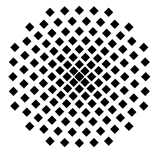


The Web-Method

K. Höllig, U. Reif, and J. Wipper



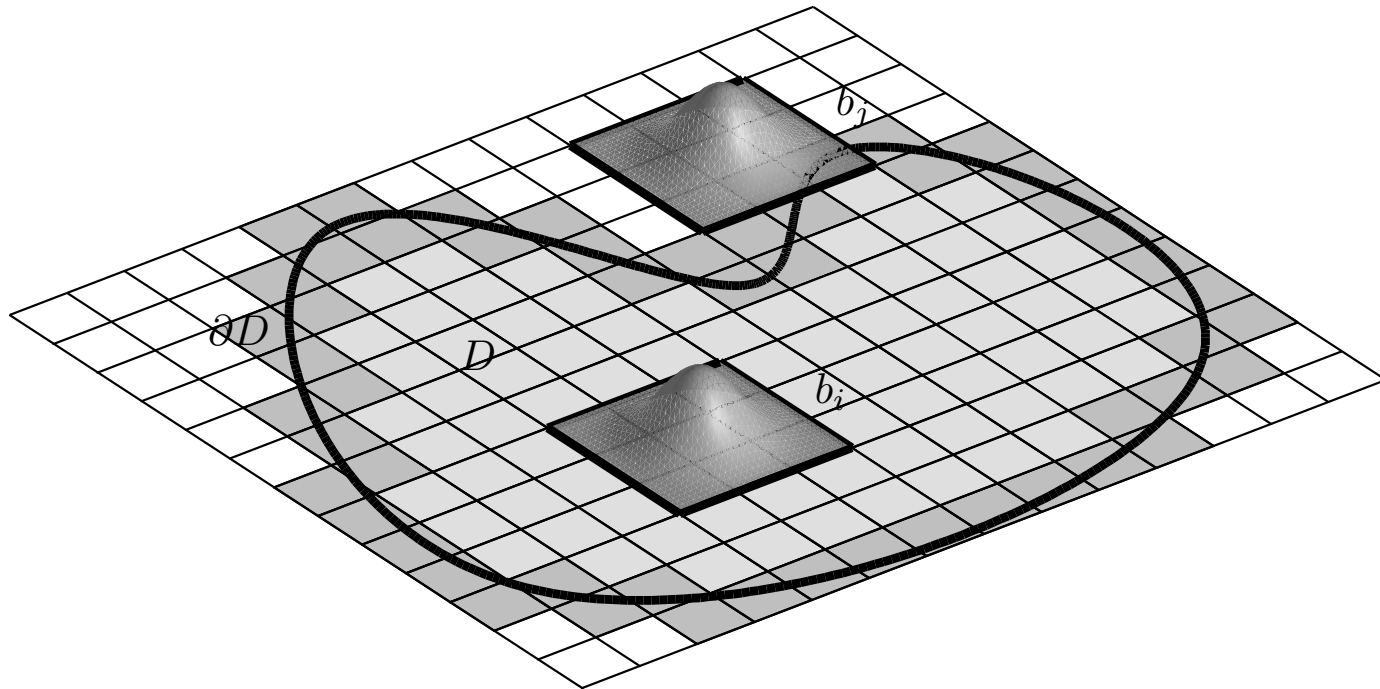
Universität Stuttgart



Technische Universität
Darmstadt

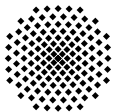
<http://www.web-spline.de>

Splines on bounded Domains

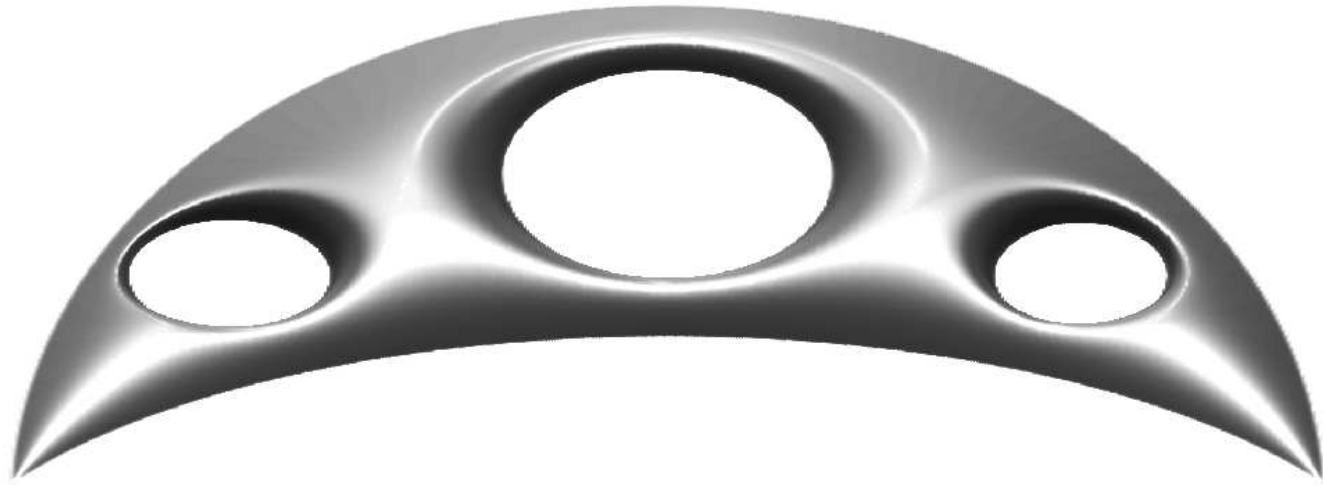


$$\mathbb{B}_h = \text{span}_{K} b_k, \quad K = I \cup J$$

boundary conditions via weight function
stability via extension of inner B-splines



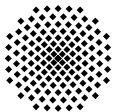
Weight function



essential boundary conditions $w|_{\mathcal{D}} > 0$, $w|_{\Gamma} = 0$, $\Gamma \subseteq \partial\mathcal{D}$

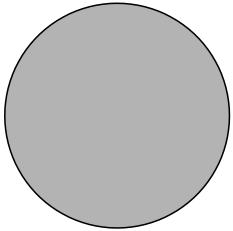
construction:

- explicit formulas
- Rvachev's Boolean expressions
- numerical distance functions
- Reif's blending technique

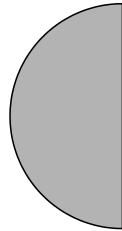


Rvachev's R-Functions

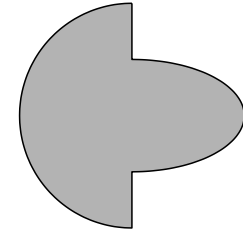
$$w_1 = 1 - x_1^2 - x_2^2$$



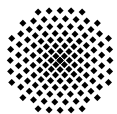
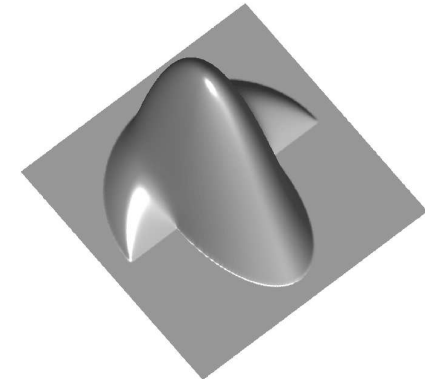
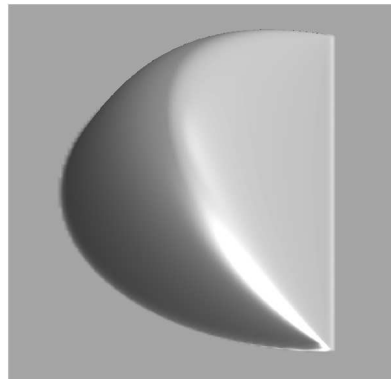
$$w_2 = -x_1$$



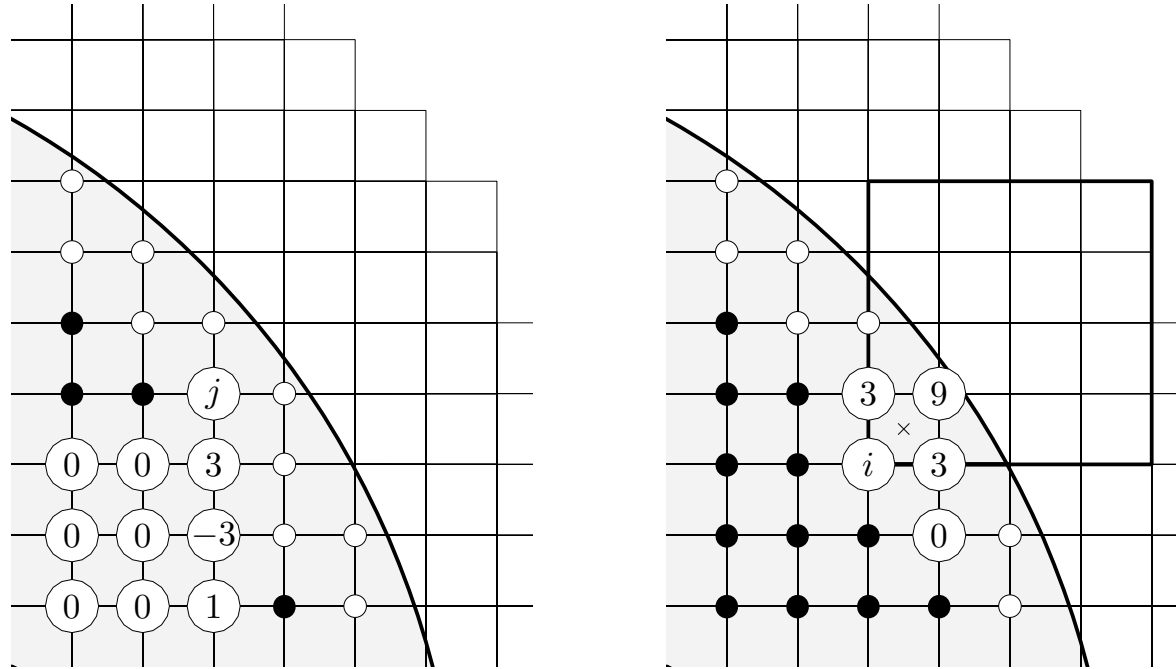
$$w_3 = 1 - x_1^2 - 4x_2^2$$



$$w_{\cap} = w_1 + w_2 - \sqrt{w_1^2 + w_2^2} \quad w_{\cup} = w_{\cap} + w_3 + \sqrt{w_{\cap}^2 + w_3^2}$$



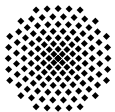
Extension



$I(j)$: nearest $(n + 1)^m$ -array of inner indices

$J(i)$: complementary sets ($i \in I(j) \Leftrightarrow j \in J(i)$)

$e_{i,j}$: values of Lagrange polynomials (uniform grid)
or de Boor functionals (nonuniform grid)

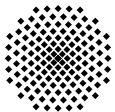


[Nonuniform] Weighted Extended B-Splines

$$B_i = \frac{w}{w(x_i)} \left(b_i + \sum_{j \in J(i)} e_{i,j} b_j \right)$$

properties:

- local support: $e_{i,j} = 0$ for $\|i - j\| \gtrsim 1$
- stability: $\|\sum c_i B_i\|_0 \asymp h^{m/2} \|C\|$
- approximation order: $\|u - P_h u\|_\ell \lesssim h^{n+1-\ell} \|u\|_{n+1}$



Ritz-Galerkin Approximation

H : Hilbert space, incorporating homogeneous boundary conditions

a : elliptic bilinear form

λ : linear functional

weak solution:

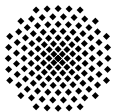
$$a(u, v) = \lambda(v), \quad v \in H$$

finite element approximation:

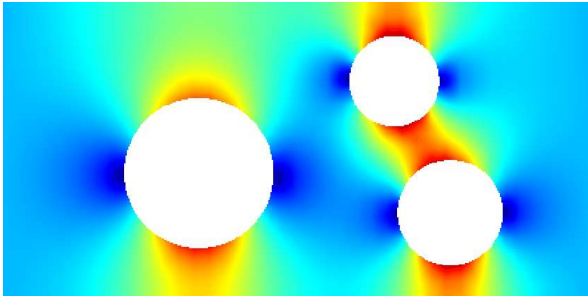
$$a(u_h, B_i) = \lambda(B_i), \quad i \in I$$

error estimate:

$$\|u - u_h\|_H \lesssim \inf_C \left\| u - \sum c_i B_i \right\|_H$$



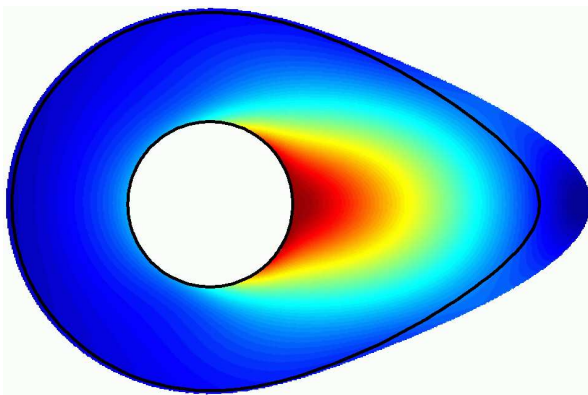
Examples of boundary value problems



incompressible flow

$$-\Delta u = 0 \text{ in } D$$

$$\frac{\partial u}{\partial n} = -v \text{ on } \Gamma_i, \quad \frac{\partial u}{\partial n} = v \text{ on } \Gamma_o, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_b$$



linear elasticity

$$-\operatorname{div} \sigma(u) = f \text{ in } D$$

$$u = 0 \text{ on } \Gamma_0, \quad \sigma(u)n = 0 \text{ on } \Gamma_1$$

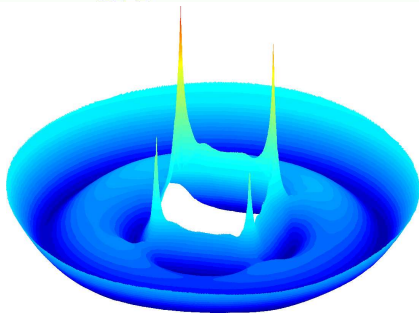
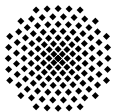


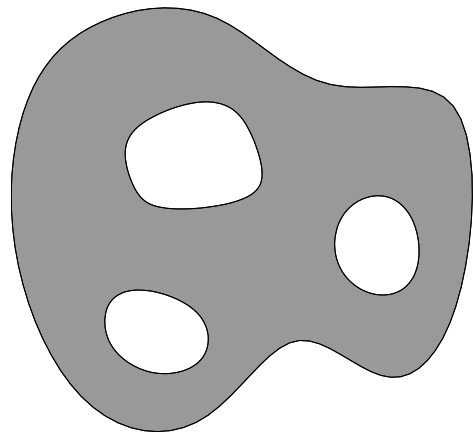
plate problem

$$\Delta^2 u = f \text{ in } D$$

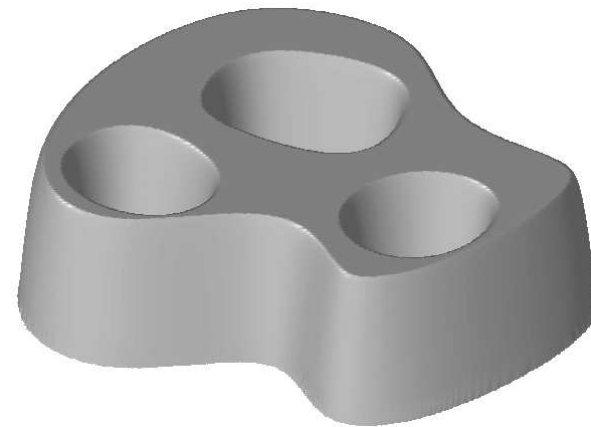
$$u = \frac{\partial}{\partial n} u = 0 \text{ on } \partial D$$



WEB-Splines on NURBS Domains



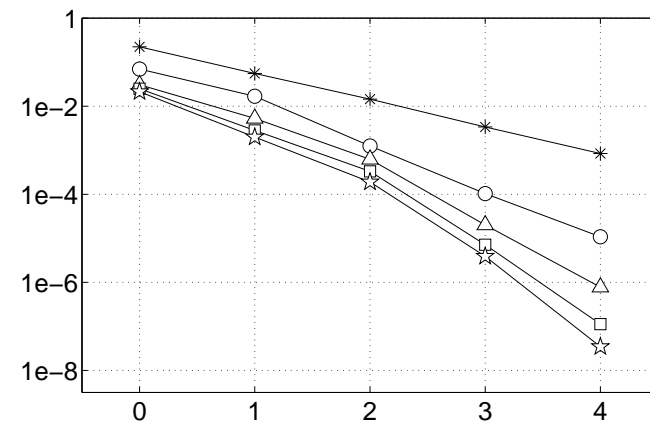
domain



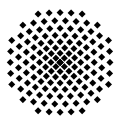
weight function



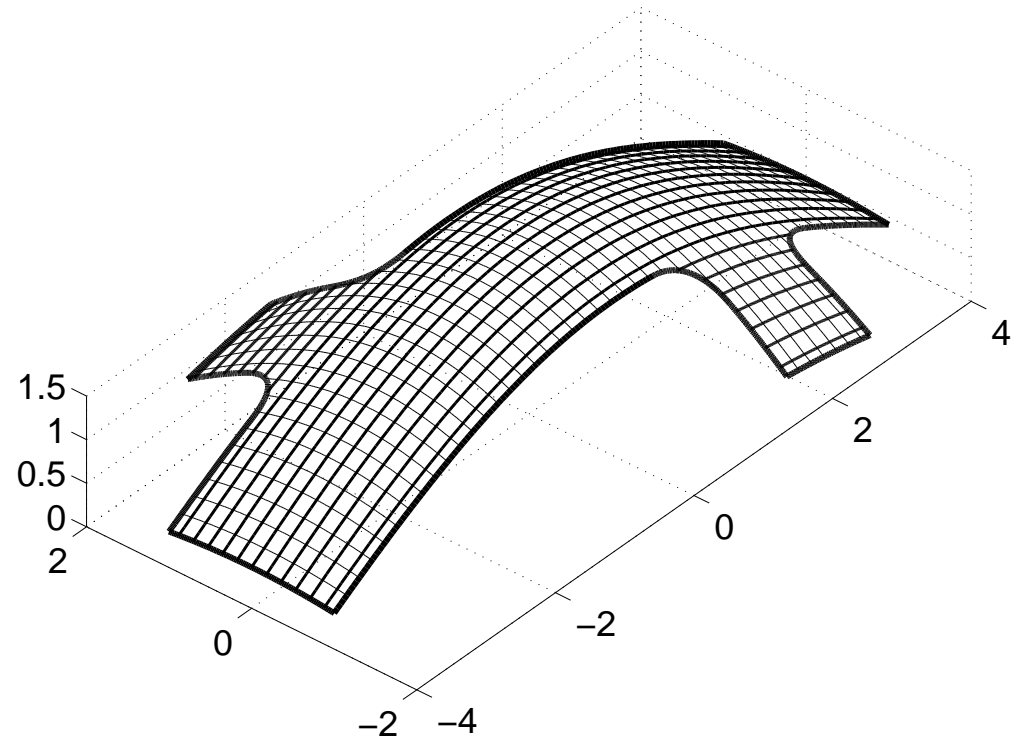
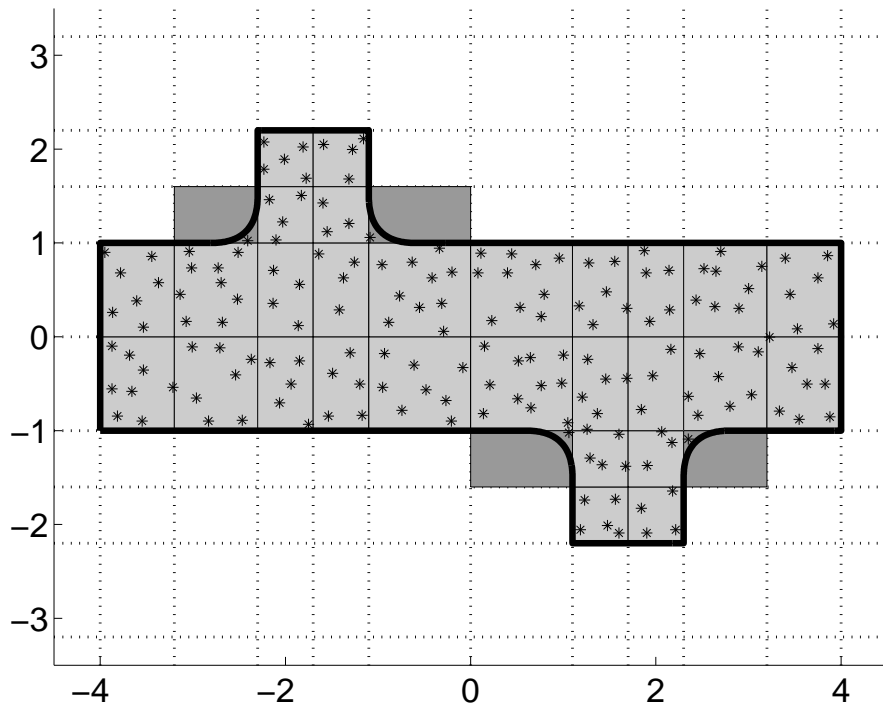
solution to $-\Delta u = 1$



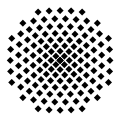
L_2 error



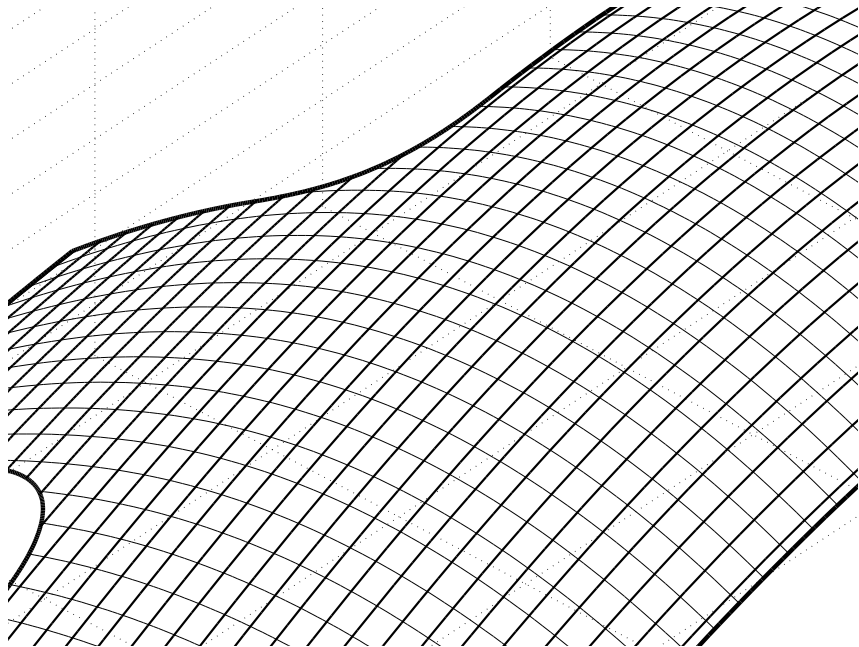
Reverse Engineering



Scattered data on a trimmed domain, sampled from a smooth function

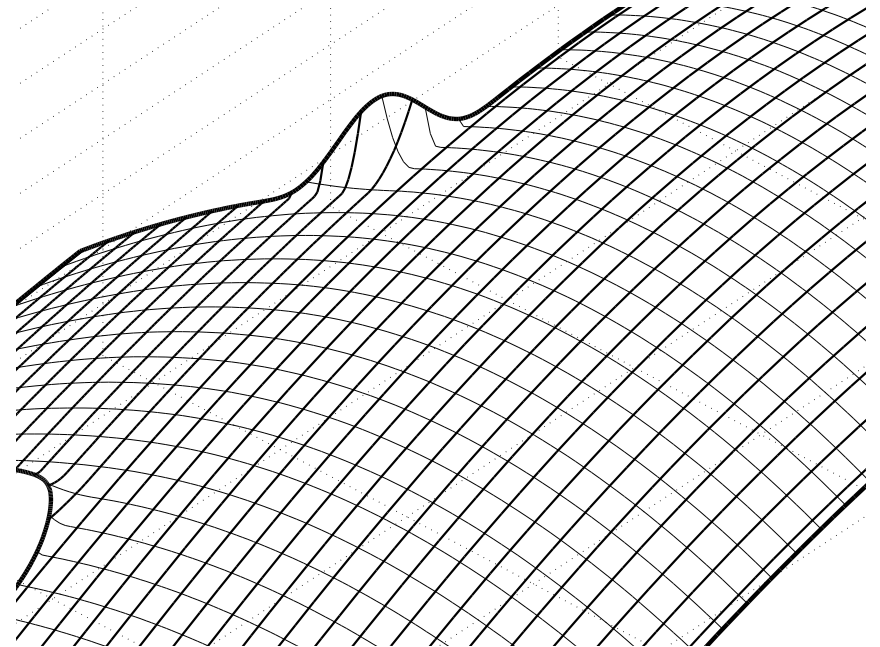


Reverse Engineering



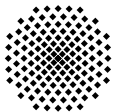
Smooth solution by web-splines

$$\|q_{\text{web}} - f\|_{\infty} \approx 2.2e-4$$

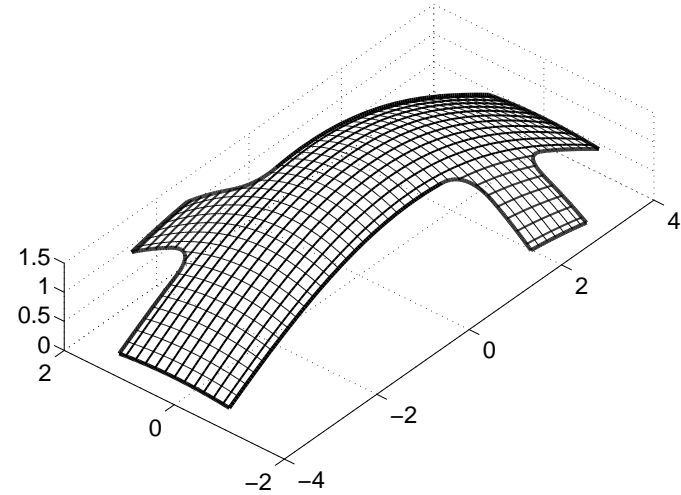
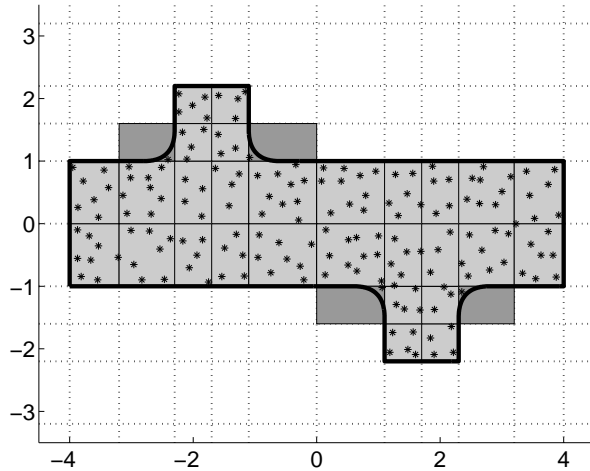


Artifacts by standard NURBS

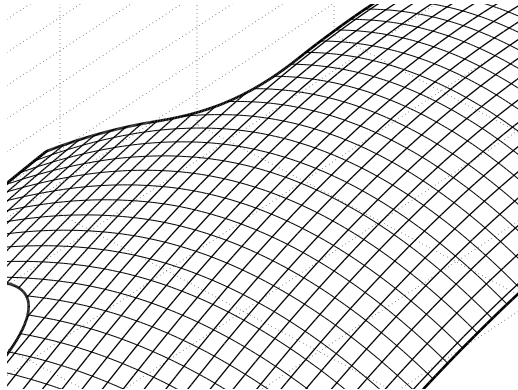
$$\|q_{\text{std}} - f\|_{\infty} \approx 2.8e-1$$



Reverse Engineering

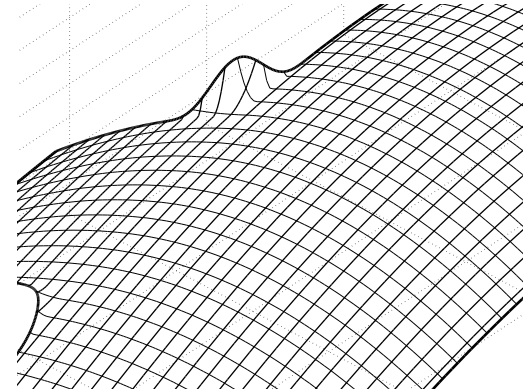


Scattered data on a trimmed domain, sampled from a smooth function



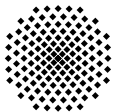
Smooth solution by web-splines

$$\|q_{\text{web}} - f\|_{\infty} \approx 2.2e-4$$



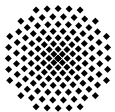
Artifacts by standard NURBS

$$\|q_{\text{std}} - f\|_{\infty} \approx 2.8e-1$$



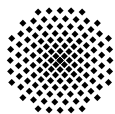
Properties of Web-Splines

- regular grid
- local support
- arbitrary degree and smoothness
- uniform stability
- high accuracy with few parameters
- exact fulfilment of boundary conditions
- adaptive refinement via local error estimators
- multigrid techniques



Multigrid history

- Fedorenko 1964: multigrid algorithm for standard five-point finite difference discretization of Poisson's equation on a square, $\mathcal{O}(n)$
- Bakvahlov 1966: generalization to difference discretizations of general elliptic pde's with variable smooth coefficients
- Brandt 1973: first practical results, important paper in 1977 outlining the main principles and practical utility of multigrid
- Bank, Dupont 1981: multigrid for finite elements
- Ries, Trottenberg, Winter 1983: multigrid reduction algorithm
- Braess, Hackbusch 1983: V-cycle convergence proof



Model Problem

Poisson equation $-\Delta u = f, \quad u \in H_0^1$

$$a(u, v) = \int_D \text{grad } u \text{ grad } v, \quad \lambda(v) = \int_D f v$$

Ritz-Galerkin Approximation

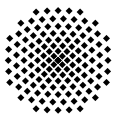
$$GU = F \quad \Longleftrightarrow \quad a\left(\sum u_i B_i, B_k\right) = \lambda(B_k)$$

regularity, approximation, and stability

$$\|u\|_2 \preceq \|f\|_0$$

$$\|u - u_h\|_\ell \preceq h^{k-\ell} \|u\|_k, \quad \ell \leq k \leq n + 1$$

$$\text{cond } G_h \asymp h^{-2}$$



Richardson Iteration

one step

$$U \leftarrow S(U, F) = U - \gamma^{-1}(GU - F)$$

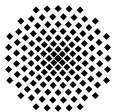
with $\gamma = \|G\|_\infty = \max_{k \in I} \sum_{i \in I} |g_{k,i}|$

error reduction of α steps $U \rightarrow V$

$$V - U_* = (E - \gamma^{-1}G)^\alpha (U - U_*)$$

where $GU_* = F$

eigenvalues $(1 - \lambda_i/\gamma)^\alpha \in [0, 1)$, slow convergence



Quasi-Interpolation

projection of coarse grid web-splines

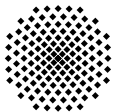
$$P_h \sum_{\ell \in \tilde{I}} \tilde{u}_\ell \tilde{B}_\ell = \sum_{i \in I} u_i B_i, \quad U = P\tilde{U}$$

with $P_h f = \sum_{i \in I} \langle \Lambda_i, f \rangle_0 B_i$

uniform boundedness

$$\|P_h f\|_0 \preceq \|f\|_0$$

sparse matrix P , explicitly defined via subdivision



Multigrid algorithm

one step

$$U \rightarrow W = \mathcal{M}(U, F, h) ,$$

which improves an approximation $U \approx U_* = G^{-1}F$

$$V = S^\alpha(U, F)$$

$$R = GV - F$$

$$\tilde{R} = P^t R$$

if $2h = h_{\max}$

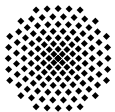
$$\tilde{W} = \tilde{G}^{-1} \tilde{R}$$

else

$$\tilde{W} = \mathcal{M}^\beta(0, \tilde{R}, 2h)$$

end

$$W = V - P\tilde{W}$$



Multigrid heuristic

smooth residuum

$$R = GV - F = G(V - U_*) \approx G(P\tilde{W})$$

related Galerkin matrices

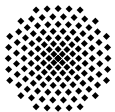
$$\tilde{g}_{k,i} = a(\tilde{B}_i, \tilde{B}_k) \approx a(P_h \tilde{B}_i, P_h \tilde{B}_k) = \sum_{\alpha, \beta \in I} p_{\alpha,i} a(B_\alpha, B_\beta) p_{\beta,k},$$

i.e., $\tilde{G} \approx P^t G P$

approximate correction

$$P^t G(P\tilde{W}) \approx \tilde{G}\tilde{W} = \tilde{R} = P^t R$$

implies $W = V - P\tilde{W} \approx V - G^{-1}R = U_*$



Smoothing

after α Richardson steps $U \rightarrow V \approx U_*$

$$\|G(V - U_*)\| \leq \frac{h^{m-2}}{\alpha + 1} \|U - U_*\|$$

Proof: estimate eigenvalues κ_i of

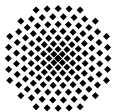
$$G(E - \gamma^{-1}G)^\alpha$$

in terms of the eigenvalues $0 < \lambda_i$ of G

(i) $\gamma \leq h^{m-2}$

(ii) $\vartheta_i = \lambda_i/\gamma \in (0, 1]$

(iii) $|\kappa_i| \leq \max_{\vartheta \in [0,1]} \gamma \vartheta (1 - \vartheta)^\alpha = \gamma \frac{1}{\alpha + 1} \left(\frac{\alpha}{\alpha + 1}\right)^\alpha$



Correction

$$\|(v - u_*) - \tilde{u}_*\|_0 \leq h^{2-m} \|r\|_0,$$

with $R = G(V - U_*)$, $\tilde{G}\tilde{U}_* = \tilde{R} = P^t R$

●●● We prove this estimate by interpreting $v - u_*$ and \tilde{u}_* as approximations to the solution φ of an auxiliary Poisson problem $-\Delta\varphi = \eta$, $\varphi|_{\partial D} = 0$.

To this end we define $\eta \in L_2$ by

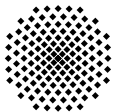
$$\langle \eta, \psi \rangle_0 = a(v - u_*, P_h \psi), \quad \forall \psi \in L_2. \quad (1)$$

By Bernstein's inequality, $\|P_h \psi\|_1 \leq h^{-1} \|P_h \psi\|_0$, i.e., the right-hand side of (1) defines indeed a continuous linear functional on L_2 . Hence, the Riesz representation theorem is applicable. From the definition (1) of η it is easily deduced that $\langle \eta, B_i \rangle_0 = r_i$, $\langle \eta, \tilde{B}_i \rangle_0 = \tilde{r}_i$.

We just set $\psi = B_i = P_h B_i$ for the first identity and note that $a(v - u_*, B_i)$ is the i -th entry of the vector $G(V - U_*)$. The second identity follows from $\tilde{r}_i = \sum_{\ell} p_{\ell,i} \underbrace{\langle \eta, B_{\ell} \rangle_0}_{r_{\ell}} = a(v - u_*, \sum_{\ell} p_{\ell,i} B_{\ell}) = a(v - u_*, P_h \tilde{B}_i)$.

Therefore, $v - u_*$ and \tilde{u}_* are both Ritz-Galerkin approximations to φ . By the standard error estimate for Poisson's equation, $\|(v - u_*) - \tilde{u}_*\|_0 \leq \|(v - u_*) - \varphi\|_0 + \|\varphi - \tilde{u}_*\|_0 \leq h^2 \|\eta\|_0$.

This completes the proof since, with $q = P_h \eta$, $\|\eta\|_0^2 = a(v - u_*, P_h \eta) = QG(V - U_*) \leq \|Q\| \|R\|$, and $\|Q\| \asymp h^{-m/2} \|q\|_0 \leq h^{-m/2} \|\eta\|_0$, $\|R\| \asymp h^{-m/2} \|r\|_0$ by the stability theorem and the boundedness of the quasi-interpolant P_h . ●●●



Grid independent convergence

for $\beta = 2$ (w -cycle)

$$\|W - U_*\| \leq \frac{1}{\alpha + 1} \|U - U_*\|$$

Proof for the two-grid iteration

$$w = v - P_h \tilde{u}_* \Leftrightarrow W = V - P\tilde{U}_*, \quad \tilde{U}_* = \tilde{G}^{-1} \tilde{R}$$

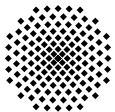
With the bound for $\|W - U_*\|$,

$$\begin{aligned} \|V - P\tilde{U}_* - U_*\| &\asymp h^{-m/2} \|P_h(v - u_*) - P_h \tilde{u}_*\|_0 \\ &\leq h^{-m/2} \|v - u_* - \tilde{u}_*\|_0, \end{aligned}$$

the results about correction and smoothing yield

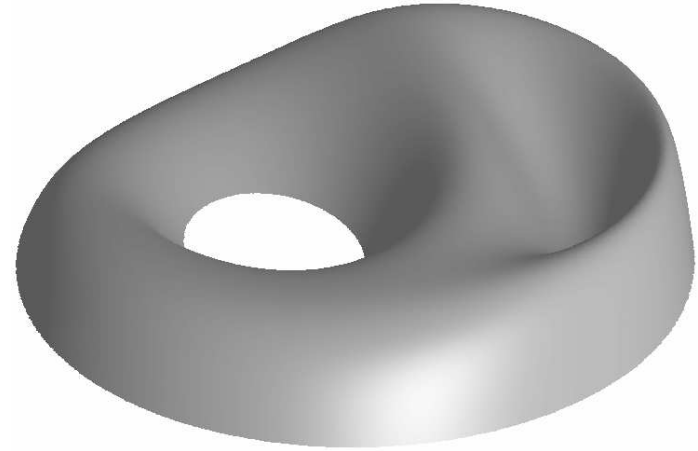
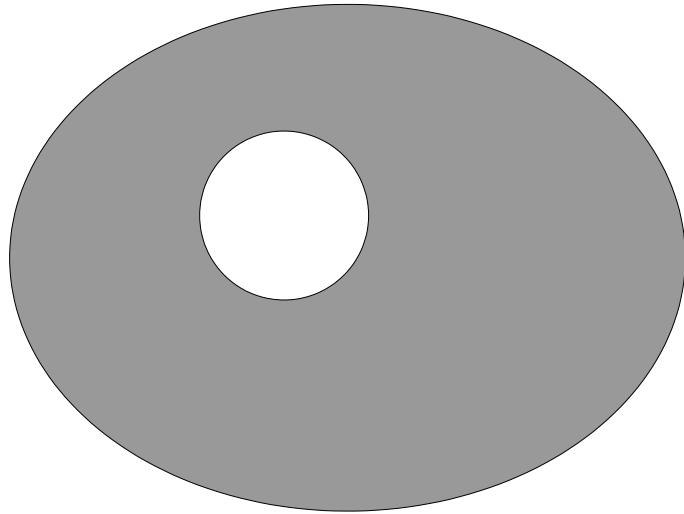
$$\|v - u_* - \tilde{u}_*\|_0 \leq \frac{h^{m/2}}{\alpha + 1} \|U - U_*\|,$$

using also $\|r\|_0 \asymp h^{m/2} \|R\|$ and $R = G(V - U_*)$.



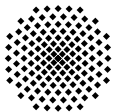
Multigrid example

Poisson Problem on Ω , bounded by an ellipse and a circle.

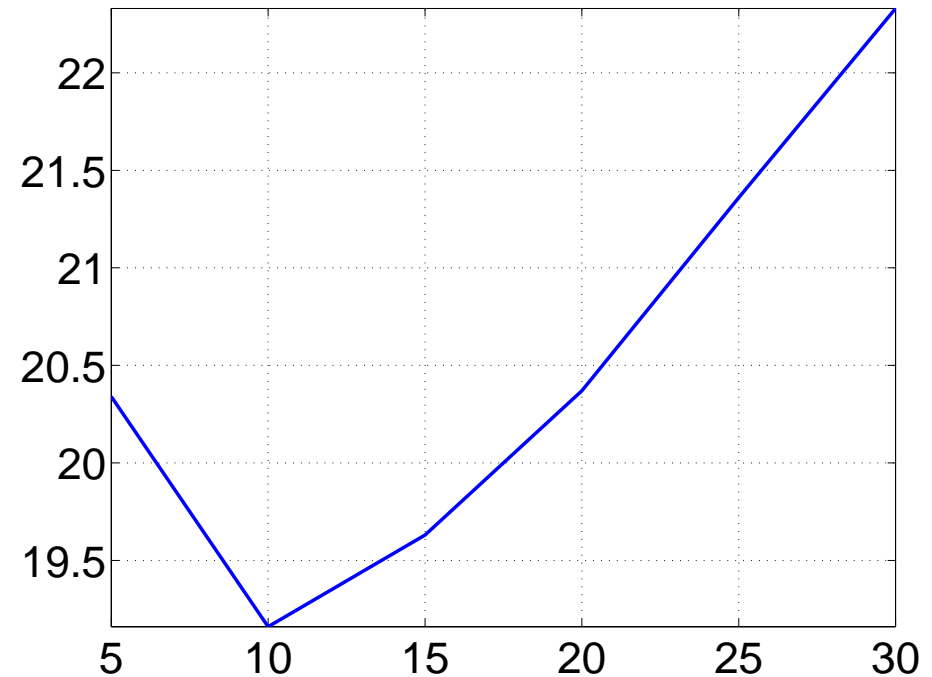
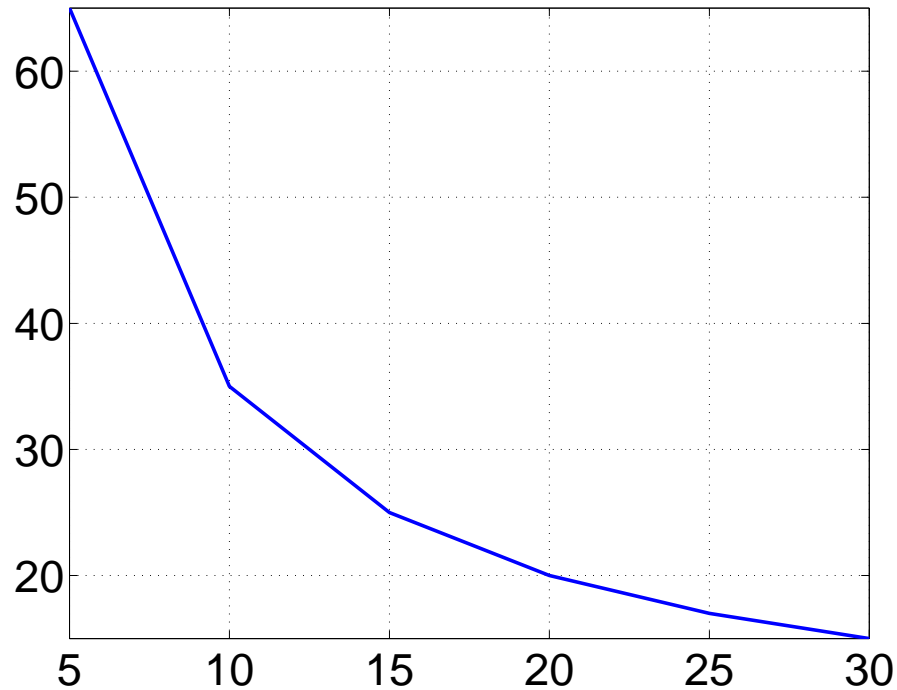


9396 web-splines of degree 3 on finest grid ($h = 2^{-4}$), 56 on coarsest ($h = 2^0$).

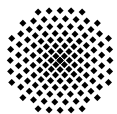
Relative errors $\|u - u_h\|_0 = 4.86e-07$, $\|u - u_h\|_1 = 2.29e-05$. Using $\beta = 1$ (V-cycle), ssor smoothing and residual norm of $1e-12$ as mg stopping criterion.



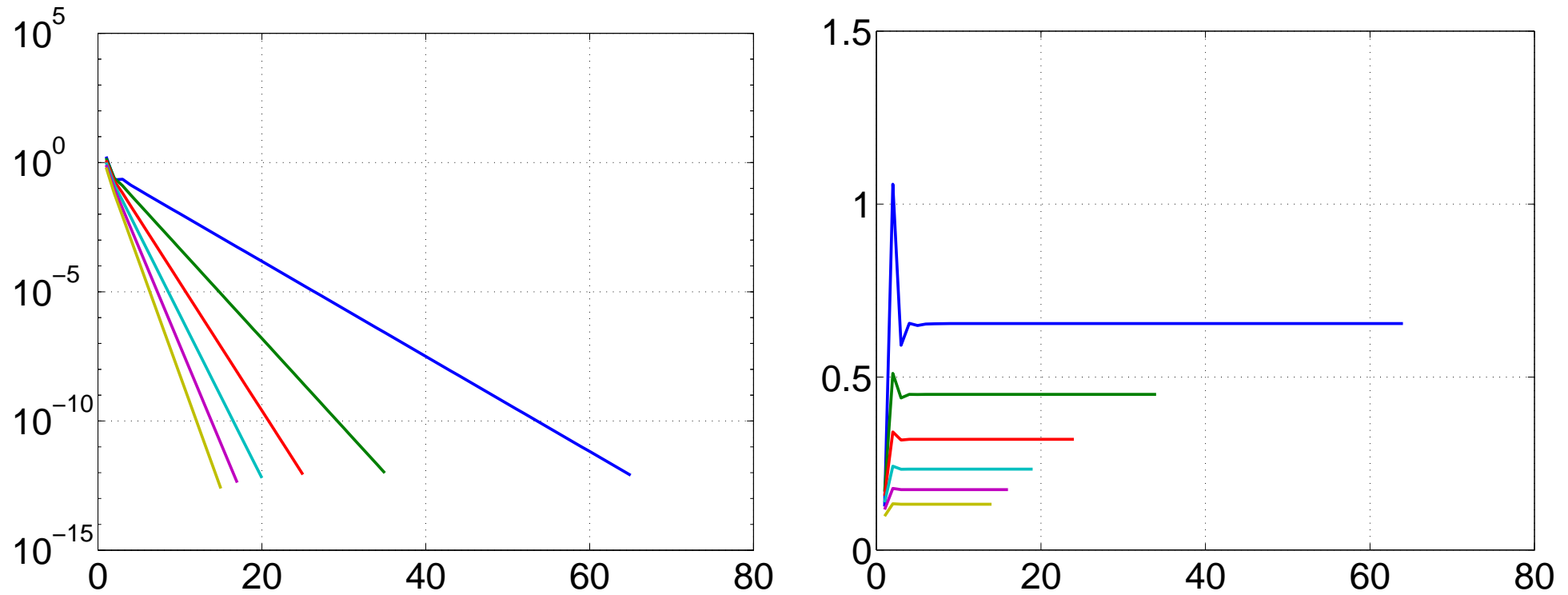
Multigrid example



Number of mg cycles versus number α of smoothing steps (left) and computation time in seconds versus number α of smoothing steps (right)



Multigrid example



Residual norm versus mg cycle number for $\alpha=[5:5:30]$ (left), and rate (right)

