Spline Methods in CAGD

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http://www.subdivision.org/subdivision/index.jsp
http://www.uio.no/studier/emner/matnat/ifi/INF-MAT5340/v04/litteratur.xml
http://www.microsoft.com/typography/ClearTypeInfo.mspx
http://www.cgal.org/

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Reference
"Spline Methods Draft"
Tom Lyche and Knut Morken
"Subdivision Methods for Geometric Design: A Constructive Approach"
Joe Warren and Henrik Weimer

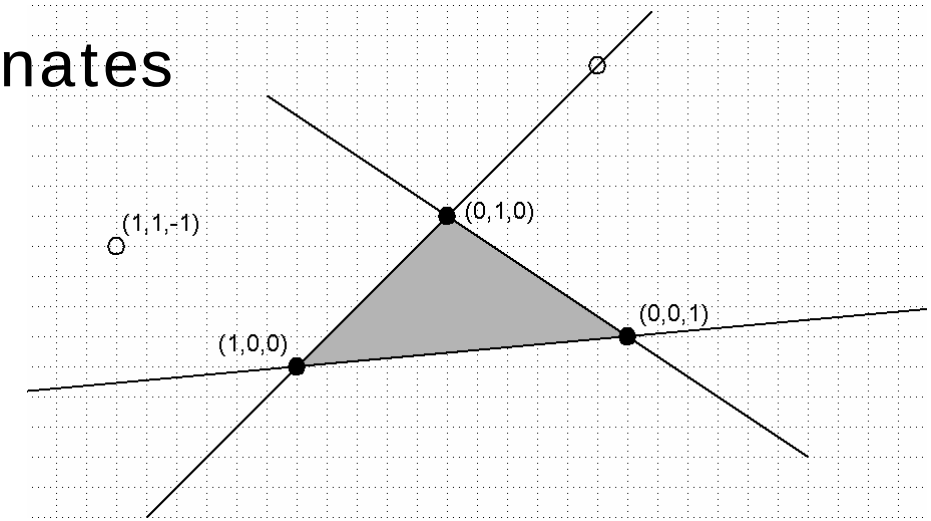
Affine combination

- Linear combinations
- Affine(Barycentric) combinations
- Convex combinations
- Barycentric coordinates

$$b = \sum_{j=0}^{n} \alpha_j b_j$$

$$\sum_{j=0}^{n} \alpha_j = 1$$

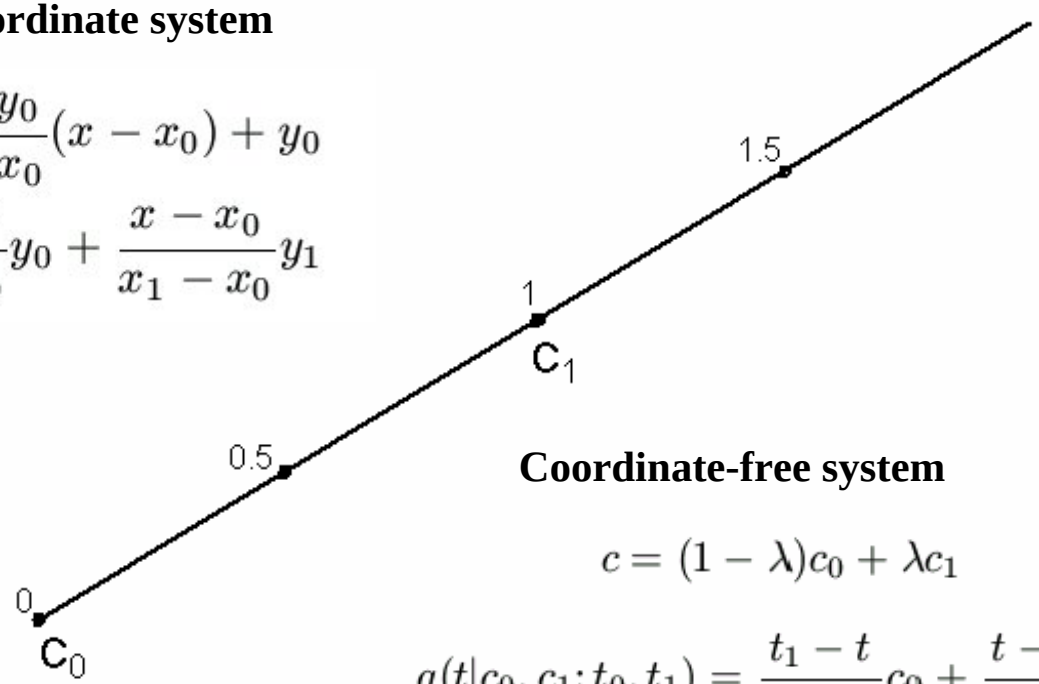
$$\alpha_j \geq 0$$



Affine combination

Euclidean coordinate system

$$y = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) + y_0$$
$$= \frac{x_1 - x}{x_1 - x_0} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$



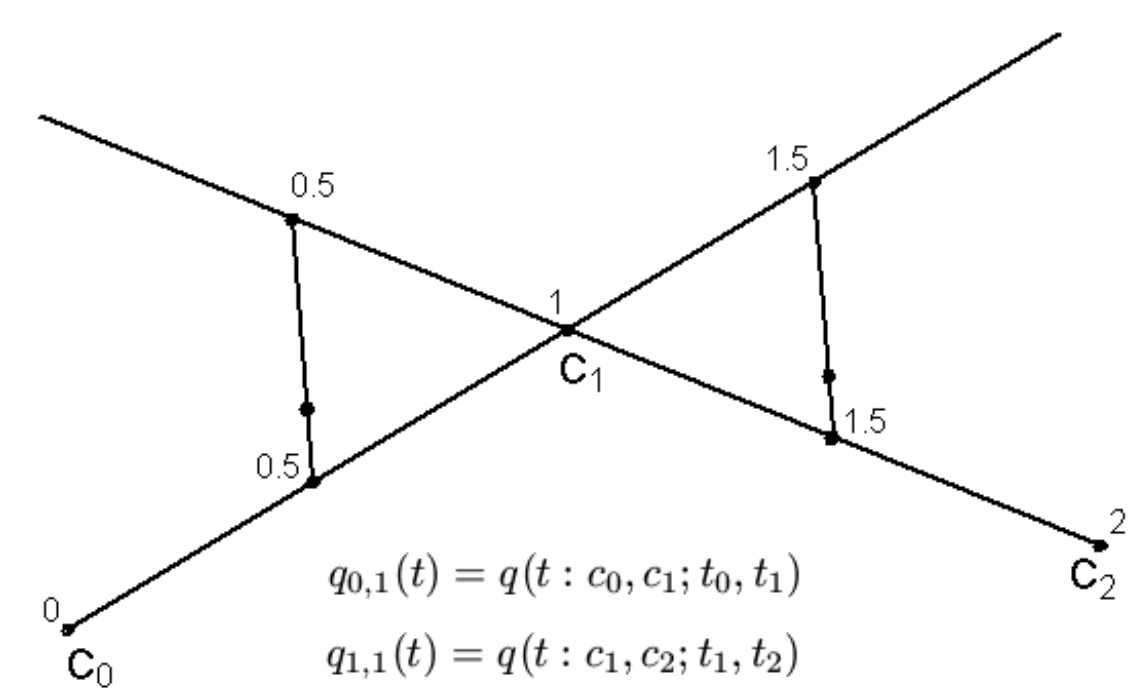
Coordinate-free system

$$c = (1 - \lambda)c_0 + \lambda c_1$$

$$q(t|c_0, c_1; t_0, t_1) = \frac{t_1 - t}{t_1 - t_0} c_0 + \frac{t - t_0}{t_1 - t_0} c_1$$

$$q'(t|c_0, c_1; t_0, t_1) = \frac{c_1 - c_0}{t_1 - t_0}$$

Polynomial interpolation



$$q_{0,2}(t) = q(t|c_0, c_1, c_2; t_0, t_1, t_2) = \frac{t_2 - t}{t_2 - t_0} q_{0,1}(t) + \frac{t - t_0}{t_2 - t_0} q_{1,1}(t)$$

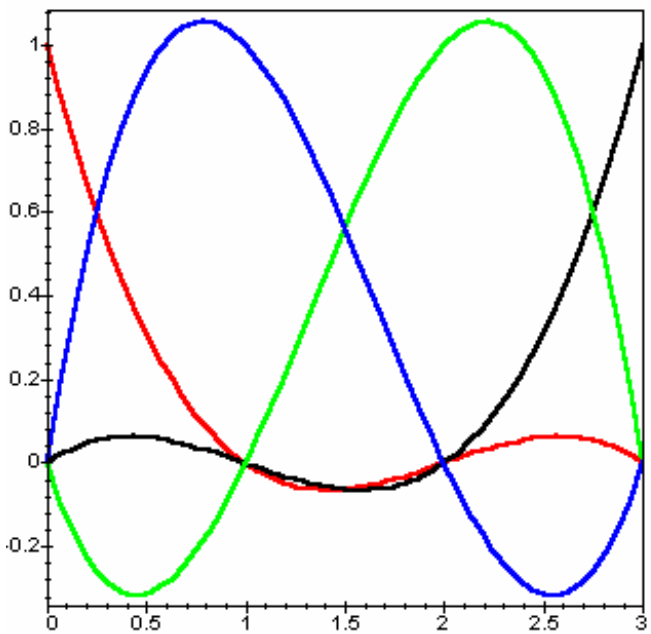
Polynomial interpolation

- Lagrange polynomials

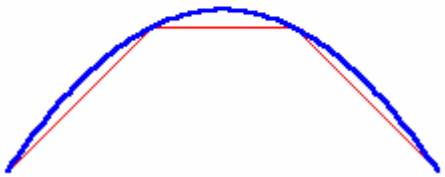
$$q_{0,d}(t) = \frac{t_d - t}{t_d - t_0} q_{0,d-1}(t) + \frac{t - t_0}{t_d - t_0} q_{1,d-1}(t)$$

$$q_{0,d}(t) = \sum_{i=0}^{d} c_i l_{i,d}(t)$$

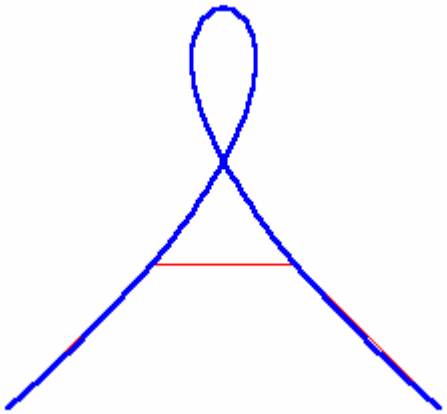
$$l_{i,d}(t) = \prod_{j \neq i} \frac{t - t_j}{t_i - t_j}$$



Examples of cubic interpolation

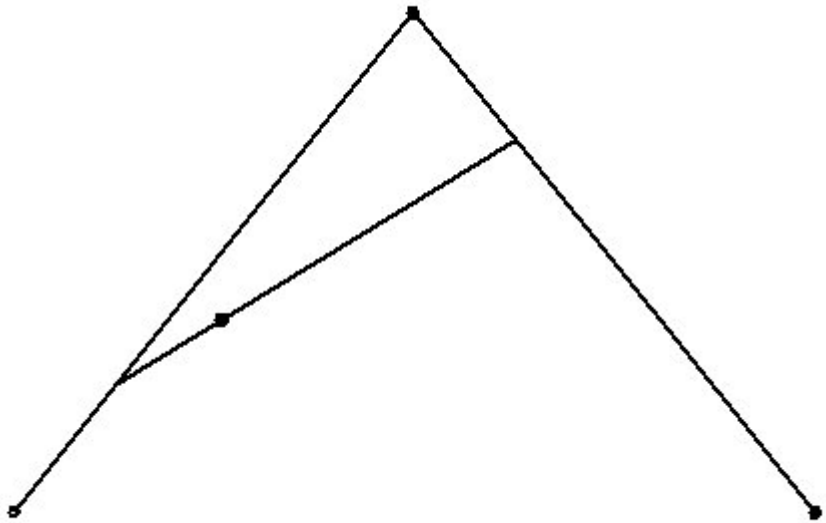


$\mathbf{t} = (0, 1, 2, 3)$



$\mathbf{t} = (0, 0.3, 2.7, 3)$

Bezier



$$p_{1,1}(t) = p(t : c_0, c_1) = (1-t)c_0 + tc_1$$
$$p_{2,1}(t) = p(t : c_1, c_2) = (1-t)c_1 + tc_2$$

$$p_{2,2}(t) = (1-t)p_{1,1}(t) + tp_{2,1}(t)$$

$$p'_{2,2}(0) = 2(c_1 - c_0)\ ,\ \ p'_{2,2}(1) = 2(c_2 - c_1)$$

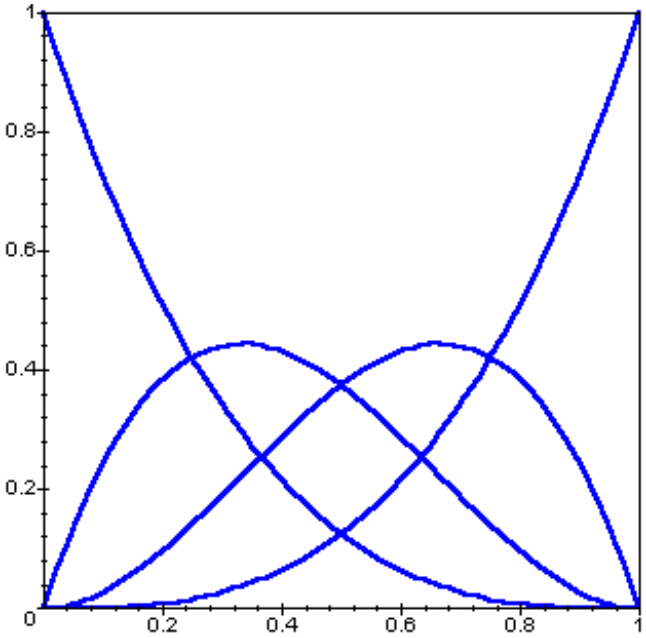
Representation Bezier

$$p_{d,d}(t) = (1-t)p_{d-1,d-1}(t) + tp_{d,d-1}(t)$$

$$p_{d,d}(t) = \sum_{i=0}^{d} c_i b_{i,d}(t)$$

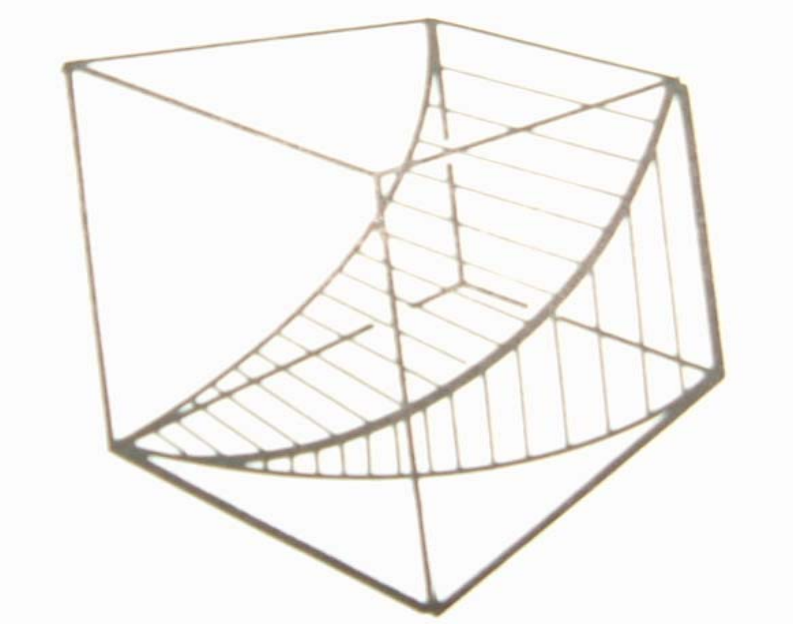
$$b_{i,d}(t) = \binom{d}{i} t^i (1-t)^{d-i}$$

ith Bernstein polynomial of degree d



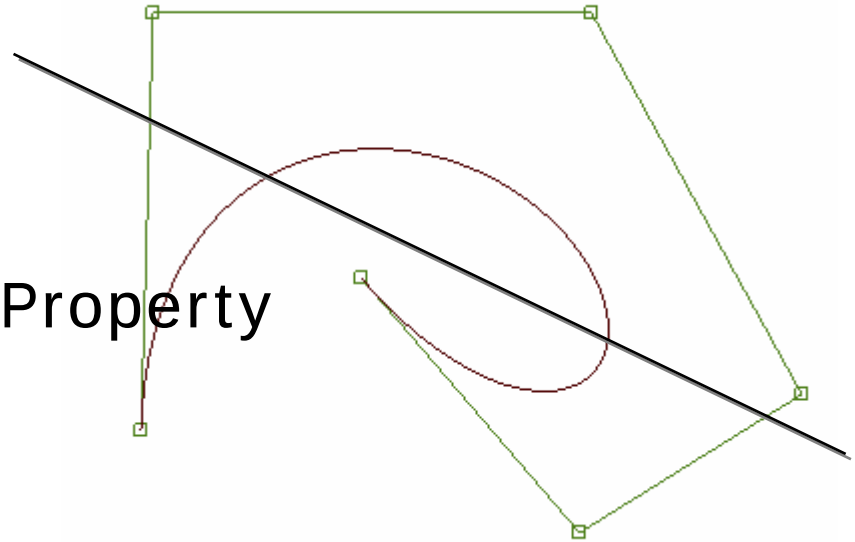
Bezier

- Paul de Faget de Casteljau, Citroen, 1959
- Pierre Bezier, Renault, UNISUF system, 1962
- A.R. Forrest, Cambridge, 1968



Properties of Bezier

- Affine invariance
- Convex hull property
- Endpoint interpolation
- Symmetry
- Linear precision
- Pseudo-local control
- Variation Diminishing Property



Spline curves

- J. Ferguson , Boeing Co., 1963
- C. de Boor, W. Gordon, General Motors, 1963

- to interpolate given data
- piecewise polynomial curves with certain differentiability constraints
- not to design free form curves

Piecewise cubic hermite interpolation

PROBLEM 1. *Let the discrete data* $(x_i, f(x_i), f'(x_i))_{i=1}^m$ *with* $a = x_1 < x_2 < \ldots < x_m = b$ *be given. Find a function* $g = Hf$ *that satisfies the following conditions:*

1. *On each subinterval* (x_i, x_{i+1}) *the function* g *is a cubic polynomial.*
2. *The given function* f *is interpolated by* g *in the sense that*

$$g(x_i) = f(x_i), \quad g'(x_i) = f'(x_i), \; i = 1, \ldots, m.$$

Cubic spline interpolation

PROBLEM 2. *Find a piecewise cubic polynomial that interpolates the given data points and belong to* $C^2[a,b]$. *One of the following boundary conditions :*

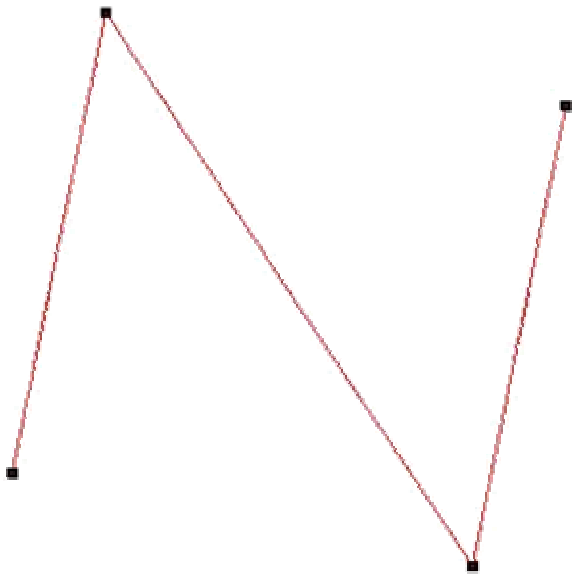
1) Hermite : $g'(a) = f'(a)$ and $g'(b) = f'(b)$
2) Natural : $g''(a) = g''(b) = 0$
3) Free : g''' is continuous at x_2 and x_{m-1}
4) Periodic : $D^j g(a) = D^j g(b)$ for $j = 1, 2$

Find a spline $g = I_Z f$ *in the spline space* S_{3,τ_Z} *such that* $g(x_i) = f(x_i)$ *for* $i = 1, 2, \ldots, n$, *and such that boundary condition* Z *holds*

$$\tau_N = \tau_H = (x_1, x_1, x_1, x_1, x_2, x_3, \ldots, x_{n-2}, x_{n-1}, x_n, x_n, x_n, x_n)$$

$$\tau_F = (x_1, x_1, x_1, x_1, x_3, x_4, \ldots, x_{n-3}, x_{n-2}, x_n, x_n, x_n, x_n)$$

Linear splines



$$p(t : c_1, c_2; t_2, t_3) = \frac{t_3 - t}{t_3 - t_2} c_1 + \frac{t - t_2}{t_3 - t_2} c_2$$

$$f(t) = \begin{cases} p_{2,1}(t) = p(t|c_1, c_2; t_2, t_3), & t \in [t_2, t_3) \\ p_{3,1}(t) = p(t|c_2, c_3; t_3, t_4), & t \in [t_3, t_4) \\ \vdots & \vdots \\ p_{n,1}(t) = p(t|c_{n-1}, c_n; t_n, t_{n+1}), & t \in [t_n, t_{n+1}) \end{cases}$$

$$f(t) = \sum_{i=2}^{n} p_{i,1}(t) B_{i,0}(t)$$

$$B_{i,0}(t) = \begin{cases} 1, & t_i \le t < t_{i+1}, \\ 0, & otherwise \end{cases}$$

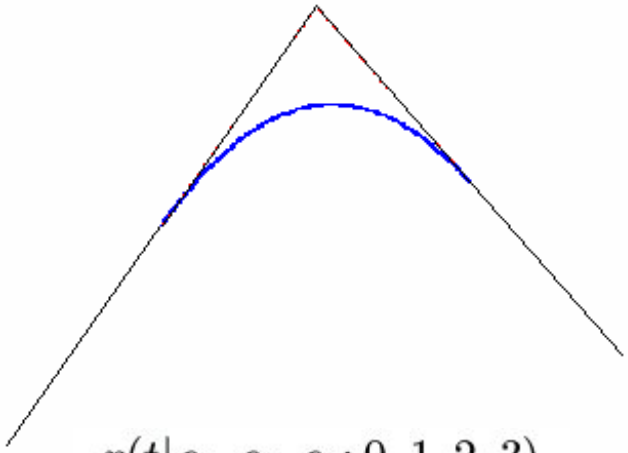
Quadratic splines

$$f(t) = \begin{cases} p_{3,2}(t) = p(t|c_1, c_2, c_3; t_2, t_3, t_4, t_5), & t \in [t_3, t_4) \\ p_{4,2}(t) = p(t|c_2, c_3, c_4; t_3, t_4, t_5, t_6), & t \in [t_4, t_5) \\ \vdots & \vdots \\ p_{n,2}(t) = p(t|c_{n-2}, c_{n-1}, c_n; t_{n-1}, t_n, t_{n+1}, t_{n+2}), & t \in [t_n, t_{n+1}) \end{cases}$$

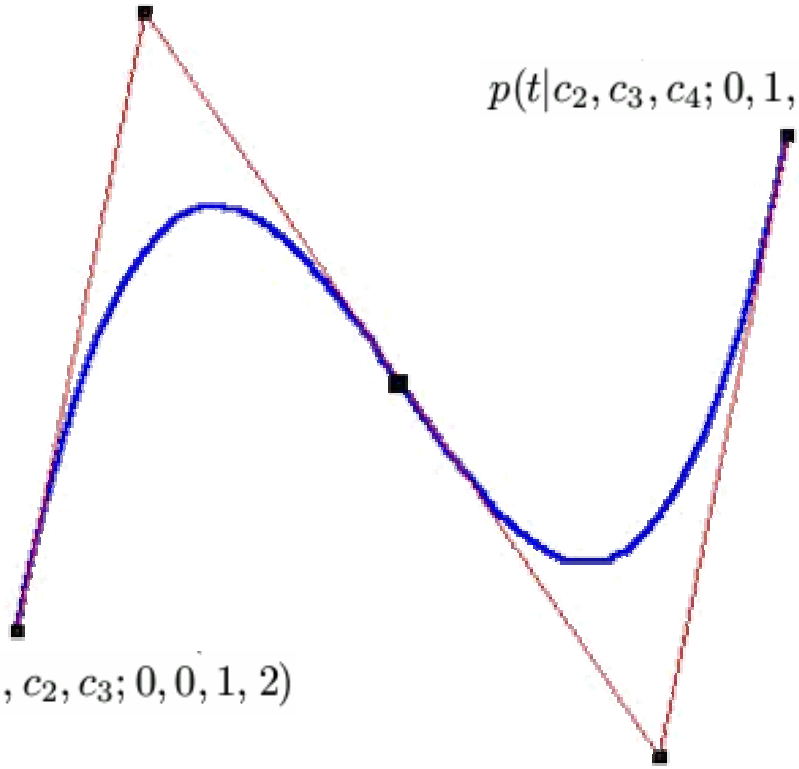
$$p(t|c_1, c_2, c_3; t_2, t_3, t_4, t_5) =$$
$$\frac{t_4 - t}{t_4 - t_3} p(t|c_1, c_2; t_2, t_4) + \frac{t - t_3}{t_4 - t_3} p(t|c_2, c_3; t_3, t_5)$$

$$f(t) = \sum_{i=3}^{n} p_{i,2}(t) B_{i,0}(t)$$

Quadratic splines



$$p(t|c_1, c_2, c_3; 0, 1, 2, 3)$$
$$= (2-t)p(t|c_1, c_2; 0, 2)$$
$$+(t-1)p(t|c_2, c_3; 1, 3)$$



with knots $t = (0, 0, 0, 1, 2, 2, 2)$

B-spline

- Recurrence Relation

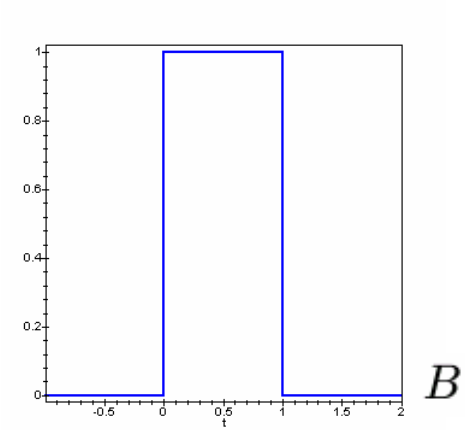
$$B_{i,d}(t) = B(t|t_i, \ldots, t_{i+d+1})$$

$$B(t|a, b, \ldots, c, d) = \frac{t-a}{c-a} B(t|a, b, \ldots, c) + \frac{d-t}{d-b} B(t|b, \ldots, c, d)$$

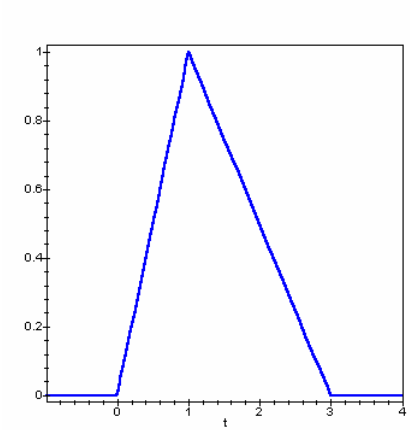
- Bernstein polynomial

$$b_{i,d}(t) = B(t|\underbrace{a, \ldots, a}_{d-i+1}, \underbrace{b, \ldots, b}_{i+1})$$

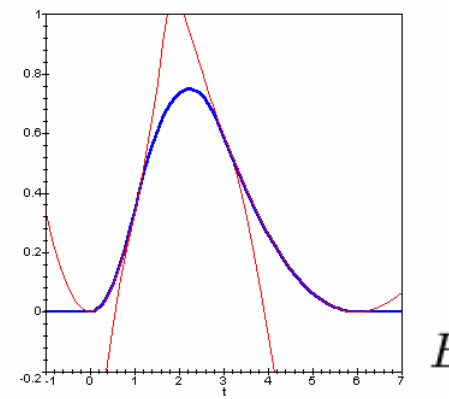
B-spline



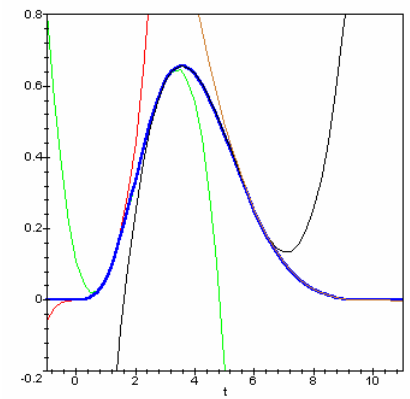
$B(t|0,1)$



$B(t|0,1,3)$



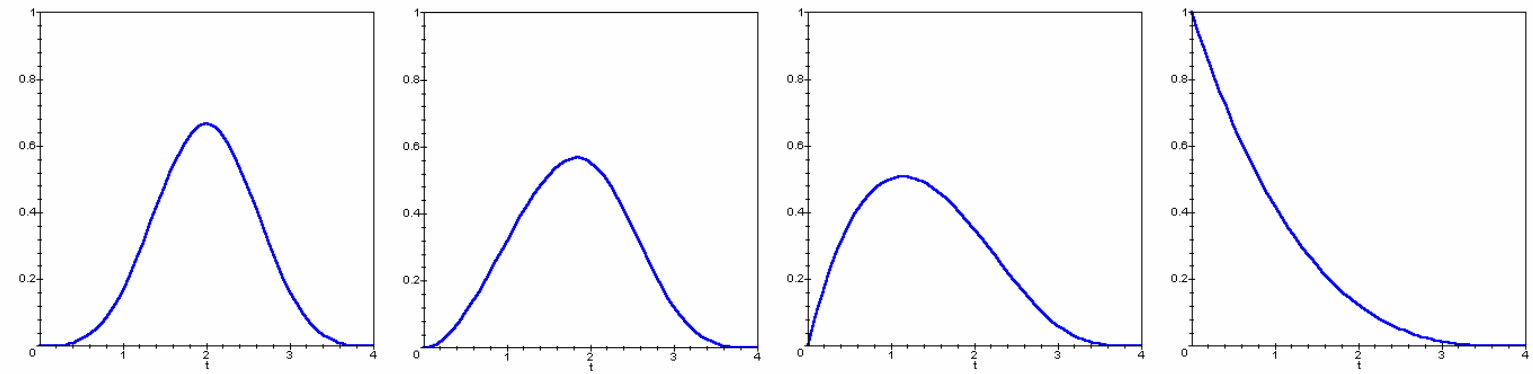
$B(t|0,1,3,6)$



$B(t|0,1,3,6,10)$

B-spline

- Smoothness=Degree-Multiplicity



$B(t|0,1,2,3,4)$ $B(t|0,0,2,3,4)$ $B(t|0,0,0,3,4)$ $B(t|0,0,0,0,4)$

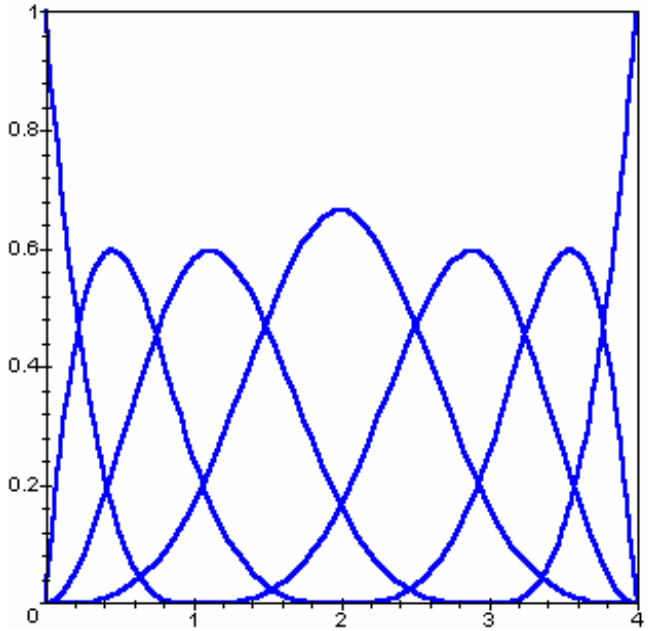
Representation splines

$$p_{i,d}(t) = \frac{t_{i+1} - t}{t_{i+1} - t_i} p_{i-1,d-1}(t) + \frac{t - t_i}{t_{i+1} - t_i} p_{i,d-1}(t)$$

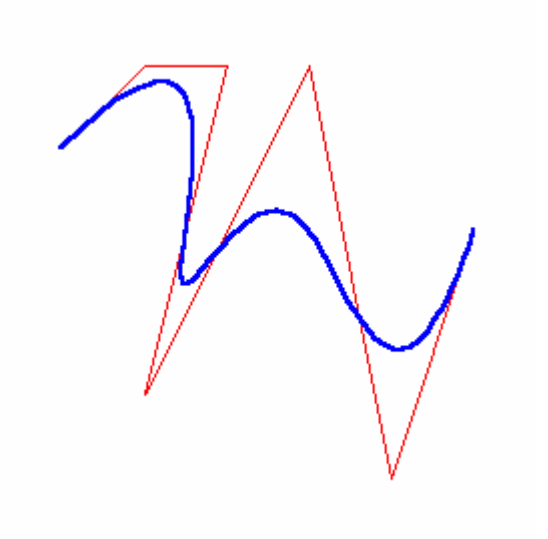
$$f(t) = \sum_{i=d+1}^{n} p_{i,d}(t) B_{i,0}(t)$$

$$f(t) = \sum_{i=1}^{n} c_i B_{i,d}(t)$$

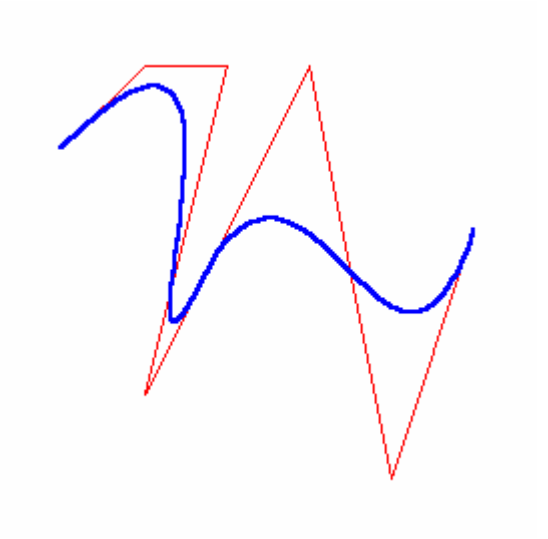
$$B_{i,d}(t) = \frac{t - t_i}{t_{i+d} - t_i} B_{i,d-1}(t) + \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}} B_{i+1,d-1}(t)$$



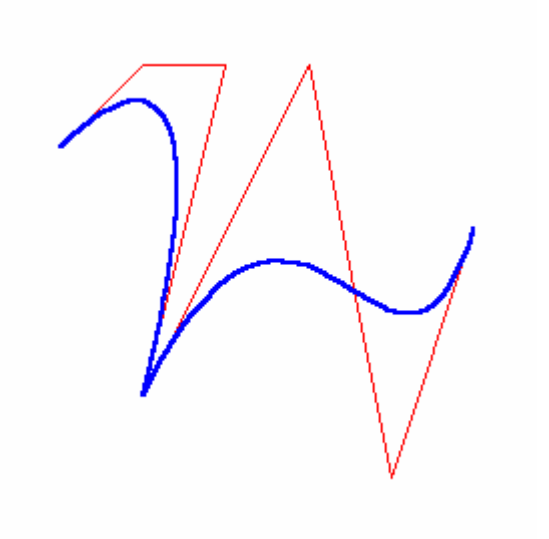
Cubic splines



$(0,0,0,0,1,2,3,4,4,4,4)$



$(0,0,0,0,1,2,2,4,4,4,4)$



$(0,0,0,0,2,2,2,4,4,4,4)$

Spline space

Let $\mathbf{t} = (t_i)_{i=1}^{n+d+1}$ *be a nondecreasing sequence of real numbers, i.e., a knot vector for a total of* n *B-splines. The linear space of all linear combinations of these B-splines is the spline space* $S_{d,t}$ *defined by*

$$S_{d,\mathbf{t}} = span\{B_{1,d}, \ldots, B_{n,d}\} = \left\{ \sum_{i=1}^{n} c_i B_{i,d} | c_i \in \mathbb{R}^q \ for \ 1 \leq i \leq n \right\}$$

An element f *of* $S_{d,\mathbf{t}}$ *is called a spline of degree* d *with knots* $\mathbf{t}$.

B-spline

- C. de Boor, 1972
- W. Gordon, Richard F. Riesenfeld, 1974

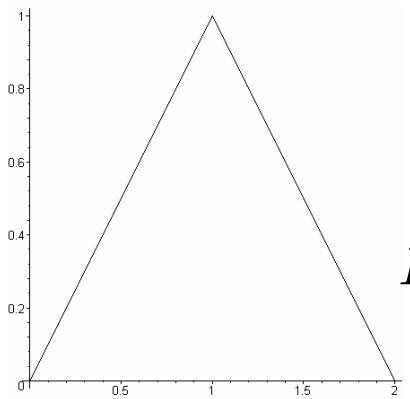
- Larry L. Schumaker
- Tom Lyche
- Nira Dyn

B-spline basis functions

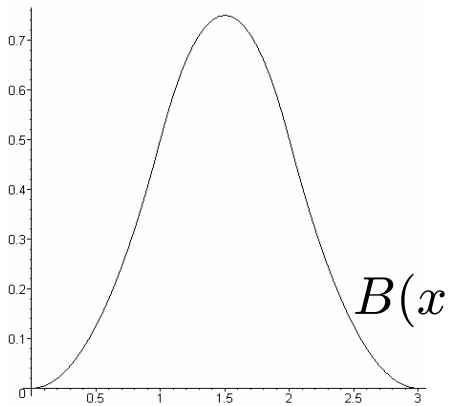
$$B_0(x) = \begin{cases} 1, & 0 \le x < 1, \\ 0, & otherwise \end{cases} \quad B(x:0,1)$$

$$B_1(x) = \begin{cases} x, & 0 \le x < 1, \\ 2-x, & 1 \le x < 2, \\ 0, & otherwise \end{cases}$$

$$B_2(x) = \begin{cases} \frac{1}{2}x^2, & 0 \le x < 1, \\ -\frac{3}{2} + 3x - x^2, & 1 \le x < 2, \\ \frac{1}{2}(-3+x)^2, & 2 \le x < 3, \\ 0, & otherwise \end{cases}$$



$B(x:0,1,2)$



$B(x:0,1,2,3)$

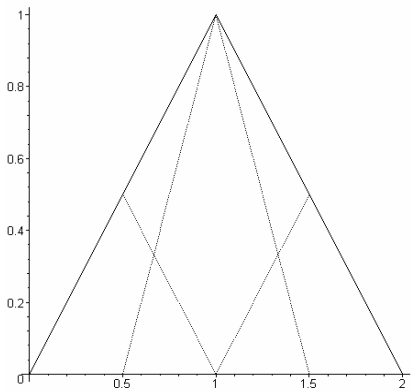
Refinement relation for B-spline

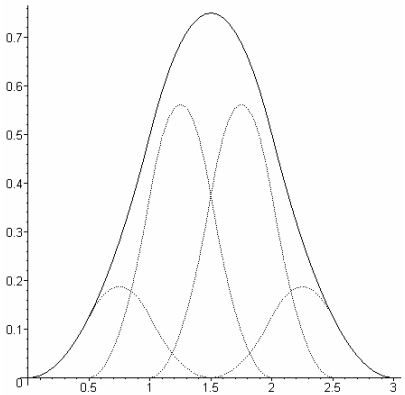
$$B_0(x) = B_0(2x) + B_0(2x-1)$$

$$B_1(x) = \frac{1}{2}B_1(2x) + B_1(2x-1) + \frac{1}{2}B_1(2x-2)$$

$$B_2(x) = \frac{1}{4}B_2(2x) + \frac{3}{4}B_2(2x-1) + \frac{3}{4}B_2(2x-2) + \frac{1}{4}B_2(2x-3)$$

$$B_n(x) = \frac{1}{2^n}\sum_{i=0}^{n+1}\binom{n+1}{i}B_n(2x-i)$$

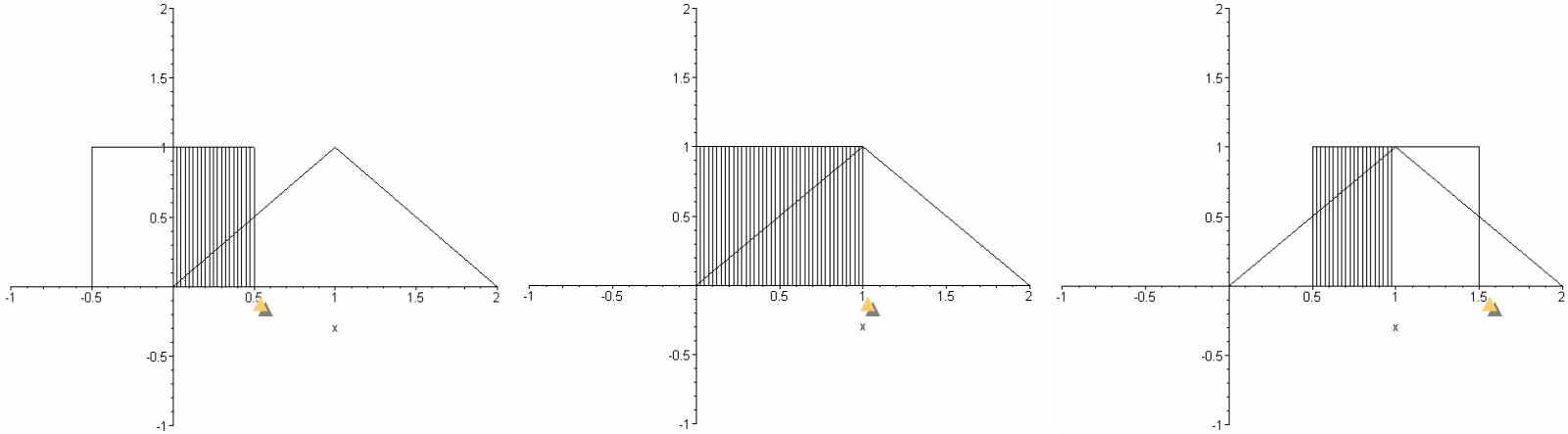




Repeated integration for B-spline

$$B_0(x) = \begin{cases} 1, & 0 \le x < 1, \\ 0, & otherwise \end{cases}$$

$$\begin{aligned} B_n(x) &= \int_0^1 B_{n-1}(x-t)dt \\ &= B_0(x) \otimes B_{n-1}(x) \quad p(x) \otimes q(x) = \int_{-\infty}^{\infty} p(t)q(x-t)dt \end{aligned}$$

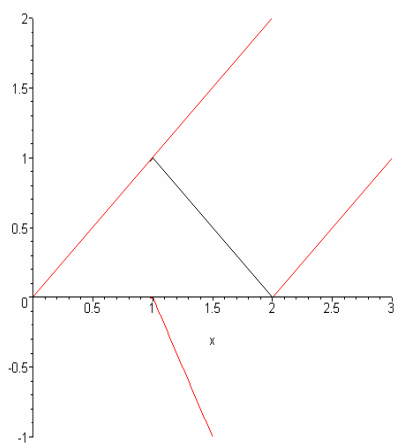


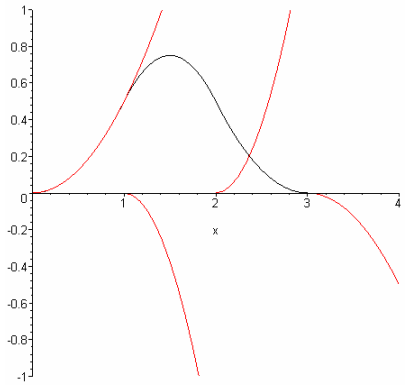
Truncated powers for B-spline

$$B_1(x) = c_1(x) - 2c_1(x-1) + c_1(x-2)$$

$$B_2(x) = c_2(x) - 3c_2(x-1) + 3c_2(x-2) - c_2(x-3)$$

$$B_n(x) = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} c_n(x-i) \quad c_n(x) = \begin{cases} \frac{1}{n!}x^n, & x \geq 0, \\ 0, & otherwise \end{cases}$$



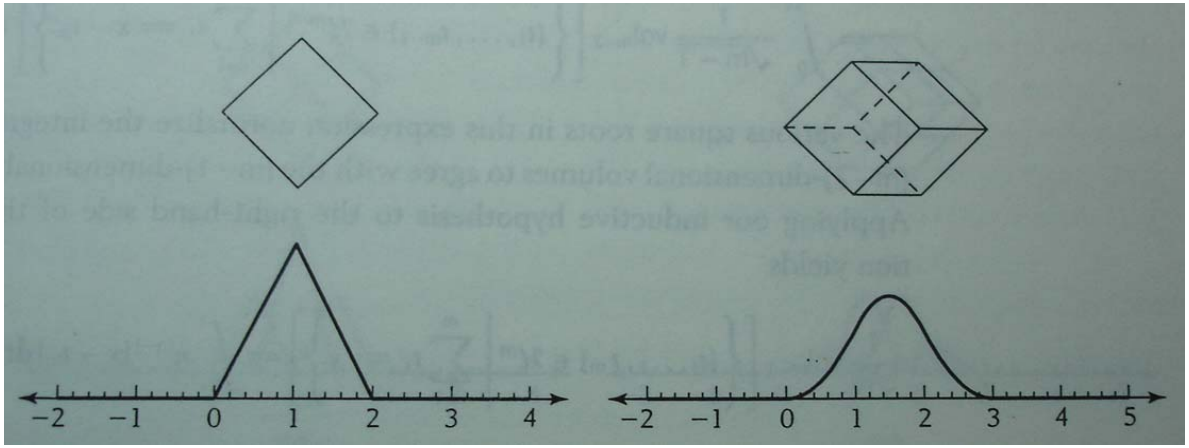


Cross-sectional Volumes

$$c_n(x) = \frac{1}{\sqrt{n+1}} vol_n \left[\left\{ \{t_0, t_1, \ldots, t_n\} \in (\mathcal{R}^+)^{n+1} | \sum_{i=0}^{n} t_i = x \right\} \right]$$

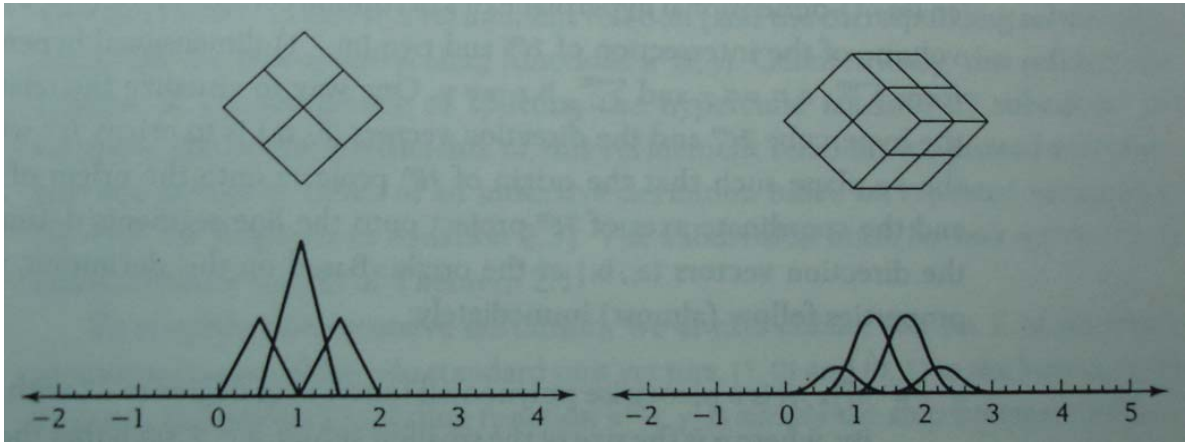
B-spline basis as Cross-sectional Volumes

$$B_n(x) = \frac{1}{\sqrt{n+1}} vol_n \left[\left\{ \{t_0, t_1, \ldots, t_n\} \in \mathcal{H}^{n+1} | \sum_{i=0}^{n} t_i = x \right\} \right]$$

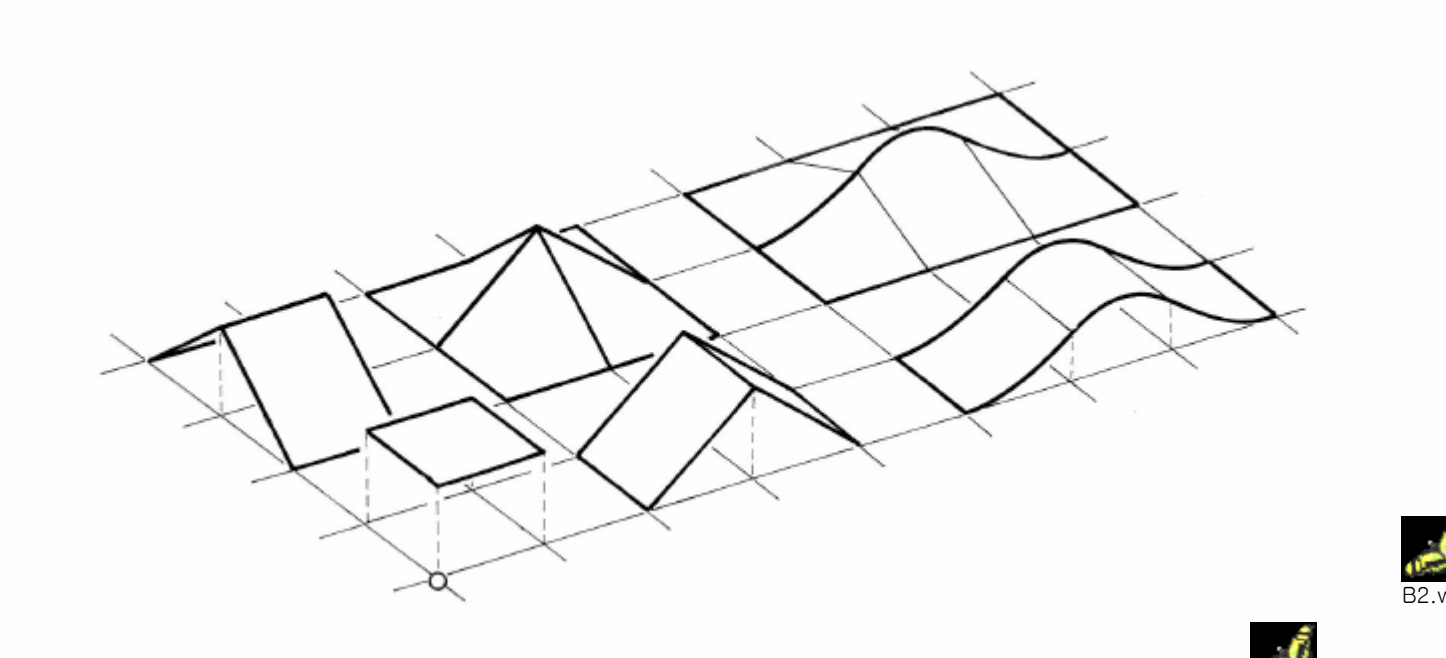


Cross-sectional Volumes for subcubes

$$B_n(x) = \frac{1}{\sqrt{n+1}} vol_n \left[\left\{ \{t_0, t_1, \ldots, t_n\} \in \mathcal{H}^{n+1} | \sum_{i=0}^{n} t_i = x \right\} \right]$$



Bivariate Tensor Product B-spline



B2.wrl

B1.wrl

Bo.wrl

B12.wrl

B01.wrl

$$B_{n,m}(x,y) = B_n(x)B_m(y)$$

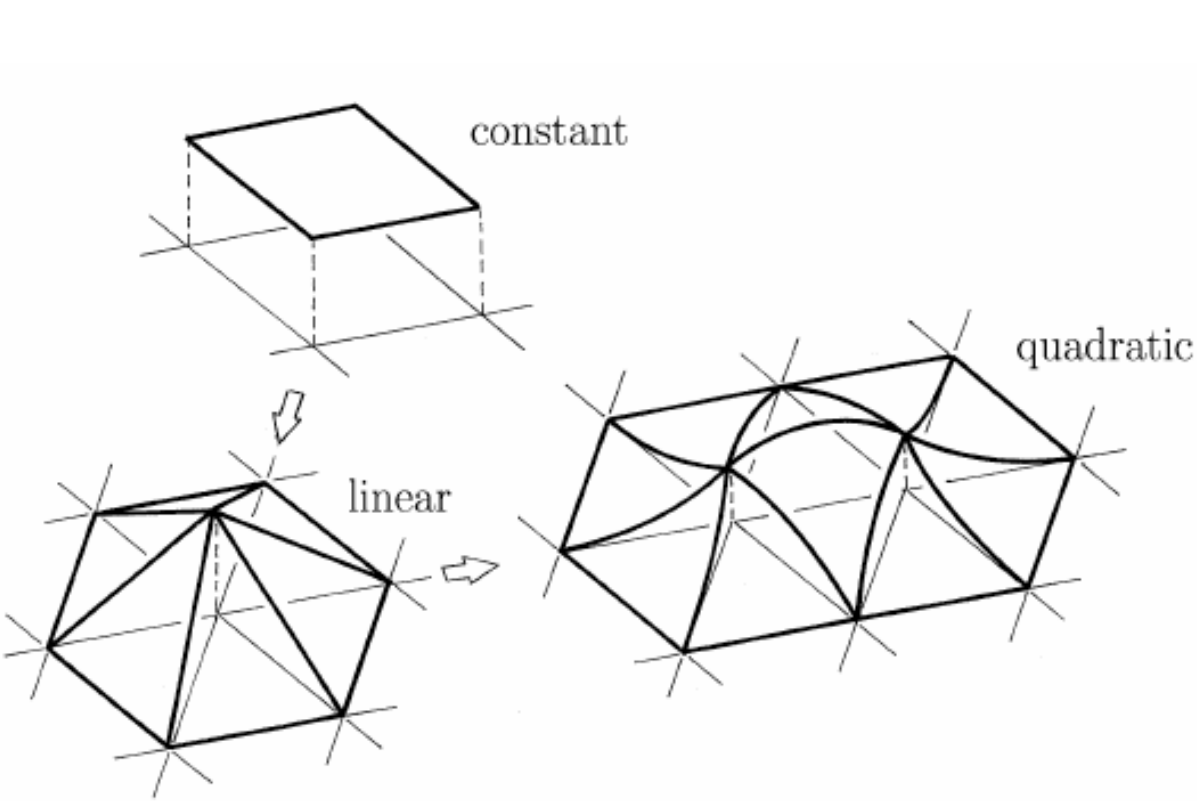
Box-spline as Cross-sectional Volumes

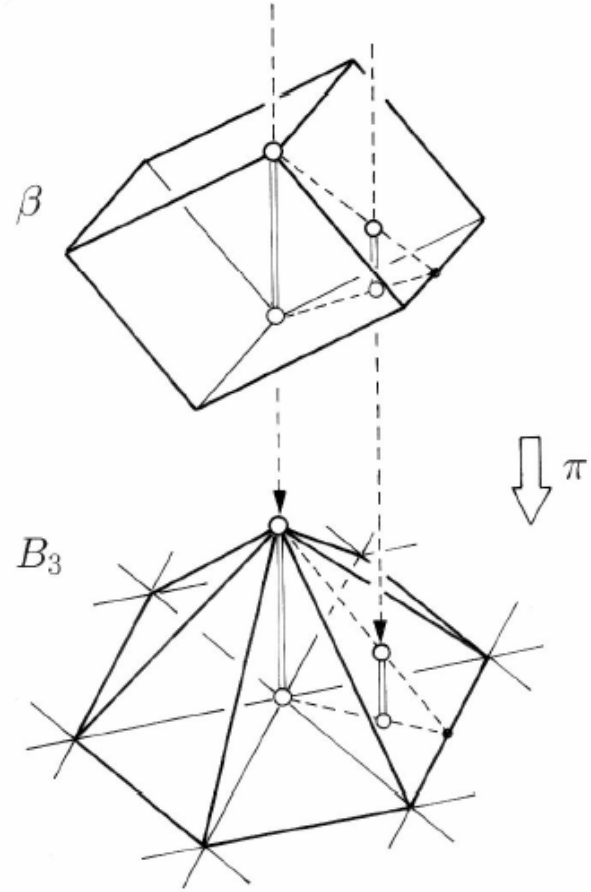
$$B_n^{\sum}(x,y) = vol_n\left[\left\{\{t_0,t_1,\ldots,t_n\} \in \mathcal{H}^{n+1} | \sum_{i=0}^{n}\{a_i,b_i\}t_i = \{x,y\}\right\}\right]$$

$$\sum = \{\{a_i,b_i\} \in z^2 | i = 0,1,\ldots,n\}$$

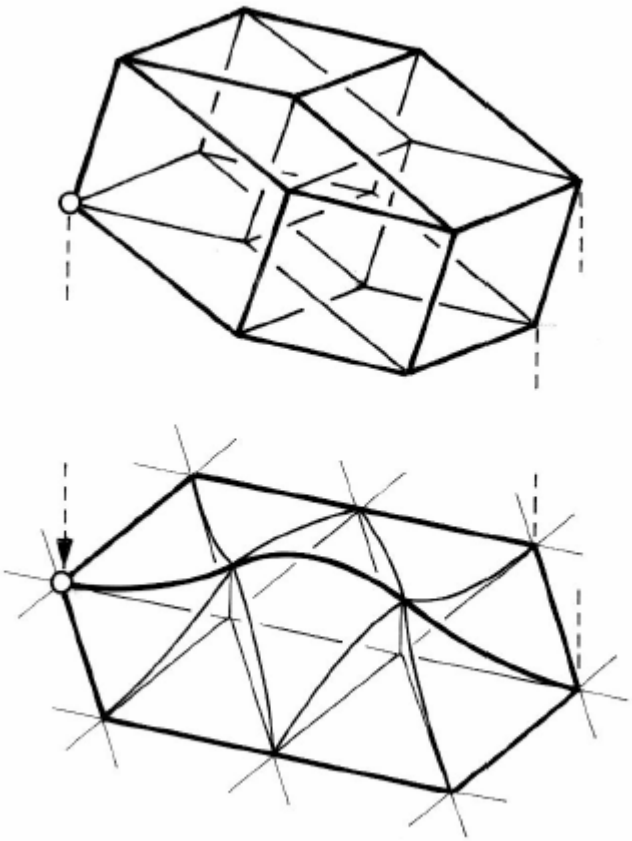
$$B_n(x) = \frac{1}{\sqrt{n+1}} vol_n\left[\left\{\{t_0,t_1,\ldots,t_n\} \in \mathcal{H}^{n+1} | \sum_{i=0}^{n} t_i = x\right\}\right]$$

Bivariate Box spline over triangular grid





Shadows of boxes



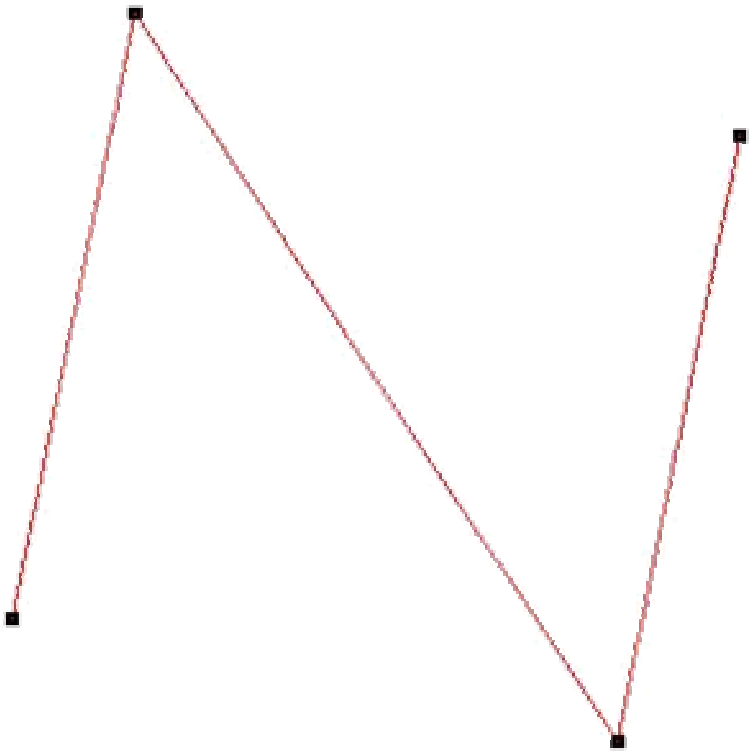
Refinement Relation for B-spline

$$S^n(x) = \frac{1}{2}(1+x)S^{n-1}(x)$$

$$S^0(x) = 1 + x$$

$$S^1(x) = \frac{1}{2}(1 + 2x + x^2)$$

$$S^2(x) = \frac{1}{4}(1 + 3x + 3x^2 + x^3)$$



Refinement Relation for Box-spline

$$B^{\Sigma}(x,y) = \begin{cases} 1, & 0 \le x, y < 1, \\ 0, & otherwise \end{cases} \qquad \sum = \{(1,0),(0,1)\}$$

$$B^{\Sigma}(x,y) =$$

$$B^{\Sigma}(2x,2y) + B^{\Sigma}(2x-1,2y) + B^{\Sigma}(2x,2y-1) + B^{\Sigma}(2x-1,2y-1)$$

$$S^{\tilde{\Sigma}}(x,y) = \frac{1}{2}(1 + x^a y^b) S^{\Sigma}(x,y) \qquad \tilde{\sum} = \sum \cup \{(a,b)\}$$

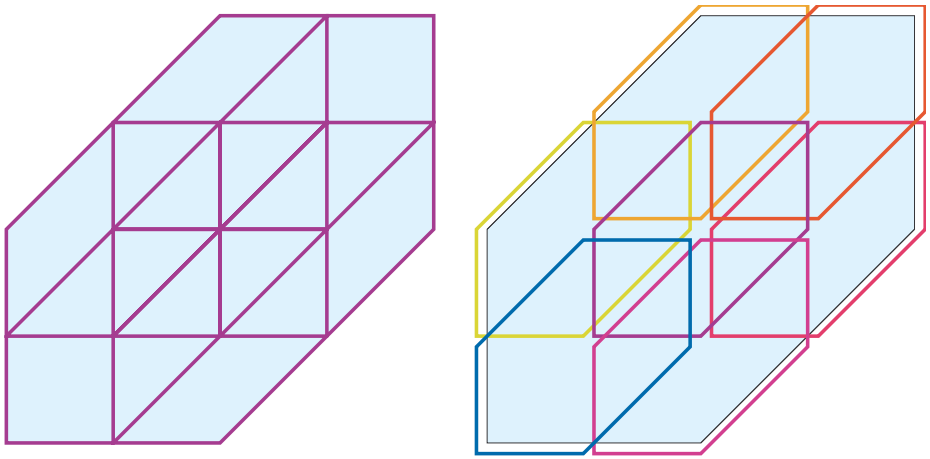
$$S^{\Sigma}(x,y) = \frac{1}{2}(1+x)(1+y)$$

Refinement Relation for Box-spline

$$S^{\tilde{\Sigma}}(x,y) = \frac{1}{2}(1+x)(1+y)(1+xy)$$

$$= \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ y \\ y^2 \end{bmatrix}$$

$$\tilde{\Sigma} = \{(1,0),(0,1),(1,1)\}$$



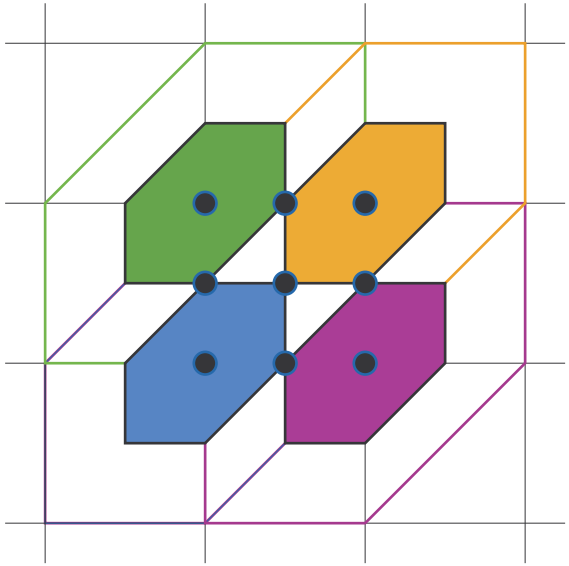
Subdivision for Box-spline

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 \end{bmatrix}$$



$$\tilde{\sum} = \{(1,0),(0,1),(1,1)\}$$

Refinement Relation for Box-spline

$$\tilde{\sum} = \{(1,0),(0,1),(1,1),(-1,1)\}$$

$$S^{\tilde{\Sigma}}(x,y) = \frac{1}{4}(1+x)(1+y)(1+xy)(1+x^{-1}y)$$

$$= \begin{bmatrix} x^{-1} & 1 & x & x^2 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ y \\ y^2 \\ y^3 \end{bmatrix}$$

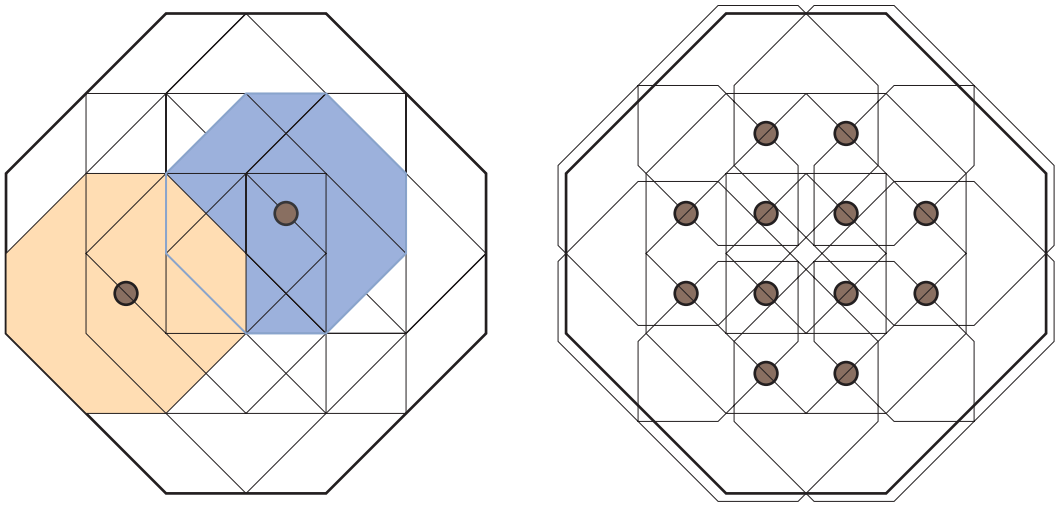
Subdivision for Box-spline

$$\begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 \end{bmatrix}$$



$$\tilde{\sum} = \{(1,0),(0,1),(1,1),(-1,1)\}$$

Refinement Relation for Box-spline

$$\tilde{\sum} = \{(1,0),(1,0),(0,1),(0,1)\}$$

$$S^{\tilde{\Sigma}}(x,y) = \frac{1}{4}(1+x)^2(1+y)^2$$

$$= \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ y \\ y^2 \end{bmatrix}$$

Refinement Relation for Box-spline

$$\tilde{\sum} = \{(1,0),(1,0),(1,0),(0,1),(0,1),(0,1)\}$$

$$S^{\tilde{\Sigma}}(x,y) = \frac{1}{16}(1+x)^3(1+y)^3$$

$$= \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \frac{1}{16} \begin{bmatrix} 1 & 3 & 3 & 1 \\ 3 & 9 & 9 & 1 \\ 3 & 9 & 9 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ y \\ y^2 \\ y^3 \end{bmatrix}$$

Refinement Relation for Box-spline

$$\tilde{\sum} = \{(1,0),(1,0),(0,1),(0,1),(1,1),(1,1)\}$$

$$S^{\tilde{\Sigma}}(x,y) = \frac{1}{16}(1+x)^2(1+y)^2(1+xy)^2$$

$$= \begin{bmatrix} 1 & x & x^2 & x^3 & x^4 \end{bmatrix} \frac{1}{16} \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 6 & 6 & 2 & 0 \\ 1 & 6 & 10 & 6 & 1 \\ 0 & 2 & 6 & 6 & 2 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ y \\ y^2 \\ y^3 \\ y^4 \end{bmatrix}$$

Condition number

Let $\mathbb{U}$ be a normed linear space. A basis (ϕ_j) for $\mathbb{U}$ is said to be stable with respect to a vector norm $||.||$ if there are small positive constants C_1 and C_2 such that

$$C_1^{-1}||(c_j)|| \leq ||\sum_j c_j \phi_j|| \leq C_2||(c_j)||,$$

for all sets of coefficients $c = (c_j)$. Let C_1^* and C_2^* denote the smallest possible values of C_1 and C_2. The condition number of the basis is then defined to be $\kappa = C_1^* C_2^*$.

Condition number of B-spline basis

There is a constant K, depending only on the polynomial degree d, such that all spline spaces $\mathbb{S}_{d,\mathbf{t}}$ and all splines $f = \sum_{i=1}^{n} b_i B_{i,d} \in \mathbb{S}_{d,\mathbf{t}}$ with B-spline coefficients $\mathbf{b} = (b_i)_{i=1}^{n}$ the inequalites

$$K^{-1}||\mathbf{b}||_p \leq ||f||_{p,[t_1,t_n+d]} \leq ||\mathbf{b}||_p$$

hold.

$$\kappa_{0,\infty} = \kappa_{1,\infty} = 1, \kappa_{2,\infty} = 3$$
$$\kappa_{3,\infty} \approx 5.5680, \kappa_{4,\infty} \approx 12.088$$
$$\kappa_{d,\infty} = O(2^d), \kappa_{d,p} < d2^d$$

Tom Lyche and Karl Scherer, *On the p-norm* condition *number of the multivariate triangular Bernstein basis*, Journal of Computational and Applied Mathematics 119(2000) 259-273

Stability

Suppose (ϕ_j) *is a stable basis for* $\mathbb{U}$. *If* $f = \sum_j c_j \phi_j$ *and* $g = \sum_j b_j \phi_j$ *are two elements in* $\mathbb{U}$ *with* $f \neq 0$, *then*

$$\frac{||f-g||}{||f||} \leq \kappa \frac{||c-b||}{||c||}$$

where κ *is the condition number of the basis.*

Blossom

Let $G(t)$ be a polynomial curve of degree n or less. The blossom $B_G(u_1, \ldots, u_n)$ of the polynomial $G(t)$ is the unique multivariate polynomial with the following properties:

1. multiaffine : for all indices i and all real numbers c

$$B_G(u_1, \ldots, u_{i-1}, cu + (1-c)v, u_{i+1}, \ldots, u_n)$$
$$= cB_G(u_1, \ldots, u_{i-1}, u, u_{i+1}, \ldots, u_n)$$
$$+(1-c)B_G(u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_n)$$

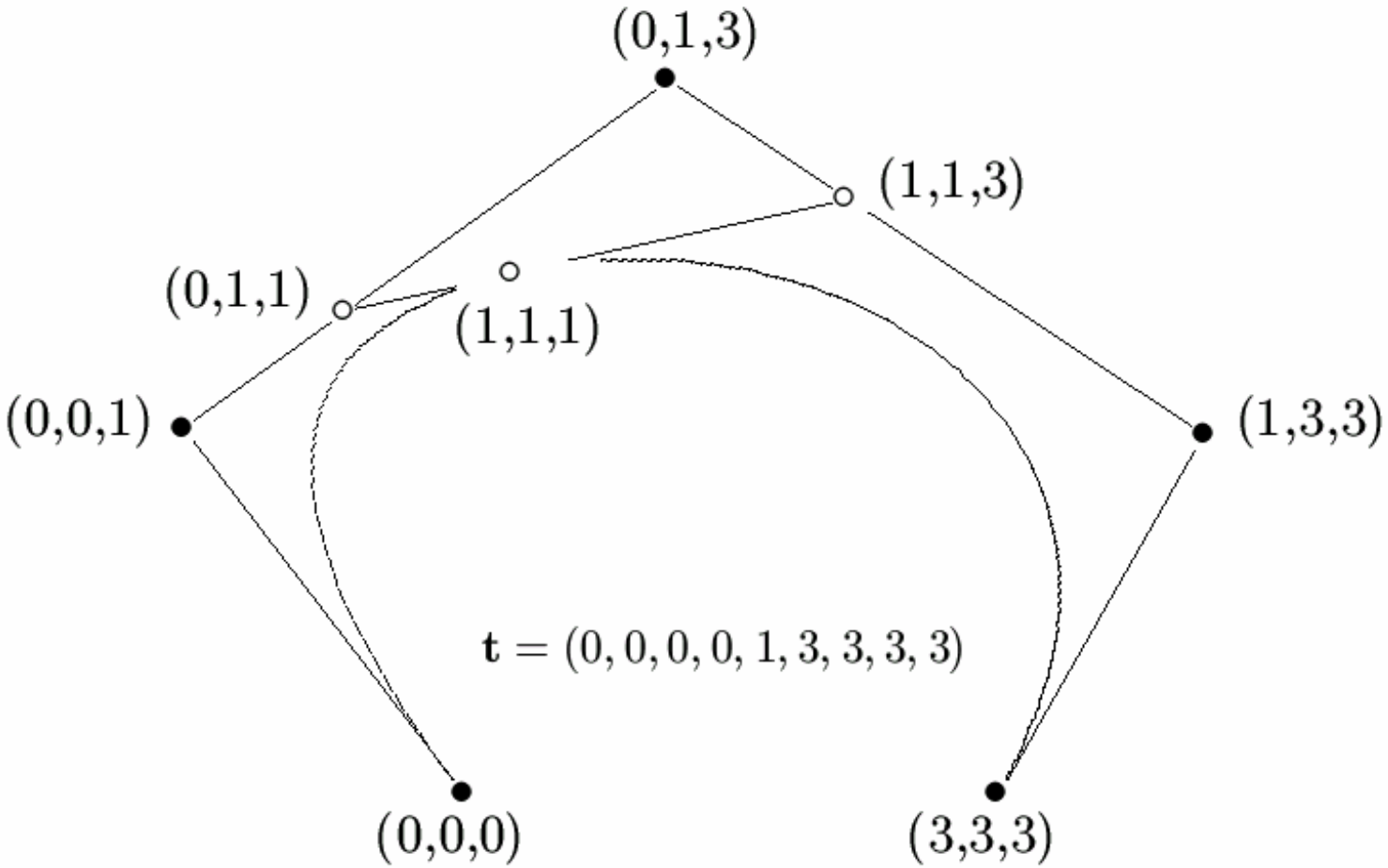
2. symmetry : for any permutation π of $\{1, 2, \ldots, n\}$

$$B_G(u_1, \ldots, u_n) = B_G(u_{\pi(1)}, \ldots, u_{\pi(n)})$$

3. diagonal : B_G reduces to G when evaluated on its diagonal; i.e,

$$B_G(t, \ldots, t) = G(t)$$

Blossom



B-spline problems

- Degree Elevation
- Degree Reduction
- Knot Insertion
- Knot Deletion

Gerald Farin, Curves and Surfaces for Computer Aided Geometric Design, 4th ed, Academic Press (1996)

Ronald N. Goldman, Tom Lyche, editors, Knot Insertion and Deletion Algorithms for B-Spline Curves and Surfaces, SIAM (1993)

Bezier Degree Reduction

PROBLEM 1 (L_2 Degree Reduction). *Let $\{c_i\}_{i=0}^n$ be a given set of control points which define the Bézier curve*

$$c^n(t) = \sum_{i=0}^{n} c_i B_i^n(t)$$

of degree n. Then find another point set $\{b_i\}_{i=0}^m$ defining the approximative Bézier curve

$$b^m(t) = \sum_{i=0}^{m} b_i B_i^m(t)$$

of lower degree $m < n$ so that an L_2-distance function $d_2(b^m, c^n)$ between b^m and c^n is minimized.

Bezier Degree Reduction

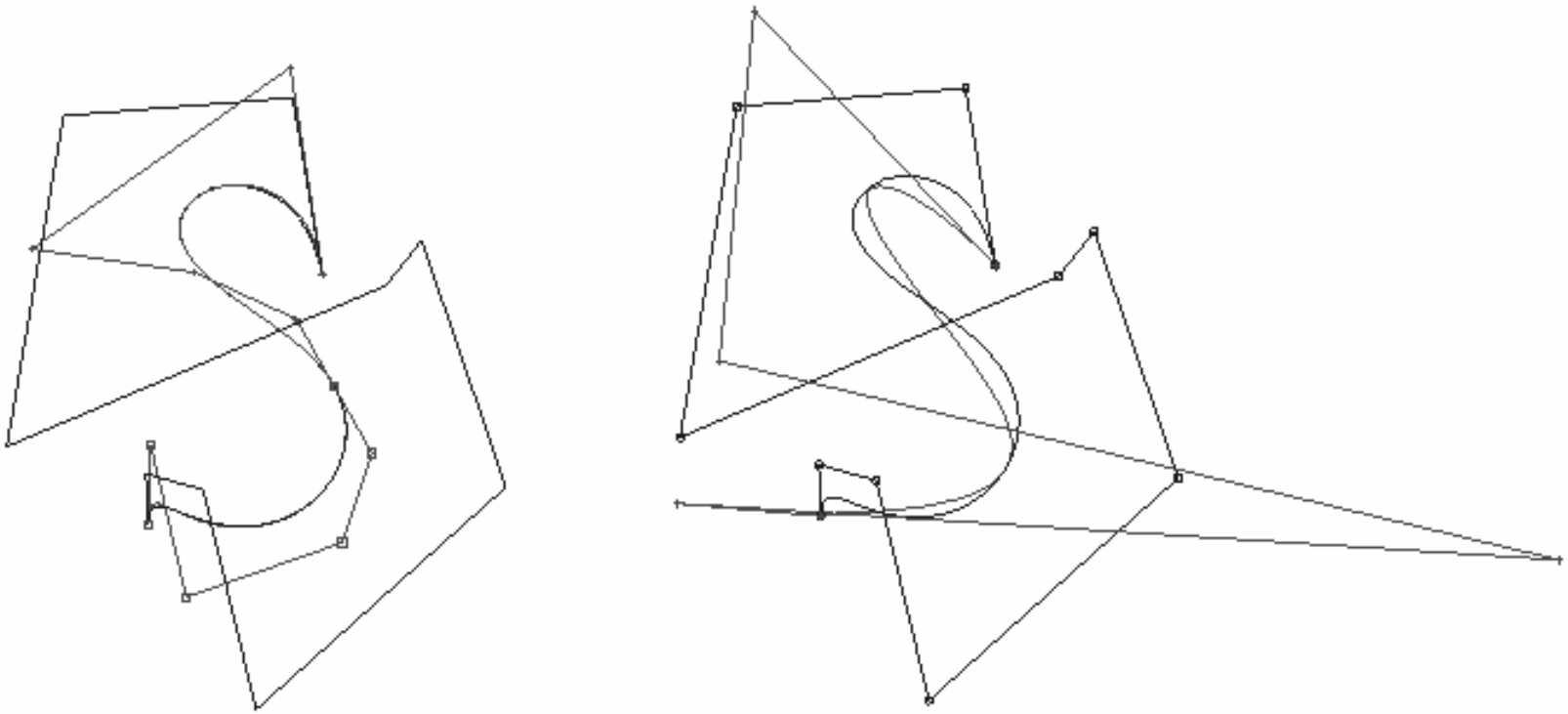
- Least square method
- Legendre-Bernstein basis transformations

$$
\begin{tikzcd}
\mathbf{c} \arrow[r, shift left, "\mathbf{M}_n^{-1}"] \arrow[d, shift right, "(\mathbf{T}_{m,r}^t \mathbf{Q}_n \mathbf{T}_{m,r})^{-1} \mathbf{T}_{m,r}^t \mathbf{Q}_n"'] & \mathbf{l} \arrow[l, shift left, "\mathbf{M}_n"] \arrow[d, shift right, "\mathbf{I}_{n,-r}"'] \\
\mathbf{c}^{(-r)} \arrow[r, shift left, "\mathbf{M}_m^{-1}"] \arrow[u, shift right, "\mathbf{T}_{m,r}"'] & \mathbf{l}^{(-r)} \arrow[l, shift left, "\mathbf{M}_m"] \arrow[u, shift right, "\mathbf{I}_{m,r}"']
\end{tikzcd}
$$

Rida T. Farouki, *Legendre-Bernstein basis transformations*, Journal of Computational and Applied Mathematics 119(2000) 145-160.

Byung-Gook Lee, Yunbeom Park and Jaechil Yoo, *Application of Legendre-Bernstein basis transformations to degree elevation and degree reduction*, Computer Aided Geometric Design 19(2002) 709-718.

Bezier Degree Reduction with constrained



Contents

- Affine combination
- Bezier curves
- Spline curves
- B-spline curves
- Condition number
- L1-norm spline
- Quasi-interpolant
- Box-spline
- Cross-sectional volumes
- Refinement relation
- Subdivision

Reference
"Spline Methods Draft"
Tom Lyche and Knut Morken
"Subdivision Methods for Geometric Design: A Constructive Approach"
Joe Warren and Henrik Weimer