all non-negative integer pairs (i,j) such that $i + j \leqslant k$. Boundary lines define three linear forms $h_1(t_1,t_2) = t_1$, $h_2(t_1,t_2) = t_2$ and $h_3(t_1,t_2) = k - t_1 - t_2$. Choosing $c_{(i,j)} = k!/(i!j!(k-i-j)!)$ we get the basis functions

$$F_{(i,j)}(t) = \frac{k!}{i!j!(k-i-j)!} t_1^i t_2^j (k - t_1 - t_2)^{k-i-j}, \quad (i,j) \in \widehat{\Delta}_k. \tag{2}$$

Hence $\mathcal{B}_{\Delta_k}$ becomes a rational Bézier triangle of degree k after the simple reparametrization $\tau_1 = t_1/k$, $\tau_2 = t_2/k$.

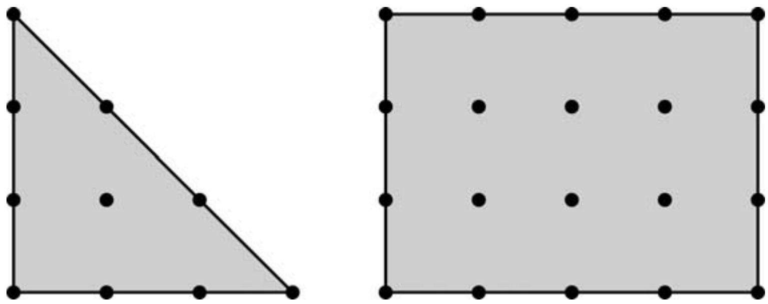

Figure 1. Lattice polygons $\triangle_3$ and $\square_{4,3}$.

(ii) Let Δ be a rectangle $\square_{k,l}$ with four vertices $(0,0)$, $(k,0)$, (k,l) and $(0,l)$. Boundary lines define four affine forms $h_1(t) = t_1$, $h_2(t) = t_2$, $h_3(t) = k - t_1$ and $h_4(t) = l - t_2$. If $c_{(i,j)} = \binom{k}{i}\binom{l}{j}$ then the basis functions are

$$F_{(i,j)}(t) = \binom{k}{i} t_1^i (k - t_1)^{k-i} \binom{l}{j} t_2^j (l - t_2)^{l-j}, \qquad (3)$$

where $i = 0, \ldots, k$, $j = 0, \ldots, l$. Hence $\mathcal{B}_{\square_{k,l}}$ becomes a rational tensor product surface of bidegree (k,l) after the reparametrization $\tau_1 = t_1/k$, $\tau_2 = t_2/l$.

2.2. *Main properties*

Toric surface patches share many properties with Bézier surfaces. In some formulas below it will be convenient to indicate control points and weights directly in the notation of a toric patch $\mathcal{B}_\Delta^{p,w}$, where p and w are maps $p : \widehat{\Delta} \to \mathbb{R}^n$, $m \mapsto p_m$ and $w : \widehat{\Delta} \to \mathbb{R}$, $m \mapsto w_m > 0$.

T1: *Affine invariance*: $A \circ \mathcal{B}_\Delta^{p,w} = \mathcal{B}_\Delta^{A\circ p,w}$, if A is an affine transformation of $\mathbb{R}^n$, i.e., a transformed patch has transformed control points $A(p_m)$.

T2: *Convex hull property*. The patch $\mathcal{B}_\Delta(\Delta)$ as subset in $\mathbb{R}^n$ is contained in the convex hull of its control points $\mathrm{Conv}\{p_m \mid m \in \widehat{\Delta}\}$.

Proof. Properties T1 and T2 follow directly from definition 1, since the control points p_m come with coefficients which sum to 1 and are non-negative when $t \in \Delta$. $\square$

If an affine transformation L of $\mathbb{R}^2$ preserves the lattice $\mathbb{Z}^2$, i.e., $L(\mathbb{Z}^2) = \mathbb{Z}^2$, then it is called an *affine unimodular transformation*. It is easy to see that L is a composition of some translation by a lattice vector and a linear transformation that has a matrix with integer entries and determinant ± 1. We denote by $e_1 = (1,0)$ and $e_2 = (0,1)$ the standard basis vectors in $\mathbb{Z}^2$.

Example 3. Let a unimodular linear transformation L is defined on the basis vectors by the formulas $L(e_1) = e_2$, $L(e_2) = -e_1 - e_2$, and let $C = \mathrm{Conv}\{e_1, e_2, -e_1\}$ and

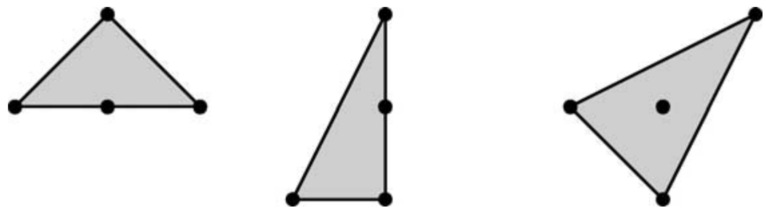

Figure 2. Lattice polygons C, $L(C)$ and $S = L(S)$.

$S = \mathrm{Conv}\{e_1 + e_2, -e_1, -e_2\}$ are two lattice triangles. From their transformations $L(C) = \mathrm{Conv}\{e_2, -e_1 - e_2, -e_2\}$ and $L(S) = S$ in figure 2 we see at once that euclidean distances between vertices are not preserved. The triangle S can be called *equilateral* in unimodular sense, because the transformation L permutes its vertices in a cyclic fashion.

Now we can formulate the property which is in some sense similar to affine invariance of the domain for Bézier surfaces.

T3: *Unimodular invariance of the domain.* If two toric patches are associated with lattice polygons that are related via an affine unimodular transformation $L(\Delta') = \Delta$, then

$$\mathcal{B}_{\Delta}^{p,w} \circ L = \mathcal{B}_{\Delta'}^{p \circ L, w \circ L}, \tag{4}$$

i.e., they are just reparametrizations of each other.

Proof. It is easy to see that L preserves inward orientation and primitivity properties of normal vectors v_i. Therefore $h_i(L(t)) = h_i'(t)$, for affine linear forms h_i and h_i' associated with Δ and Δ' respectively. Hence $F_m(L(t)) = F_{L(m)}(t)$ and formula (4) follows. □

Suppose edges ϕ_i, $i = 1, \ldots, r$, of the lattice polygon Δ are ordered counterclockwise and let v_i, $i = 1, \ldots, r$, be vertex of Δ where two edges ϕ_i and ϕ_{i+1} meet. The indices will be treated in a cyclic fashion: for instance, $\phi_{r+1} = \phi_1$, $\phi_0 = \phi_r$ and so on. For every edge ϕ_i we define its primitive directional vector

$$f_i = (v_i - v_{i-1})/l(i), \quad i = 1, \ldots, r, \tag{5}$$

where $l(i)$ is an *integer length* of ϕ_i, i.e., $l(i) + 1$ is the number of points in $\widehat{\phi_i}$. In order to satisfy property T4 below we need to choose the boundary coefficients c_m. For every edge ϕ_i we label the set $\widehat{\phi_i}$ in the natural order $m(j) = v_{i-1} + jf_i \in \widehat{\phi}$, and define $c_{m(j)} = \binom{l(i)}{j}$, $j = 0, \ldots, l(i)$.

T4: *Boundary property.* The boundary of the patch consists of rational Bézier curves $\mathcal{B}_i$, $i = 1, \ldots, r$, defined by control points p_m and weights w_m indexed by lattice points $m \in \widehat{\phi_i}$ of corresponding edges $\phi_i \subset \Delta$. In particular, $\deg \mathcal{B}_i = l(i)$ and the corner control points lie on the patch. Every $\mathcal{B}_i$ is obtained by some 1–1 reparametrization of the restricted map $\mathcal{B}_\Delta|_{\phi_i}$.

Proof. Consider a restriction $\mathcal{B}_\Delta|_\phi$ of the map $\mathcal{B}_\Delta$ to the fixed edge $\phi = \phi_i$. Denote $v = v_{i-1}$, $f = f_i$ and $l = l(i)$ for simplicity. All basis functions $F_m(t)$ with indices $m \in \widehat{\Delta} \setminus \widehat{\phi}$ will vanish, since they contain a zero term $h_i(t)^{h_i(m)} = 0$, $h_i(m) \neq 0$. Hence $\mathcal{B}_\Delta|_\phi$ depends only on control points and weights indexed by $m(j) \in \widehat{\phi}$. We evaluate h_k on lattice points $m(j) = v + jf$ (see (5)): $h_k(m(j)) = h_k(v) + j\langle \nu_k, f\rangle$. Basis functions on the edge ϕ can be expressed as follows

$$F_{v+jf}(t) = \binom{l}{j} h_1(t)^{h_1(v)} \cdots h_r(t)^{h_r(v)} \bigl(h_1(t)^{\langle \nu_1, f\rangle} \cdots h_r(t)^{\langle \nu_r, f\rangle}\bigr)^j.$$

Here the first r factors $h_k(t)^{h_k(v)}$ do not depend on j and can be canceled in formula (1). Hence we get monomial and Bézier forms of the patch by introducing new variables

$$s = h_1(t)^{\langle \nu_1, f\rangle} \cdots h_r(t)^{\langle \nu_r, f\rangle}, \quad u = s/(1+s). \tag{6}$$

Indeed,

$$\mathcal{B}_\Delta|_{\phi_k}(t) = \frac{\sum_{j=0}^{l} w_{m(j)} p_{m(j)} \binom{l}{j} s^j}{\sum_{j=0}^{l} w_{m(j)} \binom{l}{j} s^j} = \frac{\sum_{j=0}^{l} w_{m(j)} p_{m(j)} \binom{l}{j} (1-u)^{l-j} u^j}{\sum_{j=0}^{l} w_{m(j)} \binom{l}{j} (1-u)^{l-j} u^j}.$$

In order to prove that this reparametrization is 1–1 we choose a natural parameter τ on the edge, $t = v + \tau l f \in \phi$, and calculate derivatives

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}\tau} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\bigl(h_1(t)^{\langle \nu_1, f\rangle}\bigr) \cdots h_r(t)^{\langle \nu_r, f\rangle} + \cdots + h_1(t)^{\langle \nu_1, f\rangle} \cdots \frac{\mathrm{d}}{\mathrm{d}\tau}\bigl(h_r(t)^{\langle \nu_r, f\rangle}\bigr) \\ &= l \prod_{k=1}^{r} h_k(t)^{\langle \nu_k, v\rangle} \sum_{j=1}^{r} \frac{\langle \nu_j, f\rangle^2}{h_j(t)} > 0, \quad 0 < \tau < 1,\end{aligned}$$

and

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{s}{1+s}\right) = \frac{\mathrm{d}s/\mathrm{d}\tau}{(1+s^2)^2} > 0.$$

Hence the reparametrization $\tau \mapsto u$ is monotonic. Also it is easy to check that it preserves endpoints. Therefore it is 1–1. □

Using the notation of the toric patch (1), we define a rational map in the monomial form $\mathcal{M}_\Delta : \mathrm{Int}\ \Delta \to \mathbb{R}^n$

$$\mathcal{M}_\Delta(s_1, s_2) = \frac{\sum_{m(i,j) \in \widehat{\Delta}} w_{m(i,j)} c_{m(i,j)} p_{m(i,j)} s_1^i s_2^j}{\sum_{m(i,j) \in \widehat{\Delta}} w_{m(i,j)} c_{m(i,j)} s_1^i s_2^j}, \tag{7}$$

where lattice points $m(i, j) = m_0 + ie_1 + je_2$ are expressed in the standard basis $\{e_1, e_2\}$ for any fixed $m_0 \in \mathbb{Z}^2$.

T5: *Monomial parametrization.* There exists a 1–1 reparametrization (in fact an analytic isomorphism) $\mathcal{R} : \mathrm{Int}\ \Delta \to \mathbb{R}_+^2$ such that $\mathcal{B}_\Delta|_{\mathrm{Int}\ \Delta} = \mathcal{M}_\Delta \circ \mathcal{R}$.

Proof. At first we evaluate h_k, $k = 1, \dots, r$, on lattice points $h_k(m(i, j)) = h_k(m_0) + i\langle \nu_k, e_1\rangle + j\langle \nu_k, e_2\rangle$ and express the basis functions in monomial form

$$F_{m(i,j)}(t) = c_{m(i,j)} h_1(t)^{h_1(m_0)} \cdots h_r(t)^{h_r(m_0)} s_1^i s_2^j$$

with

$$s_1 = h_1(t)^{\langle \nu_1, e_1\rangle} \cdots h_r(t)^{\langle \nu_r, e_1\rangle}, \qquad s_2 = h_1(t)^{\langle \nu_1, e_2\rangle} \cdots h_r(t)^{\langle \nu_r, e_2\rangle}. \tag{8}$$

After substitution of this formula into (1) the factor $h_1(t)^{h_1(m_0)} \cdots h_r(t)^{h_r(m_0)}$ cancels and we have exactly (7). Therefore we define $\mathcal{R}: \mathrm{Int}\ \Delta \to \mathbb{R}^2_+$, $t \mapsto (s_1, s_2)$. The proof that this map is an analytic isomorphism follows from more general lemma 21 in section 5. □

Corollary 4. Warren's polygonal surfaces [12,13] are reparametrized toric patches.

Proof. Consider a Bézier triangular surface of degree k with some zero weights, such that the corresponding lattice triangle $\triangle_k$ contains the inscribed lattice polygon $\Delta = \mathrm{Conv}\{m \in \widehat{\triangle}_k \mid w_m > 0\}$. Using property T5 we can reparametrize the Bézier triangle and $\mathcal{B}_\Delta$ to the monomial form $\mathcal{M}_\Delta$ via $\mathcal{R}: \mathrm{Int}\ \Delta \to \mathbb{R}^2_+$. Then we get the Bézier triangle after the simple projective transformation

$$\mathbb{R}^2_+ \to \triangle_k, \qquad (s_1, s_2) \mapsto \left(\frac{ks_1}{1 + s_1 + s_2}, \frac{ks_2}{1 + s_1 + s_2}\right).$$

See also example 6(ii) (section 3). □

An affine unimodular transformation L preserves area, since $\det L = \pm 1$. Here we use the so-called *normalized area* Vol_2 which is twice as large as than standard area in $\mathbb{R}^2$. $\mathrm{Vol}_2 \Delta$ is an integer for every lattice polygon Δ, as is easy to check. This number is tightly related with implicit degree of a toric surface.

T6: *Implicit degree*. The implicit degree of an algebraic surface corresponding to a toric patch $\mathcal{B}_\Delta(\Delta)$ does not exceed $\mathrm{Vol}_2(\Delta)$. It is equal to $\mathrm{Vol}_2(\Delta)$ when the control points are in general position.

Proof. Corollary 4 allows us to refer the reader to a relatively elementary proof in [13, theorem 4]. Also this is a particular case of theorem 24. □

From figure 1 we see that $\mathrm{Vol}_2 \triangle_k = k^2$ and $\mathrm{Vol}_2 \square_{k,l} = 2kl$. These numbers coincide with well-known estimates for the implicit degree of Bézier triangles and tensor product surfaces [6]. Note that implicit equations of several low-degree toric patches are calculated in [14], where monomial parametrizations are used.

Let $\mathcal{B}^{\mathrm{id}}_\Delta$ be a toric surface patch with control points $p_m = m$, $m \in \widehat{\Delta}$ and some weights. This defines a rational map from Δ to itself. In case of Bézier surfaces with appropriate coefficients c_m (see example 2) and unit weights we get the identity map. This is the so-called linear precision property. In the general toric case we have a weaker analog of this property, which is natural to call an *analytic precision property*.

T7: *Analytic precision*. Let all weights $w_m \geqslant 0$ and $w_m > 0$ for the corner points $m \in \widehat{\Delta}$. Then $\mathcal{B}^{\mathrm{id}}_{\Delta} : \Delta \to \Delta$ defines a 1–1 map which is an analytic isomorphism on subsets: Int Δ and Int ϕ, for all edges $\phi \subset \Delta$.

Proof. This is a particular case of a more general theorem 25, which is proved in section 5.3. □

On a toric surface patch $\mathcal{B}_{\Delta}$ singular points can occur at corners points. Consider a lattice triangle Θ_i with vertices in v_i and the two nearest lattice points on the adjacent boundary edges, i.e., $v_i + f_i$ and $v_i - f_{i-1}$ according to the notation (5). We call Θ_i a *corner triangle*.

T8: *Singular points*. A corner point corresponding to a vertex v_i is non-singular if and only if the corner triangle Θ_i has unit area $\mathrm{Vol}_2(\Theta_i) = 1$.

See theorem 13 in section 3.5 for the proof and more details.

2.3. Examples

Simple examples of lattice polygons having normalized area less than or equal to 3 are shown in figure 3. In fact this is a complete list: any other lattice polygon with this property will be unimodular equivalent to one of these polygons. Also they are not equivalent to each other, since they have different area or different number of singular points, which are specially marked in figure 3. In the first row we see $\triangle_1$, $\square_{1,1}$ and $C_2 = \mathrm{Conv}\{e_1, e_2, -e_1\}$. The associated toric patches are pieces of the following surfaces: plane, double ruled quadric and quadratic cone. Lattice polygons on the second row H, S, C_3 correspond to three kinds of cubic surfaces: a kind of Hirzebruch surface, a cubic with 3 lines intersecting in 3 singular points, and a cone over a rational cubic.

The first polygon in figure 4 is a lattice square $D = \mathrm{Conv}\{e_1, e_2, -e_1, -e_2\}$ with area $\mathrm{Vol}_2(D) = 4$. The corresponding full quartic surface is shown in figure 5. This surface is in the form of a pillow with 'antennas', and has 4 lines intersecting in 4 singular

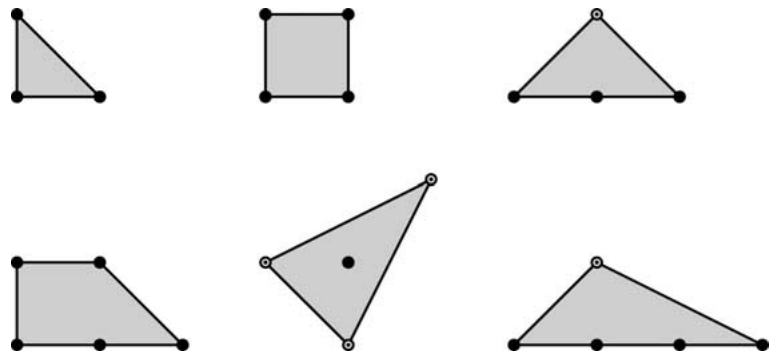

Figure 3. Lattice polygons with $\mathrm{Vol}_2 \leqslant 3$.

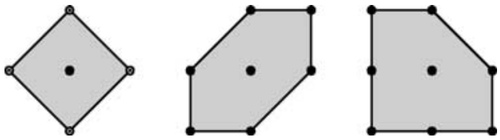

Figure 4. Lattice polygons D, W_1 and Z.

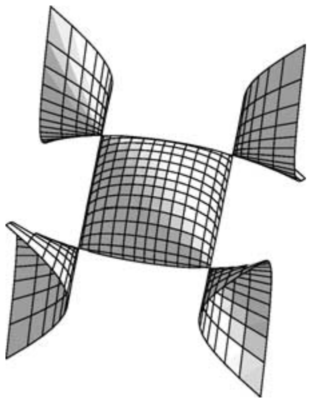

Figure 5. A full toric surface associated with lattice polygon D.

points. This quartic appears in the context of Laguerre geometry as a bisector of two cylinders [9].

The hexagon W_1 and the pentagon Z in figure 4 define Warren's hexagon $\mathcal{B}_{W_1}$ [12] and Zubė's pentagon $\mathcal{B}_Z$ [14].

3. Subdivision and singular points

3.1. Homogeneous control points

As usual, we represent points in real projective space $\mathbb{R}P^n$ via homogeneous coordinates using the natural projection

$$\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n, \quad (x_0, \dots, x_n) \mapsto [x_0, \dots, x_n]. \tag{9}$$

We call elements of $\mathbb{R}^{n+1}$ *homogeneous points* and denote them by underlined letters. Any map of the type $F : X \to \mathbb{R}P^n$ we usually define via its homogeneous form $\underline{F} : X \to \mathbb{R}^{n+1}$, i.e., $F = \pi \circ \underline{F}$.

Affine space $\mathbb{R}A^n$ and its associated vector space $\mathbb{R}V^n$ can be identified with subsets $\{x_0 = 1\}$ and $\{x_0 = 0\}$ in $\mathbb{R}^{n+1}$. Usually both spaces $\mathbb{R}V^n$ and $\mathbb{R}A^n$ we denote by $\mathbb{R}^n$ when the meaning is clear from the context. Elements of the complement $\mathbb{R}^{n+1} \setminus \mathbb{R}V^n$ are treated as points with weights $\underline{p} = (w, wp)$, $p \in \mathbb{R}^n$, $w \neq 0$. Then $\pi(w, wp) = (1, p)$ and π defines the central projection

$$\pi : \mathbb{R}^{n+1} \setminus \mathbb{R}V^n \to \mathbb{R}A^n \subset \mathbb{R}P^n, \quad (x_0, x_1, \dots, x_n) \mapsto (1, x_1/x_0, \dots, x_n/x_0). \tag{10}$$

Hence there are two types of homogeneous points: weighted points and vectors, including the zero vector $\underline{0}$ (which is also called zero point). They both are useful for a description of rational curves and surfaces in a control point setting.

For example, the map $\mathcal{B}_\Delta$ in definition 1 can be rewritten in the following homogeneous form

$$\underline{\mathcal{B}}_\Delta(t) = \sum_{m\in\widehat{\Delta}} \underline{p}_m h_1(t)^{h_1(m)} \cdots h_r(t)^{h_r(m)}, \tag{11}$$

where $\underline{p}_m = (w_m c_m, w_m c_m p_m) \in \mathbb{R}^{n+1}$ are homogeneous control points. The same formula with some control vectors also works as one can see in the example below.

Example 5. Let the toric patch $\underline{\mathcal{B}}_D$ associated with $\Delta = D$ (see figure 4) has the following weighted control points in the corners

$$\underline{p}_1 = (1, 1, 0, 0), \qquad \underline{p}_2 = (1, 0, 1, 0),$$
$$\underline{p}_3 = (1, -1, 0, 0), \qquad \underline{p}_4 = (1, 0, -1, 0)$$

and a control vector $\underline{p}_0 = (0, 0, 0, 4)$ in the center. One can check that the parametrized patch $\mathcal{B}_D(D)$ satisfies the implicit equation $(x_1^2 - x_2^2)^2 - 2x_1^2 - 2x_2^2 - x_3^2 + 1 = 0$. The corresponding full surface is shown in figure 5: $\mathcal{B}_D(D)$ is the upper part of the "pillow".

3.2. *Different parametrizations*

It will be convenient to fix a notation that differentiates between vectors and dual vectors. Let M be the standard lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ and let M^* be the *dual lattice* of linear forms on M with integer values. The basis $E = \{e_1, e_2\}$ in M defines the *dual basis* $E^* = \{e_1^*, e_2^*\}$ in M^* as usual: $\langle e_i^*, e_j\rangle = 1$ if $i = j$, else it is zero.

Any finite subset of vectors in M^* will be called a *collection*. For a lattice polygon Δ a *normal collection* $\nu(\Delta)$ is defined to be the set $\{\nu_1, \dots, \nu_r\} \subset M^*$ of primitive normals of Δ.

Let $\Sigma = \{\sigma_1, \dots, \sigma_q\}$, rank $\Sigma = 2$, be some collection. We generalize the formula of a toric patc $\mathcal{B}_\Delta$ in homogeneous form (11) as follows

$$\underline{\mathcal{B}}_{\Delta,\Sigma}(t) = \sum_{m\in\widehat{\Delta}} \underline{p}_m g_1(t)^{g_1(m)} \cdots g_q(t)^{g_q(m)}, \tag{12}$$

where $g_i(t) = \langle\sigma_i, t\rangle + b_i$ are affine forms that define supporting lines of Δ with normals $\sigma_i \in \Sigma$, $i = 1, \dots, q$. Thus every inequality $g_i(t) \geqslant 0$ defines the smallest half-plane containing Δ. The system of all such inequalities defines a polygon $P(\Sigma)$. In general $P(\Sigma)$ is not a lattice polygon, since it may be infinite and its vertices are not necessarily points of the lattice M.

A map $\mathcal{B}_{\Delta,\Sigma} = \pi \circ \underline{\mathcal{B}}_{\Delta,\Sigma}$ is correctly defined on the whole polygon $P(\Sigma)$ except perhaps at its vertices, where $\underline{\mathcal{B}}_{\Delta,\Sigma}$ may attain zero value $\underline{0}$. The latter points are so-called basepoints of the parametrization $\mathcal{B}_{\Delta,\Sigma}$.

Example 6.

(i) In case $\Sigma = E^*$ we have $\mathcal{B}_{\Delta,E^*} = \mathcal{M}_\Delta$, when we choose m_0 in (13) to be the vertex at the corner of $P(E^*)$.

(ii) If $\Sigma = \{e_1^*, e_2^*, -e_1^* - e_2^*\}$ then $P(\Sigma)$ is the circumscribed triangle $\triangle_k$ (may be translated) for some $k \geqslant 1$. Then $\mathcal{B}_{\Delta,\Sigma}$ coincides with a Bézier triangular patch $\mathcal{B}_{\triangle_k}$ with zero weights $w_m = 0$ for all $m \in \widehat{\triangle}_k \setminus \widehat{\Delta}$. Hence there are basepoints in vertices of $\triangle_k$, which are not in $\widehat{\Delta}$. This is exactly the Warren's construction of multisided Bézier patches [12,13].

(iii) If $\Lambda_i = \{\nu_i, \nu_{i+1}\}$ is a collection of two adjacent normals then $P(\Lambda_i)$ is an angle bounded by inequalities $h_i(t) \geqslant 0$ and $h_{i+1}(t) \geqslant 0$. In skew coordinates $s_1 = h_i(t)$, $s_2 = h_{i+1}(t)$ of $P(\Lambda_i)$ the parametrization $\mathcal{B}_{\Delta,\Lambda_i}$ has the monomial form

$$\underline{\mathcal{B}}_{\Delta,\Lambda_i}(s_1, s_2) = \sum_{m \in \widehat{\Delta}} \underline{p}_m s_1^{h_i(m)} s_2^{h_{i+1}(m)}. \tag{13}$$

3.3. *Lattice extensions*

Sometimes it is useful to consider a given toric patch $\mathcal{B}_\Delta$ with respect to some bigger lattice $\widetilde{M}$, $M \subset \widetilde{M}$. Consider the homogeneous form $\underline{\mathcal{B}}_\Delta$ as defined in (11). We define a toric patch $\underline{\widetilde{\mathcal{B}}}_\Delta$ with respect to the extended lattice $\widetilde{M}$ using the formula (11), where the sum is extended to a bigger set of lattice points $\Delta \cap \widetilde{M}$ as follows: $\underline{p}_m$ is the old control point if $m \in \widehat{\Delta}$ and it is zero $\underline{0}$ if $m \notin \widehat{\Delta}$. If the corresponding affine forms h_i and $\widetilde{h}_i$ coincide, for all $i = 1, \ldots, r$, then the maps $\underline{\mathcal{B}}_\Delta$ and $\underline{\widetilde{\mathcal{B}}}_\Delta$ are equal. At the same time the polygon Δ may have a simpler structure in the extended lattice $\widetilde{M}$. Two such cases we can see in figure 6 (where void circles mean additional lattice points). They are considered in the example below.

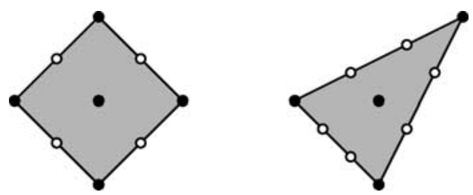

Figure 6. Polygons D and S in extended lattices.

Example 7.

(i) Let D be a lattice square $\mathrm{Conv}\{\pm e_1, \pm e_2\}$ and let $\widetilde{M}$ be an extended lattice with basis vectors $\widetilde{e}_1 = (e_1 - e_2)/2$, $\widetilde{e}_2 = (e_1 + e_2)/2$. It is clear from figure 6 that D in the lattice $\widetilde{M}$ is equivalent to $\square_{2,2}$. Therefore $\mathcal{B}_D$ is a special case of a Bézier biquadratic patch with 4 zero control points. It is shown in figure 5 as an upper part of the "pillow".

(ii) A lattice triangle $S = \mathrm{Conv}\{e_1 + e_2, -e_1, -e_2\}$ considered in an extended lattice $\widetilde{M}$ with basis vectors $\widetilde{e}_1 = (2e_1 + e_2)/3$, $\widetilde{e}_2 = (e_1 + 2e_2)/3$ is equivalent to $\triangle_3$. Therefore, a cubic patch $\mathcal{B}_S$ is a special case of a Bézier triangular patch of degree 3 with 6 zero control points.

With an arbitrary vertex v_i we are going to associate a special lattice extension. The idea is to find a lattice such that the corner triangle Θ_i with vertices $m_0 = v_i$, $m_1 = v_i + f_i$, $m_2 = v_i - f_{i-1}$ will have the type of some $\triangle_k$ as in example 7. We define the lattice $\widetilde{M}_i$ by fixing its basis

$$\widetilde{E}_i = \{\widetilde{e}_1, \widetilde{e}_2\}, \quad \widetilde{e}_1 = (m_1 - m_0)/D_i, \ \widetilde{e}_2 = (m_2 - m_0)/D_i, \tag{14}$$

where $D_i = \mathrm{Vol}_2(\Theta_i)$.

Lemma 8. The lattice M is a sublattice of $\widetilde{M}_i$ and $h_k = \widetilde{h}_k$, $k = i, i+1$, where $\widetilde{h}_s$ are affine forms corresponding to edges of Δ with respect to the lattice $\widetilde{M}_i$.

Proof. Without loss of generality we can assume that $i = 1$, and $m_0 = v_1$ is the origin. Since the vertices m_0, m_1, m_2 are in counter-clockwise order, we can calculate

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad \det \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = D_1 > 0,$$

where α_{11}, α_{12} and α_{21}, α_{22} are mutually prime integer pairs. The basis E can be expressed via integer combinations of $\widetilde{E}$:

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}^{-1} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \frac{1}{D_1} \begin{pmatrix} \alpha_{22} & -\alpha_{12} \\ -\alpha_{21} & \alpha_{11} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_{22} & -\alpha_{12} \\ -\alpha_{21} & \alpha_{11} \end{pmatrix} \begin{pmatrix} \widetilde{e}_1 \\ \widetilde{e}_2 \end{pmatrix}.$$

Hence $M \subset \widetilde{M}_1$. In order to check $h_k = \widetilde{h}_k$, $k = 1, 2$, we calculate the case $k = 1$ explicitly. Since v_1 is the origin, $h_1(t) = \langle \nu_1, t \rangle$ for some $\nu_1 \in M^*$. In fact $\nu_1 = \alpha_{22} e_1^* - \alpha_{21} e_2^*$, because ν_1 is primitive, inward oriented ($\langle \nu_1, m_1 \rangle = D > 0$) and normal to the edge $\overline{v_1 m_2}$ ($\langle \nu_1, m_2 \rangle = 0$). Similarly $\widetilde{h}_1(t) = \langle \widetilde{e}_1^*, t \rangle$. It remains to prove the equation $\nu_1 = \widetilde{e}_1^*$, which we check easily

$$\langle \nu_1, e_1 \rangle = \alpha_{22} = \langle \widetilde{e}_1^*, e_1 \rangle, \qquad \langle \nu_1, e_2 \rangle = -\alpha_{21} = \langle \widetilde{e}_1^*, e_2 \rangle. \qquad \square$$

Lemma 9. For any r positive numbers $\lambda_1, \ldots, \lambda_r$ there exists a reparametrization $\mathcal{R}(\lambda_1, \ldots, \lambda_r) : \Delta \to \Delta$ that affects only the weights of a given toric patch, i.e.,

$$\mathcal{R}(\lambda_1, \ldots, \lambda_r) \circ \mathcal{B}_\Delta^{p,w} = \mathcal{B}_\Delta^{p,w'}, \quad w'_m = \lambda_1^{h_1(m)} \cdots \lambda_r^{h_r(m)} w_m.$$

Moreover, any interior point of Δ can be moved to any other one using such reparametrization.

Proof. At first we consider only the interior part Int Δ and suppose that $\lambda_k = 1$ for all indices except two $k = i, i+1$. Notice that $\mathcal{B}_{\Delta,\Lambda_i}$ (where $\Lambda_i = \{v_i, v_{i+1}\}$, example 6(iii)) is a monomial parametrization with respect to the extended matrix $\widetilde{M}_i$. We denote by

$$\mathcal{R}_i : \Delta_i \to P(\Lambda_i) = \mathbb{R}^2_{\geqslant 0} \tag{15}$$

the corresponding reparametrization $\widetilde{\mathcal{M}}_\Delta$, where Δ_i is the subset in Δ defined by strict inequalities $h_k(t) > 0$, for all $k \neq i, i+1$. According to property T5 we reduce our proof to the monomial case, where it becomes obvious. The general case of arbitrary $\lambda_1, \ldots, \lambda_r$ can be obtained step by step using different Λ_k. □

3.4. *Subdivision into tensor product patches*

For every vertex v_i of Δ consider a monomial parametrization $\widetilde{\mathcal{B}}_{\Delta,\widetilde{E}_i^*} : \mathbb{R}^2_{\geqslant 0} \to \mathbb{R}^n$ associated with the extended lattice $\widetilde{M}_i$. According to lemma 8 this map has the same formula (13) as $\mathcal{B}_{\Delta,\Lambda_i}$. Here we will use notations $\mathcal{R}_i$ and Δ_i from (15).

Lemma 10. The 1–1 reparametrizations $\mathcal{R}_i : \Delta_i \to \mathbb{R}^2_{\geqslant 0}$, $i = 1, \ldots, r$, define a subdivision of Δ into r preimage quadrangles $Q_i = \mathcal{R}_i^{-1}(\square_{1,1})$ of the unit square $\square_{1,1} \subset \mathbb{R}^2_{\geqslant 0}$.

Proof. At first define cutting curves γ_i of Int Δ by the equations $\rho_i(t) = 1$, where

$$\rho(t) = \prod_{i=1}^{r} h_i(t)^{\langle v_i, f_i \rangle}, \quad i = 1, \ldots, r. \tag{16}$$

In order to calculate $\mathcal{R}_i$ explicitly we express any point $m \in \widehat{\Delta}$ in the basis $\widetilde{E}_i = \{\widetilde{e}_1, \widetilde{e}_2\}$ of the extended lattice $\widetilde{M}_i$ as follows $m = m_0 + h_i(m)\widetilde{e}_1 + h_{i+1}(m)\widetilde{e}_2$. After substitution to the formula (11) and obvious cancellations we get (13), where $s_k = \prod_{j=1}^{r} h_j(t)^{\langle v_j, \widetilde{e}_k \rangle}$, $k = 1, 2$. Since $\widetilde{e}_1 = f_{i+1}/D_i$ and $\widetilde{e}_2 = -f_i/D_i$ (cf. (14)), it follows that $\mathcal{R}_i(t) = (\rho_{i+1}(t)^{1/D_i}, \rho_i(t)^{-1/D_i}) = (s_1, s_2)$. We see that the isoparametric lines $s_1 = 1$, $s_2 = 1$ correspond to curves γ_{i+1}, γ_i and the unit square $s_1, s_2 \leqslant 1$ in $\mathbb{R}^2_{\geqslant 0}$ has a preimage $Q_i = \{t \in \Delta \mid \rho_{i+1}(t) < 1, \ \rho_i(t) > 1\}$ in Δ. Furthermore, all points $\mathcal{R}_i^{-1}(1, 1)$ coincide with some point $q \in$ Int Δ. Indeed, according to (13) the image of (1, 1) does not depend on i and is equal to the "centroid" of control points $\sum_{m \in \Delta} \underline{p}_m$. Therefore all curves γ_i intersect in q and in no other point, since $\mathcal{R}_i$ is 1–1 according to property T5. As a consequence all Q_i meet in the point q and adjacent Q_{i-1} and Q_i have a common arc of curve γ_i. Also from the proof of the boundary property T4 follows that γ_i intersects

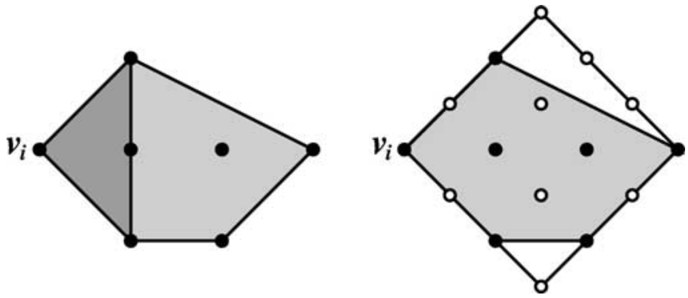

Figure 7. A corner triangle Θ_i and a corner parallelogram Π_i.

the edge ϕ_i in the midpoint. So Q_i is a quadrangle bounded by two halves of edges ϕ_i, ϕ_{i+1} and two arcs of curves γ_i, γ_{i+1} meeting in the point q. □

For every vertex v_i define a *corner parallelogram* Π_i as a lattice polygon $P(\{\widetilde{e}_1^*, \widetilde{e}_2^*, -\widetilde{e}_1^*, -\widetilde{e}_2^*\})$ (see (14)) with respect to the extended lattice $\widetilde{M}_i$. This is the minimal circumscribed lattice parallelogram with a corner vertex v_i. Notice that it may not be a lattice polygon with respect to M: one can see such an example in figure 7, where additional lattice points are shown as void circles. From lemma 8 it follows that Π_i is equal to $\square_{k(i),k(i+1)}$, where $k(j) = \max_{m\in\Delta} h_j(m)$. Therefore on one hand $\widetilde{\mathcal{B}}_{\Pi_i}$ defines a Bézier tensor product patch of bidegree $(k(i), k(i+1))$. On the other hand it is related to the earlier considered monomial map $\widetilde{\mathcal{B}}_{\Delta,\widetilde{E}_i} : \mathbb{R}^2_{\geqslant 0} \to \mathbb{R}^n$ via the simple projective transformation (cf. (6))

$$\mathcal{T} : \mathbb{R}^2_{\geqslant 0} \to [0, 1)^2, \qquad (s_1, s_2) \mapsto \left(\frac{s_1}{1+s_1}, \frac{s_2}{1+s_2}\right).$$

Finally we obtained a reparametrization $\mathcal{R}'_i = \mathcal{T} \circ \mathcal{R}_i : \Delta_i \to [0, 1)^2$, $\mathcal{B}_\Delta|_{\Delta_i} = \mathcal{R}'_i \circ \widetilde{\mathcal{B}}_{\Pi_i}|_{[0,1)^2}$. Here we suppose that Bézier tensor product surfaces are defined on the unit square $\square_{1,1}$ as usual (in contrast to example 2(ii)). For any point $\tau = (\tau_1, \tau_2) \in \square_{1,1}$ denote by $\square_\tau$ the rectangular $[0, \tau_1] \times [0, \tau_2]$. The midpoint $(1/2, 1/2)$ of the square will be denoted by μ. The following theorem is a direct consequence of lemma 10.

Theorem 11. The r-sided domain polygon Δ of a toric patch $\mathcal{B}_\Delta$ can be subdivided into r quadrangular pieces Q_i, $i = 1, \ldots, r$, such that $\mathcal{B}_\Delta(Q_i) = \widetilde{\mathcal{B}}_{\Pi_i}(\square_\mu)$. The algorithm of the subdivision: for every $i = 1, \ldots, r$

(1) calculate the control points of the corner parallelogram Π_i: all $\underline{q}_{ij} = \underline{0}$ except $\underline{q}_{h_i(m),h_{i+1}(m)} = \underline{p}_m$;

(2) apply the de Casteljau algorithm for subdivision of the tensor product surface $\widetilde{\mathcal{B}}_{\Pi_i}$ in four smaller patches at the midpoint $(1/2, 1/2)$ of the parameter square;

(3) choose the patch with the corner labeled by v_i.

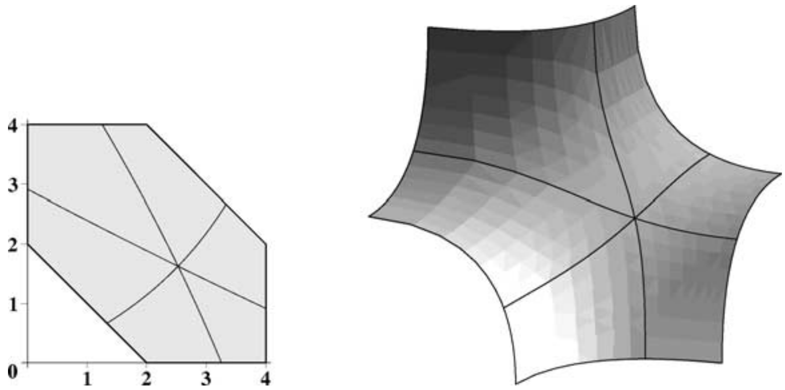

Figure 8. Subdivision of a toric patch and its domain polygon.

Corollary 12. For every interior point q of the polygon Δ there exists a subdivision $\Delta = \bigcup_{i=1}^{r} Q_i(q)$ into quadrangular regions with a common vertex q, such that $\mathcal{B}_\Delta(Q_i(q)) = \widetilde{\mathcal{B}}_{\Pi_i}(\square_{\tau(i)})$, where $\tau(i) = \mathcal{R}_i'(q)$.

Proof. Notice that for any fixed point $q \in \operatorname{Int} \Delta$ the algebraic curve $\rho_i(t) - \rho_i(q) = 0$ goes through it. These curves will serve for cutting the domain polygon into quadrangular pieces $Q_i(q)$. The rest of the proof follows directly from previous constructions using lemma 9. □

3.5. Singular points

General toric surface patches can have singular points in contrast to Bézier surfaces. Here we mean singularities in normal forms of these surfaces (see (28)) when control points are in general positions. So we do not consider singular points, which appear as a result of projection from higher dimensional spaces.

Since the tangent plane is not defined in a singular point, we use a more subtle construction. Define the *tangent cone* of a point on a surface as the union of tangent lines to all possible curves lying on the surface that go through the point.

Theorem 13. Let $v = v_i$ be a vertex of the lattice polygon Δ, let Θ_i be its corner triangle and let θ_i be its opposite edge to v. Then the tangent cone at a corner point p_v of the toric patch $\mathcal{B}_\Delta$ is a cone over a union of Bézier curves $\mathcal{B}_\theta$ associated with a set of edges θ (i.e., with control points $\underline{p}_m$, $m \in \theta$), depending on the cases:

(1) if Θ_i has no interior lattice points then $\theta = \theta_i$;

(2) otherwise, θ runs through all edges of a polygon $\operatorname{Conv}(\widehat{\Theta}_i \setminus v)$, excluding θ_i.

Proof. It will be convenient to use the monomial parametrization $\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}}_{\Delta,\widetilde{E}_i}$ associated with the vertex v_i and the extended lattice $\widetilde{M}_i$ (see section 3.4). We represent any curve on the patch going through the corner p in the form of a composition $\widetilde{\mathcal{B}} \circ \gamma$ with some curve $\gamma(\tau) = (\lambda_1 \tau^{\alpha_1} + o(\tau^{\alpha_1}), \lambda_2 \tau^{\alpha_2} + o(\tau^{\alpha_2}))$ in $\mathbb{R}^2_{\geqslant 0}$. Then we collect all terms with lowest nonzero degree of the parameter τ in the homogeneous form of this composition $\underline{\widetilde{\mathcal{B}}}(\gamma(\tau)) = \underline{p}_v + \underline{q}_1 \tau^c + \underline{q}_2 o(\tau^c)$, where

$$\underline{q}_1 = \sum_{\langle \alpha, m \rangle = c} \underline{p}_m \lambda_1^j \lambda_2^k, \quad \alpha = \alpha_1 \widetilde{e}_1^* + \alpha_2 \widetilde{e}_2^*, \ m = j\widetilde{e}_1 + k\widetilde{e}_2. \tag{17}$$

Thus a line $p_v q_1$ is a tangent line to this curve. For a fixed α the equation $\langle \alpha, t \rangle = c$ defines a supporting line of the polygon $\Theta' = \mathrm{Conv}(\widehat{\Theta}_i \setminus v)$ (figure 9). Hence the sum in (17) contains all control points $\underline{p}_m$ labeled by lattice points $m \in \widehat{\theta}$ of some edge θ of Θ' (or m is just a single vertex of Θ'). If we change the ratio $\lambda_1 : \lambda_2$ then q_1 runs along a Bézier curve $\mathcal{B}_\theta$ defined by these control points. Therefore, we obtain a cone over $\mathcal{B}_\theta$ with the apex p_v. It is clear that the case $\theta = \theta_i$ can occur only when Θ_i has no interior lattice points. □

Singular points may be useful for specific purposes in geometric modeling. For example, a toric patch associated with a triangle $S = \mathrm{Conv}\{e_1 + e_2, -e_1, -e_2\}$ can be applied for rounding a 3-sided corner of a cube as shown in figure 10. Here three singular points are endpoints of sharp edges: their tangent cones are pairs of planes.

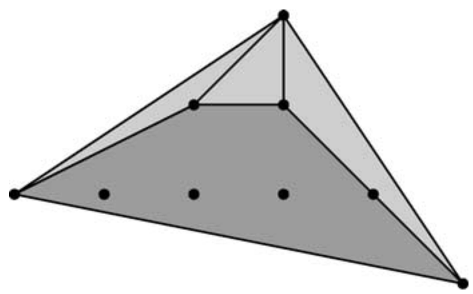

Figure 9. The corner triangle Θ_i and its subpolygon $\mathrm{Conv}(\widehat{\Theta}_i \setminus v)$ in dark grey.

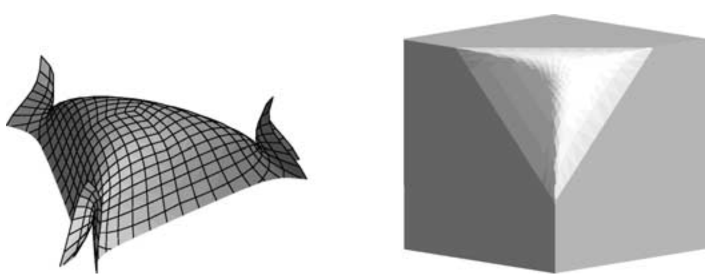

Figure 10. Rounding corner with the singular cubic patch.

4. Projective toric varieties

Here we extend our topics from 2- to d-dimensions. With just a little more effort it is possible to consider toric varieties of arbitrary dimension, and put our results from previous chapters in a wider perspective. Also this will be useful for blossoming toric surface patches.

A d-dimensional *lattice* $M = \mathbb{Z}^d \subset \mathbb{R}^d$ contains all points with integer coordinates. Define a *lattice polytope* $\Delta \subset \mathbb{R}^d$ as the convex hull of some finite subset in M. Let $\dim \Delta = d$, i.e., Δ is not contained in a hyperplane. Then *facets* (i.e., $(d-1)$-dimensional faces) ϕ_i of Δ are intersections with hyperplanes $h_i(t) = 0$, $i = 1, \dots, r$. Here we also suppose the affine linear forms $h_i(t) = \langle \nu_i, t\rangle + a_i$ to be normalized: vectors ν_i are primitive and inward oriented. We denote $\widehat{\Delta} = \Delta \cap M$ a set of lattice points of Δ.

Definition 14. A *real projective toric variety* $\mathbb{R}T_\Delta$ associated with a lattice polytope Δ, $\widehat{\Delta} = \{m_0, m_1, \dots, m_N\}$, is a subset in $\mathbb{R}P^N$ parametrized by the following formula

$$G_\Delta(u_1, \dots, u_r) = \left[u^{h(m_0)}, u^{h(m_1)}, \dots, u^{h(m_N)}\right], \tag{18}$$

where $u^{h(m)} = u_1^{h_1(m)} u_2^{h_2(m)} \cdots u_r^{h_r(m)}$. The variables $u_i \in \mathbb{R}$, $i = 1, \dots, r$, are called *facet variables* [2] or *global coordinates* [1].

Remark 15. In some singular cases the range of G_Δ does not coincide with the whole toric variety, and it is necessary to use complex values of facet variables u_i or to introduce some additional variables. For instance, $\Delta = D$ is a bad case, because G_D covers only "pillow" but not "antennas" (figure 5). Fortunately many classical cases (example 16) are good. On the other hand we are mostly interested in the non-negative part of a toric variety, which is always covered by G_Δ.

The parametrization G_Δ is a composition $\pi \circ \underline{G}_\Delta$, where $\underline{G}_\Delta : \mathbb{R}^r \to \mathbb{R}^{N+1}$ is defined by substituting square brackets by ordinary ones in (18). Since π is undefined at the origin, the map G_Δ is undefined on the set $\mathrm{Ex}(\Delta) = \underline{G}^{-1}(0)$, which is called an *exceptional subset* [1]. Hence $\mathbb{R}T_\Delta$ is parametrized by $\mathbb{R}U_\Delta = \mathbb{R}^r \setminus \mathrm{Ex}(\Delta)$. One can check that $\mathrm{Ex}(\Delta)$ is contained in a union of intersections of some pairs of coordinate hyperplanes $u_i = 0$, i.e., $\mathrm{Ex}(\Delta)$ has at least codimension 2.

Let $E = \{e_1, \dots, e_d\}$ be a basis in the lattice M. The standard d-dimensional simplex Δ^d is a convex hull $\mathrm{Conv}\{0, e_1, \dots, e_d\}$. Let $k\Delta = \{kx \mid x \in \Delta\}$ be a k-times scaled polytope Δ and define a product $\Delta_1 \times \Delta_2$ as usual.

Some well-known classical projective rational varieties are toric. Each of the examples (i)–(iv) below are associated with simplices or their products.

Example 16.

(i) In case of an interval $\mathrm{I}_k = \mathrm{Conv}\{0, ke_1\}$ (in fact $\mathrm{I}_k = k\Delta^1$) $\mathrm{Ex}(\mathrm{I}_k) = \{0\}$ and equation (18) looks like

$$G_{\mathrm{I}_k}(u_1, u_2) = \left[u_2^k, u_1 u_2^{k-1}, \ldots, u_1^{k-1} u_2, u_1^k\right]. \tag{19}$$

This is exactly a homogeneous parametrization $\mathbb{R}P^1 \to \mathbb{R}P^k$ of a rational normal curve.

(ii) Consider a triangle $\Delta_k = k\Delta^2$ (cf. example 2(i)). Then $\mathrm{Ex}(\Delta_k) = \{0\}$ and the homogeneous coordinates

$$G_{\Delta_k}(u_1, u_2, u_3) = \left[\ldots, u_1^i u_2^j u_3^{k-i-j}, \ldots\right], \quad i, j \geqslant 0,\ i + j \leqslant k, \tag{20}$$

coincide with a list of all monomials of total degree k. Hence this is the classical *Veronese embedding* $\mathbb{R}P^2 \to \mathbb{R}P^N$, $N = \binom{k+2}{2} - 1$, and its image $\mathbb{R}T_{\Delta_k}$ is the *Veronese surface*.

(iii) Let Δ be a rectangle $\Box_{k,l} = I_k \times I_l$ (cf. example 2(ii)). Then homogeneous coordinates of

$$G_{\Box_{k,l}}(u_1, u_2, u_3, u_4) = \left[\ldots, u_1^i u_2^j u_3^{k-i} u_4^{l-j}, \ldots\right], \quad 0 \leqslant i \leqslant k,\ 0 \leqslant j \leqslant l, \tag{21}$$

coincide with a list of all monomials of total degree k (resp. l) in a pair of variables u_1, u_3 (resp. u_2, u_4). Since $\mathrm{Ex}(\Box_{k,l}) = (V_1 \cap V_3) \cup (V_2 \cap V_4)$, the domain $\mathbb{R}U_{\Box_{k,l}}$ can be identified with the product $(\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}^2 \setminus \{0\})$ (just swap u_2 and u_3). Therefore, treating each pair of variables as homogeneous coordinates of a separate copy of a projective line $\mathbb{R}P^1$ we get the classical *Segre embedding* $\mathbb{R}P^1 \times \mathbb{R}P^1 \to \mathbb{R}P^N$, $N = kl + k + l$. Hence $\mathbb{R}T_{\Box_{k,l}}$ is the *Segre surface*.

(iv) Let Δ be a d-dimensional simplex Δ^d. In this case it will be convenient to use also zero indices. The linear forms $h_0(t) = 1 - t_1 - \cdots - t_d$, $h_i(t) = t_i$, $i = 1, \ldots, d$, define facets of the simplex, and the map $G_{\Delta^d}(u_0, u_1, \ldots, u_d) = [u_0, u_1, \ldots, u_d]$ coincides with the projection π from equation (9). Hence $\mathbb{R}T_{\Delta^d} = \mathbb{R}P^d$.

Here we do not go into details of the structure of $\mathbb{R}T_\Delta$. We concentrate our attention on the *non-negative part* $\mathbb{R}_{\geqslant 0}T_\Delta$, which is defined as the subset of all points with non-negative coordinates – the image of the non-negative domain $\mathbb{R}_{\geqslant 0}U_\Delta = \mathbb{R}^r_{\geqslant 0} \setminus \mathrm{Ex}(\Delta) \subset \mathbb{R}U_\Delta$. It is easy to see that these non-negative parts are disjoint unions (indexed by all faces $\delta \subset \Delta$)

$$\mathbb{R}_{\geqslant 0}U_\Delta = \bigcup_{\delta \subset \Delta} \mathbb{R}_+ U_\delta, \qquad \mathbb{R}_{\geqslant 0}T_\Delta = \bigcup_{\delta \subset \Delta} \mathbb{R}_+ T_\delta, \tag{22}$$

of the corresponding *positive* subsets

$$\begin{aligned} &\mathbb{R}_+ U_\delta = \left\{(u_1, \ldots, u_r) \in \mathbb{R}^r_{\geqslant 0} \,\middle|\, u_i = 0 \Longleftrightarrow \delta \subset \phi_i\right\}, \\ &\mathbb{R}_+ T_\delta = \mathbb{R}_{\geqslant 0} T_\Delta \cap \mathbb{R}_+ P_\delta, \end{aligned} \tag{23}$$

where $\mathbb{R}P_\delta = \{[x_0, \dots, x_N] \in \mathbb{R}P_N \mid x_j = 0,\ m_j \notin \delta\}$ are coordinate subspaces. These subdivisions are compatible with the map $G_\Delta : \mathbb{R}_{\geqslant 0}U_\Delta \to \mathbb{R}_{\geqslant 0}T_\Delta$, i.e., every $\mathbb{R}_+U_\delta$ is mapped to $\mathbb{R}_+T_\delta$.

As in 2-dimensional case we denote by M^* the *dual lattice* of linear forms on the lattice M with integer values. The basis $E = \{e_1, \dots, e_d\}$ in $M = \mathbb{Z}^d$ defines the *dual basis* $E^* = \{e_1^*, \dots, e_d^*\}$ in M^*. Any finite subset of vectors in M^* will be called a *collection*. For a lattice polytope Δ a *normal collection* $\nu(\Delta)$ is defined to be the set $\{\nu_1, \dots, \nu_r\} \subset M^*$ of primitive normals as in definition 14 .

In order to deal with restrictions of the map G_Δ to various faces of Δ we need to generalize the formula (18).

Definition 17. Let $\Sigma = \{\sigma_1, \dots, \sigma_q\}$ be some collection and let $g_i(t) = \langle \sigma_i, t\rangle + b_i$ be affine forms, such that $g_i(t) \geqslant 0$, $i = 1, \dots, q$, for all $t \in \Delta$. We define a map $G_{\Delta,\Sigma} : \mathbb{R}_+^q \to \mathbb{R}P^N$ as follows

$$G_{\Delta,\Sigma}(s) = \left[s^{g(m_0)}, s^{g(m_1)}, \dots, s^{g(m_N)}\right], \quad s^{g(m)} = s_1^{g_1(m)} \cdots s_q^{g_q(m)}. \tag{24}$$

Lemma 18. For any collection $\Sigma = \{\sigma_1, \dots, \sigma_q\} \subset M^*$, $\operatorname{rank} \Sigma = d$, the image $G_{\Delta,\Sigma}(\mathbb{R}_+^q)$ coincides with $\mathbb{R}_+T_\Delta$ which is a positive part of some d-dimensional algebraic variety in $\mathbb{R}P_N$.

Proof. Since the image of $G_{\Delta,\Sigma}$ is contained in an affine part of $\mathbb{R}P^N$, the restriction of G_Δ on $\mathbb{R}_+^r$ is easily calculated in affine coordinates

$$\begin{aligned} G_{\Delta,\Sigma}(s_1, \dots, s_q) &= \left(s^{g(m_1)}/s^{g(m_0)}, \dots, s^{g(m_N)}/s^{g(m_0)}\right) \\ &= \left(s^{g(m_1-m_0)}, \dots, s^{g(m_N-m_0)}\right) \\ &= \left(s^{a_1*}, \dots, s^{a_N*}\right), \end{aligned}$$

where a_i* are rows of an $(N \times q)$-matrix A_Σ with entries $a_{ij} = g_j(v_i) = \langle \sigma_j, v_i\rangle$, $v_i = m_i - m_0$, $i = 1, \dots, N$. In fact, the matrix A_Σ defines a linear map $A_\Sigma \widetilde{s} = \widetilde{x}$ which represents $G_{\Delta,\Sigma}$ in logarithmic coordinates $\widetilde{u}_i = \log u_i$ and $\widetilde{x}_i = \log x_i$. Let all σ_j are substituted in entries $\langle \sigma_j, v_i\rangle$ of A_Σ by their expressions in the dual basis $\sigma_j = \sum_{k=1}^d b_{kj} e_k^*$. An easy computation shows that A_Σ is a product $A_{E^*}B$, where $B = (b_{kj})$ is $d \times q$ matrix. Also $\operatorname{rank} A_{E^*} = \operatorname{rank} B = d$, since $\dim \Delta = d$ (so vectors $v_i = m_i - m_0$ span the vector space $\mathbb{R}^d$) and $\operatorname{rank} \Sigma = d$. Hence B defines a surjective linear map $\mathbb{R}^q \to \mathbb{R}^d$. Therefore, d-dimensional images of linear maps A_Σ and A_{E^*} coincide and can be defined by some system of linear equations in $\mathbb{R}^N$.

If we go back from logarithmic coordinates to the ordinary ones then we conclude that the images $G_{\Delta,\Sigma}(\mathbb{R}_+^q)$ and $G_{\Delta,E^*}(\mathbb{R}_+^d)$ coincide and are defined by some system of binomial algebraic equations in affine space. Taking in particular $\Sigma = \nu(\Delta)$ we see that this is exactly $\mathbb{R}_+T_\Delta = G_\Delta(\mathbb{R}_+^r)$. $\square$

Corollary 19. Let $\delta \subset \Delta$ be a face then $\mathbb{R}_+T_\delta$ is a positive part of some algebraic variety of dimension $\dim \delta$ in the coordinate subspace $\mathbb{R}P_\delta$.

Proof. Notice that a restriction $G_\Delta|_\delta : \mathbb{R}^r \to \mathbb{R}P_\delta$ coincides with $G_{\delta,\Sigma}$, where $\Sigma = \{\nu_1|_\delta, \ldots, \nu_r|_\delta\}$. Now the proof follows directly from lemma 18. □

5. Bézier polytopes

Definition 1 of toric surface patches has straightforward generalization to arbitrary dimensions. Let $\Delta \subset \mathbb{R}^d$ be a lattice polytope, $\dim \Delta = d$, and let equations $h_i(t) = 0$, $i = 1, \ldots, r$, define facets of Δ as earlier.

Definition 20. A *Bézier polytope* $\mathcal{B}^p_\Delta$ associated with a lattice polytope Δ with homogeneous control points $p : \widehat{\Delta} \to \mathbb{R}^{n+1}$, $m \mapsto \underline{p}_m$, is a rational map

$$\mathcal{B}^p_\Delta : \Delta \to \mathbb{R}P^n, \quad \underline{\mathcal{B}}^p_\Delta(t) = \sum_{m \in \widehat{\Delta}} \underline{p}_m h_1(t)^{h_1(m)} \cdots h_r(t)^{h_r(m)}. \tag{25}$$

It is easy to check that well-known Bézier like constructions are particular cases of Bézier polytopes listed in table 1, where the concept of Bézier simploids [3] includes all other examples.

5.1. Properties

Similar to the 2-dimensional case, an affine transformation L of $\mathbb{R}^d$ is called an *affine unimodular transformation* if it preserves the lattice $L(\mathbb{Z}^d) = \mathbb{Z}^d$. Now properties T1–T3 of toric surface patches from section 2.2 can be word for word reformulated and are valid for Bézier polytopes. The proofs are straightforward.

A monomial parametrization $\mathcal{M}_\Delta : \text{Int } \Delta \to \mathbb{R}^n$ of the Bézier polytope $\mathcal{B}_\Delta$ can be defined as the following rational map in homogeneous form

$$\underline{\mathcal{M}}_\Delta(s_1, \ldots, s_d) = \sum_{m(i) \in \widehat{\Delta}} \underline{p}_{m(i)} s_1^{i_1} \cdots s_d^{i_d}, \tag{26}$$

where lattice points $m(i) = m(i_1, \ldots, i_d) = m_0 + i_1 e_1 + \cdots + i_d e_d$ are expressed in the standard basis E of for some fixed $m_0 \in M$. Similarly one can define a map

Table 1
Lattice polytopes and associated Bézier polytopes.

dim	Lattice polytope Δ	Bézier polytope $\mathcal{B}_\Delta$
1	Segment I_k	Bézier curve of degree k
2	Triangle $\triangle_k$	Bézier triangle of degree k
2	Rectangle $\square_{k,l}$	Tensor product surface of bidegree (k, l)
3	$\mathrm{I}_k \times \mathrm{I}_l \times \mathrm{I}_m$	Bézier volume
d	simplex $k\triangle^d$	Bézier simplex
$\sum_i d_i$	$k_1\triangle^{d_1} \times \cdots \times k_n\triangle^{d_n}$	Bézier simploid

$\mathcal{M}_\delta : \mathrm{Int}\ \delta \to \mathbb{R}^n$ for every face $\delta \subset \Delta$. It is enough to use some basis E_δ of the sublattice $M_\delta \subset M$ corresponding to the affine span of δ.

Lemma 21. For every face $\delta \subset \Delta$ there exists an analytic isomorphism $\mathcal{R}_\delta : \mathrm{Int}\ \delta \to \mathbb{R}_+^{\dim \delta}$ with such that $\mathcal{B}_\Delta|_{\mathrm{Int}\ \delta} = \mathcal{M}_\delta \circ \mathcal{R}_\delta$.

Proof. Similarly to the 2-dimensional case we evaluate affine the forms h_k, $k = 1, \ldots, r$, on lattice points $h_k(m(i)) = h_k(m_0) + i_1\langle \nu_k, e_1\rangle + \cdots + i_d\langle \nu_k, e_d\rangle$ and substitute into (25). Then we collect terms with the same powers $i_1, \ldots, i_d$ and get (26) up to some constant terms, where

$$s_j = h_1(t)^{\langle \nu_1, e_j\rangle} \cdots h_r(t)^{\langle \nu_r, e_j\rangle}, \quad j = 1, \ldots, d. \tag{27}$$

Therefore we define $\mathcal{R}_\Delta : \mathrm{Int}\ \Delta \to \mathbb{R}_+^d$, $t \mapsto (s_1, \ldots, s_d)$, and the formula $\mathcal{B}_\Delta = \mathcal{M}_\Delta \circ \mathcal{R}_\Delta$ is satisfied. For any face $\delta \subset \Delta$ one can define $\mathcal{R}_\delta$ similarly using the basis E_δ. The proof that this map is an analytic isomorphism we postpone to section 5.3. □

The non-negative part $\mathbb{R}_{\geqslant 0}T_\Delta$ of the toric variety is contained in the projective space $\mathbb{R}P^N$ with homogeneous coordinates labeled by lattice points $m_0, \ldots, m_r$ of the polytope Δ. We denote the corresponding basis vectors of $\mathbb{R}^{N+1}$ by $\underline{e}_m$, $m \in \widehat{\Delta}$, i.e., $e_{m_0} = (1, 0, \ldots, 0)$, $e_{m_1} = (0, 1, 0, \ldots, 0)$ and so on. We call a map $\mathcal{B}_\Delta^e : \Delta \to \mathbb{R}P^N$ with these control points $\underline{e}_m$ a *normal form* of a Bézier polytope.

Define an affine map $h : \Delta \to \mathbb{R}_{\geqslant 0}U$, $h(t) = (h_1(t), \ldots, h_r(t))$. It is clear that $\mathcal{B}_\Delta^e = G_\Delta \circ h$. Hence the image of $\mathcal{B}_\Delta^e$ is contained in $\mathbb{R}_{\geqslant 0}T_\Delta$. In fact they coincide.

Corollary 22. $\mathcal{B}_\Delta^e(\delta) = \mathbb{R}_{\geqslant 0}T_\delta$ for every face $\delta \subset \Delta$. In particular, the image of the normal form of Bézier polytope $\mathcal{B}_\Delta^e$ coincides with the non-negative part $\mathbb{R}_{\geqslant 0}T_\Delta$ of the toric variety.

Proof. The proof directly follows from lemmas 18 and 21 if we notice that $G_{\Delta, E^*} = \mathcal{M}_\Delta^e$ in this normal case, where $\mathcal{M}_\Delta^e$ has control points $\underline{p}_m = \underline{e}_m$. □

On the other hand every Bézier polytope $\mathcal{B}_\Delta^p : \Delta \to \mathbb{R}P^n$ can be obtained from its normal form via the unique projection $\mathcal{P}^p : \mathbb{R}P^N \to \mathbb{R}P^n$, $\underline{e}_m \mapsto \underline{p}_m$, i.e.,

$$\mathcal{B}_\Delta^p = \mathcal{P}^p \circ \mathcal{B}_\Delta^e. \tag{28}$$

Now the boundary property directly follows from corollary 22.

Corollary 23. For any face $\delta \subset \Delta$ the restriction $\mathcal{B}_\Delta|_\delta$ of a Bézier polytope with control points $\underline{p}_m$, $m \in \widehat{\Delta}$ has the same image as the Bézier polytope $\mathcal{B}_\delta$ with a subset of the same control points $\underline{p}_m$, $m \in \widehat{\delta}$.

Any unimodular transformation L preserves volume Vol_d, which will be convenient to normalize assuming $\mathrm{Vol}_d(\Delta^d) = 1$ for the standard d-dimensional simplex Δ^d.

For example, Vol_1 has a meaning of integer length of a segment with lattice endpoints (see (5)).

Theorem 24. The implicit degree of $\mathcal{B}_\Delta(\Delta)$ does not exceed $\mathrm{Vol}_d(\Delta)$. It is equal to $\mathrm{Vol}_d(\Delta)$ when the control points are in general position.

Proof. There is a classical result that $\deg \mathbb{R}T_\Delta = \mathrm{Vol}_d(\Delta)$ (cf. [5]). Hence in case of normal form when $\mathcal{B}^e_\Delta(\Delta) = \mathbb{R}_{\geqslant 0}T_\Delta$ we have the equation. In a general case $\mathcal{B}_\Delta(\Delta)$ is a projection of $\mathbb{R}_{\geqslant 0}T_\Delta$ to lower dimensional space, according to (28). Hence the degree can only drop as it is explained, for example, in [7, example 18.16]. □

The analytic precision property also can be generalized. Let $\mathcal{B}^{\mathrm{id}}_\Delta$ be a Bézier polytope with control points $p_m = m$, $m \in \widehat{\Delta}$ and some weights w_m.

Theorem 25. Let all weights $w_m \geqslant 0$ and $w_m > 0$ on the corner points $m \in \widehat{\Delta}$. Then $\mathcal{B}^{\mathrm{id}}_\Delta : \Delta \to \Delta$ defines a 1–1 map which is an analytic isomorphism on the interior Int δ of every face $\delta \subset \Delta$.

Proof. Postponed to section 5.3. □

The projection $\mathcal{P}^m : \mathbb{R}T_\Delta \to \Delta$, which maps $\underline{e}_m$ to m, is called a *moment map*. From corollary 22 and the previous theorem we derive easily the following classical result (see [5, p. 82] or [4, Chap. VII]).

Theorem 26. The moment map $\mathcal{P}^m : \mathbb{R}_{\geqslant 0}T_\Delta \to \Delta$ is 1–1, and every restriction $\mathbb{R}_{>0}T_\delta \to \mathrm{Int}\ \delta$ is an analytic isomorphism.

5.2. Singularities

Singularities, however, can have more complicated structure than in 2-dimensional case.

Example 27. Let Γ be 3-dimensional lattice polytope $\mathrm{Conv}\{\pm e_1, \pm e_2 + e_3\} \subset \mathbb{R}^3$. In figure 11 we see that this is a tetrahedron bounded by four inequalities $h_1(t) = 1 - t_1 - t_3 \geqslant 0$, $h_2(t) = 1 + t_1 - t_3 \geqslant 0$, $h_3(t) = -t_2 + t_3 \geqslant 0$, $h_4(t) = t_2 + t_3 \geqslant 0$, and containing six lattice points $\widehat{\Gamma} = \{e_1, 0, -e_1, e_2 + e_3, e_3, -e_2 + e_3\}$. Denote the corresponding control points of a Bézier polytope $\mathcal{B}_\Gamma$ by $\underline{p}_0, \underline{p}_1, \underline{p}_2, \underline{q}_0, \underline{q}_1, \underline{q}_2$. From the explicit formula for $\mathcal{B}_\Gamma(t)$

$$\underline{p}_0 h_1^2(t) + \underline{p}_1 h_1(t)h_2(t) + \underline{p}_2 h_2^2(t) + \underline{q}_0 h_3^2(t) + \underline{q}_1 h_3(t)h_4(t) + \underline{q}_2 h_4^2(t)$$

it follows that $\mathcal{B}_\Gamma(\Gamma)$ is a union of line segments joining Bézier quadratic curves $\mathcal{B}^p_{\mathrm{I}_2}$ and $\mathcal{B}^q_{\mathrm{I}_2}$, which are curves consisting only of singular points. Exactly this situation

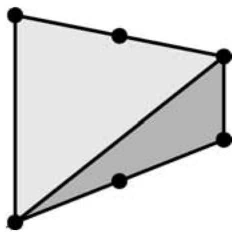

Figure 11. Lattice polytope Γ.

appears in the construction of quartic hypersurfaces in $\mathbb{R}^4$ which contain Dupin cyclides as hyperplane sections [11]. Note that Γ is a join of its edges $\mathrm{Conv}\{\pm e_1\}$ and $\mathrm{Conv}\{\pm e_2 + e_3\}$ in the sense of polytope theory [4, Chap. 3] and $\mathrm{Vol}_3(\Gamma) = 4$.

5.3. Technical proofs

This section is devoted to the proof of lemma 21 and theorem 25. We will use induction on dimension of faces $\delta \subset \Delta$.

Consider the following propositions:

(I_k) If $\dim\delta = k$ then the map $\mathcal{B}^{\mathrm{id}}_\Delta$ restricted to $\mathrm{Int}\ \delta$ is surjective;

(II_k) If $\dim\delta = k$ then the map $\mathcal{B}^{\mathrm{id}}_\Delta$ restricted to δ is 1–1;

(III) The Jacobian of $\mathcal{R}_\delta : \mathrm{Int}\ \delta \to \mathbb{R}_+^{\dim\delta}$ is positive;

(IV) The Jacobian of $\mathcal{M}^{\mathrm{id}}_\delta : \mathbb{R}_+^{\dim\delta} \to \mathrm{Int}\ \delta$ is positive.

Propositions (III) and (IV) are proved below using straightforward computations that do not depend on dimensions. For the initial step (II_0) of the induction it is enough to notice that $\mathcal{B}^{\mathrm{id}}_\Delta$ keeps vertices fixed. Then we proceed as follows.

(II_{k-1}) $\Rightarrow$ (I_k). Consider the restriction $\mathcal{B}^{\mathrm{id}}_\Delta|_\delta$ on any face δ, $\dim\delta = k+1$. Hence this map $\delta \to \delta$ is continuous, and it is 1–1 on the boundary according to (II_k). Note that δ is a k-dimensional topological ball and its boundary is a $k-1$-dimensional topological sphere. It is a well-known result from elementary topology that $\mathcal{B}^{\mathrm{id}}_\Delta$ must be surjective on δ. It is also surjective on $\mathrm{Int}\ \delta$, since $\mathcal{B}^{\mathrm{id}}_\Delta$ preserves faces. This proves (I_k).

(I_k) $\Rightarrow$ (II_k). The restriction $\mathcal{B}^{\mathrm{id}}_\Delta|_{\mathrm{Int}\ \delta}$ is a composition $\mathcal{M}^{\mathrm{id}}_\delta \circ \mathcal{R}_\delta$. ($\mathrm{I}_k$) means that $\mathcal{M}^{\mathrm{id}}_\delta$ is also surjective. Since this is a map between convex sets, from (IV) follows it is an analytic isomorphism, in particular 1–1. Then $\mathcal{R}_\delta$ is also surjective. From (III) we derive similarly that it is an analytic isomorphism and in particular 1–1. This proves lemma 21 for $\dim\delta = k$. Now theorem 25 for $\dim\delta = k$ and also (II_k) easily follows.

Proof of (III). Let $\delta = \Delta$ then according to (27)

$$\mathcal{R}_\Delta(t) = (s_1, \dots, s_d), \quad s_i = \prod_{k=1}^{r} h_k(t)^{\langle \nu_k, e_i \rangle}.$$

For any $t = t_1 e_1 + \cdots + t_d e_d \in \operatorname{Int} \Delta$ we have

$$\frac{\partial s_i}{\partial t_j} = \frac{\partial}{\partial t_j} \prod_{k=1}^{r} h_k(t)^{\langle \nu_k, e_i \rangle} = s_i \sum_{k=1}^{r} \frac{\langle \nu_k, e_i \rangle \langle \nu_k, e_j \rangle}{h_k(t)}.$$

Now we can see that the Jacobian of $\mathcal{R}_\Delta$

$$\det\left(\frac{\partial s_i}{\partial t_j}\right) = s_1 \cdots s_d \det\left(\sum_{k=1}^{r} \frac{1}{h_k(t)} \langle \nu_k, e_i \rangle \langle \nu_k, e_j \rangle\right)$$

is positive. Indeed, all s_i are positive and the last determinant is positive according to proposition 28 applied with $\lambda_k = 1/h_k(t) > 0$ and $a_{ki} = \langle \nu_k, e_i \rangle$. In the case of arbitrary face δ the proof is the same: just notice that the rank of the corresponding matrix a_{ki} will be $\dim \delta$. □

Proof of (IV). We check first that for any collection of vectors $\underline{m}_0, \dots, \underline{m}_N \in \mathbb{R}^{d+1}$ of rank d and any $w_0, \dots, w_N > 0$ the map $f : \mathbb{R}_+^{d+1} \to \mathbb{R}^{d+1}$ defined by

$$f(s_0, \dots, s_d) = \sum_{k=0}^{N} w_k s_0^{m_{k0}} \cdots s_d^{m_{kd}} \underline{m}_k, \quad \underline{m}_k = (m_{k0}, \dots, m_{kd}),$$

has positive Jacobian. This follows from calculations

$$\det\left(\frac{\partial f_i(s)}{\partial s_j}\right) = \frac{1}{s_1 \cdots s_d} \det\left(\sum_{k=0}^{N} \left(w_k s_0^{m_{k0}} \cdots s_d^{m_{kd}}\right) m_{ki} m_{kj}\right)$$

and proposition 28 applied with parameters $\lambda_k = w_k s_0^{m_{k0}} \cdots s_d^{m_{kd}}$ and $a_{ki} = m_{k+1,i+1}$, $i = 0, \dots, d$.

Let $\underline{m}_k = (1, m_k)$, $k = 0, \dots, N$, where m_i are lattice points of the polygon Δ, and let $W(s) = \sum_{k=0}^{N} w_k s_1^{m_{k1}} \cdots s_d^{m_{kd}}$. Then it is easy to see that the map $\widetilde{f}(s_0, s_1, \dots, s_d) = f(s_0/W(s), s_1, \dots, s_d)$ also has positive Jacobian and maps the affine subspace $\{x_0 = 1\}$ to itself. The proof is completed by noticing that $\mathcal{M}_\Delta^{\mathrm{id}}(s_1, \dots, s_d) = \widetilde{f}(1, s_1, \dots, s_d)$. □

Proposition 28. Let $A = (a_{ki})$ be $(N \times n)$-matrix, $\operatorname{rank} A = n$, and let λ_k, $k = 1, \dots, N$, be any positive numbers. Then the $(n \times n)$-matrix B with entries $b_{ij} = \sum_{k=1}^{N} \lambda_k a_{ki} a_{kj}$ has a positive determinant.

Proof. Consider a euclidean structure in $\mathbb{R}^N$ defined by the scalar product $\langle x, y \rangle := \sum_{k=1}^{N} \lambda_k x_k y_k$. We see that the matrix B has entries in a form of scalar products $\langle a_{*i}, a_{*j} \rangle$

of colums of matrix A, which are linearly independent. Hence $\det B > 0$, since this is exactly the Gram determinant. □

6. Summary and further work

We have proposed a concept of a toric surface patch associated with a lattice polygon. In particular cases of lattice triangles $\triangle_k$ and rectangles $\square_{k,l}$ the construction gives Bézier triangular and tensor product Bézier surfaces. It appears that toric patches share many important properties with these classical Bézier surfaces. In particular, any r-sided toric patch can be subdivided to r smaller tensor product pieces. Therefore one can easily include this construction into popular surface modeling software. At the same time toric patches demonstrate new shape possibilities and richer geometries: multisided forms, singular corner points, and a wider variety of implicit degrees. We have also developed a multidimensional variant of the theory by introducing a Bézier polytope, which is a free-form analog of a lattice polytope of arbitrary dimension.

Further work will be devoted to more detailed studies of the simplest cases: Hirzebruch and hexagonal surface patches. The general theory will be developed further, including blossoming, De Casteljau algorithm, and degree raising for arbitrary toric patches.

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