Finite Element Methods with B-Splines

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http://www.mathematik.uni-stuttgart.de/mathA/lst2/

http://www.web-spline.de (with U. Reif, J. Wipper)

History of Finite Elements and Splines

Engineering

Turner, Clough, Martin,
and Topp (1956)
Argyris (1960)
Clough (1960)

de Casteljau (1959),
Bézier (1966)

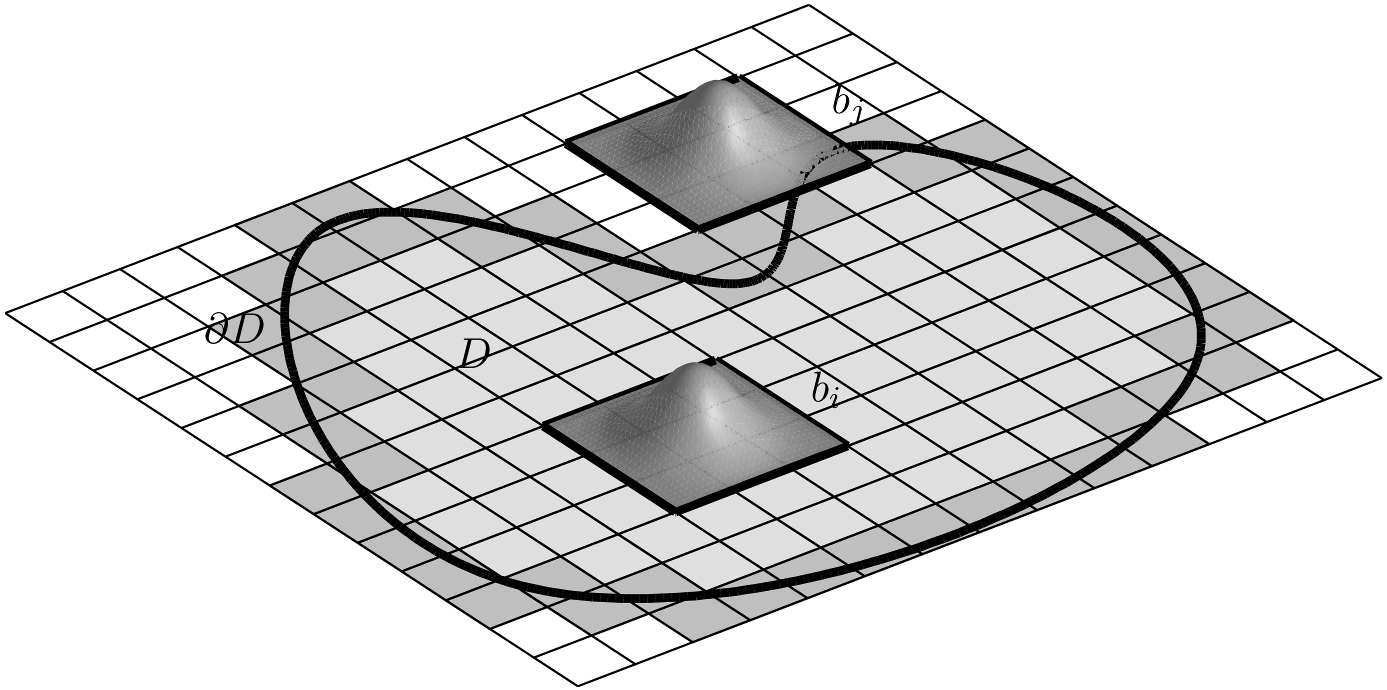
FEM ← Splines

Rayleigh (1870)
Ritz (1908)
Galerkin (1915)
Courant (1943)
Strang and Fix (1973)

Schoenberg (1946)
de Boor (1972)

Mathematics

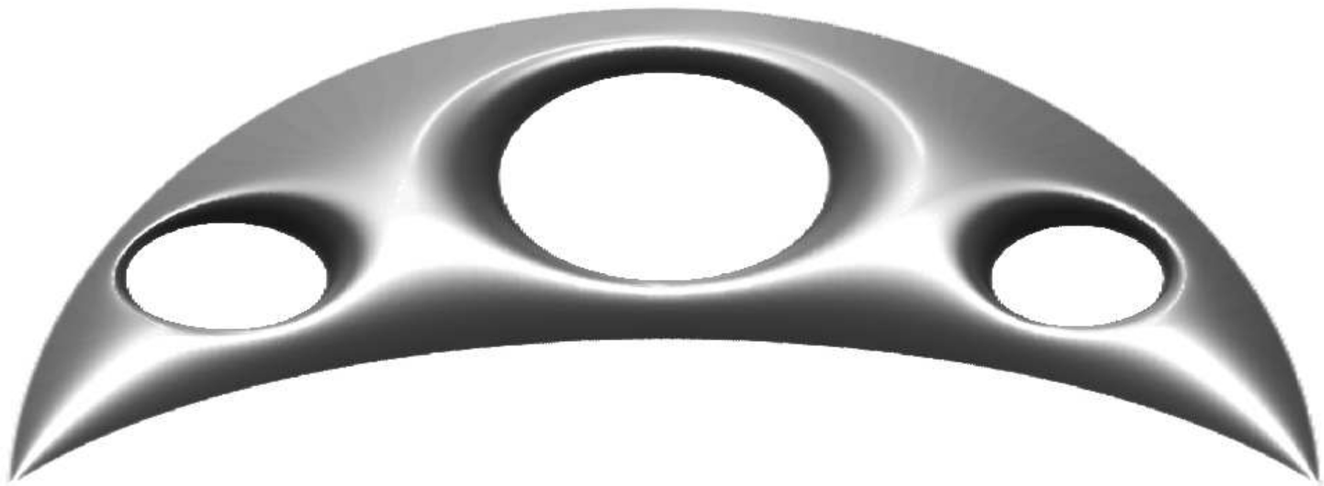
Splines on bounded Domains



$$\mathbb{B}_h = \operatorname*{span}_K b_k\,,\; K = I \cup J$$

problems: boundary conditions, stability

Weight function



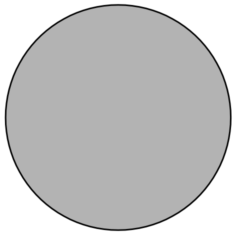
essential boundary conditions $w_{|_{\mathcal{D}}} > 0\,,\ w_{|_{\partial\mathcal{D}}} = 0$

construction:

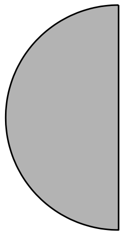
- explicit formulas
- Rvachev's Boolean expressions
- numerical distance functions

Rvachev's R-Functions

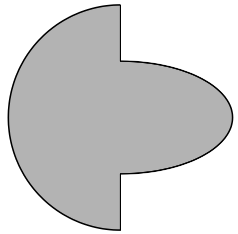
$w_1 = 1 - x_1^2 - x_2^2$



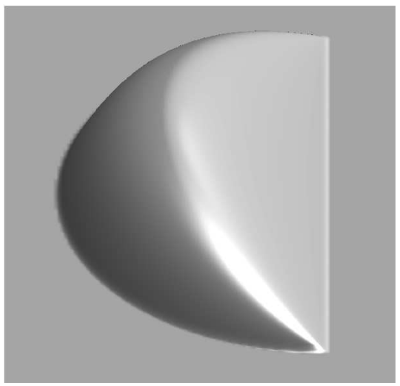
$w_2 = -x_1$



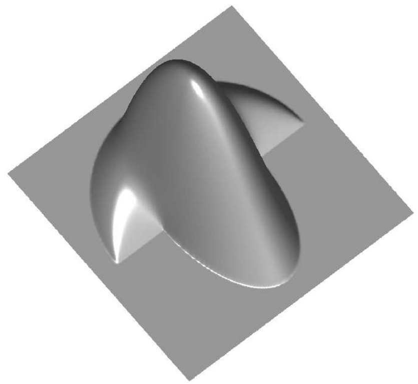
$w_3 = 1 - x_1^2 - 4x_2^2$



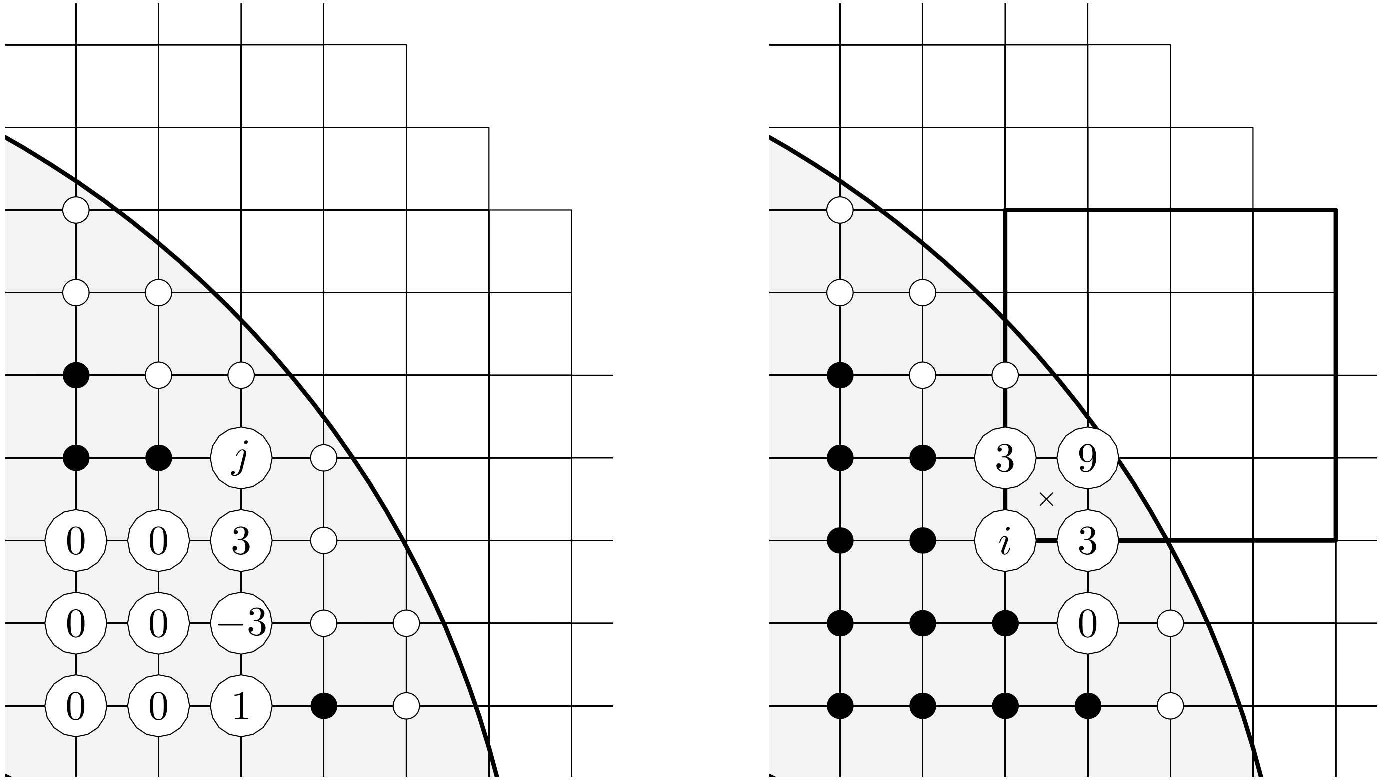
$w_\cap = w_1 + w_2 - \sqrt{w_1^2 + w_2^2}$



$w_\cup = w_\cap + w_3 + \sqrt{w_\cap^2 + w_3^2}$



Extension



$I(j)$: nearest $(n+1)^m$-array of inner indices

$e_{i,j}$: value at j of the Lagrange-polynomial to i

$J(i)$: complementary sets $(i \in I(j) \Leftrightarrow j \in J(i))$

Weighted-Extended-B-Splines

(with U. Reif and J. Wipper)

$$B_i = \frac{w}{w(x_i)} \left(b_i + \sum_{j \in J(i)} e_{i,j} b_j \right)$$

properties:

- local support: $e_{i,j} = 0$ for $||i - j|| \gtrsim 1$
- stability: $|| \sum c_i B_i ||_0 \asymp ||C||$
- approximation order: $||u - P_h u||_\ell \lesssim h^{n+1-\ell} ||u||_{n+1}$

Ritz Galerkin Approximation

H : Hilbert space, incorporating homogeneous boundary conditions

a : elliptic bilinear form

λ : linear functional

weak solution:

$$a(u,v) = \lambda(v)\,,\; v \in H$$

finite element approximation:

$$a(u_h, B_i) = \lambda(B_i)\,,\; i \in I$$

error estimate:

$$||u - u_h||_H \lesssim \inf_C \left\| u - \sum c_i B_i \right\|_H$$

Linear Elasticity

displacement:

$$(u_1, u_2, u_3) \in \left(H^1_\Gamma\right)^3$$

strain tensor:

$$\varepsilon_{k,l} = \frac{1}{2}\left(\partial_k u_\ell + \partial_\ell u_k\right)$$

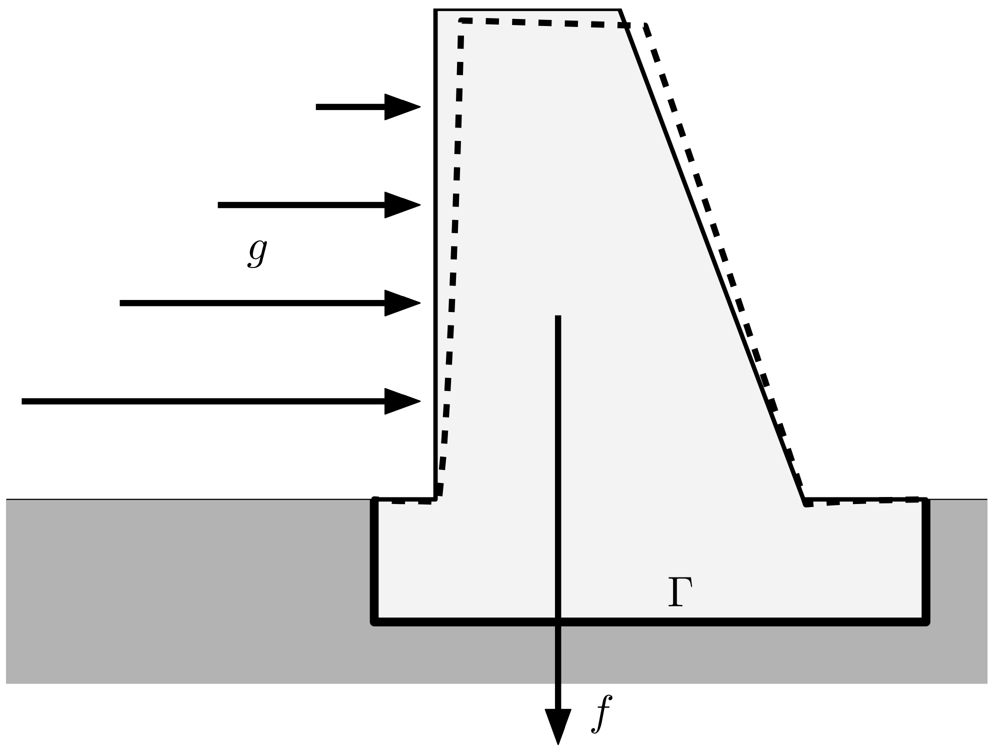
stress tensor;

$$\sigma_{k,l} = \lambda(\mathrm{trace}\,\varepsilon)\delta_{k,l} + 2\mu\varepsilon$$

variational formulation:

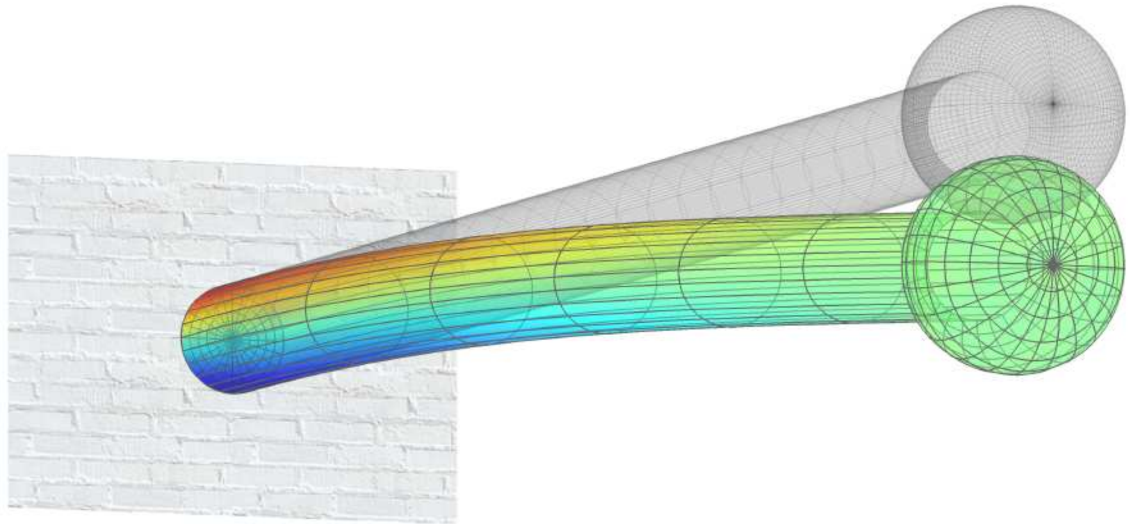
$$a(u,v) = \int_{\mathcal{D}} \sigma : \varepsilon$$

$$\lambda(v) = \int_{\mathcal{D}} fv + \int_{\partial\mathcal{D}\setminus\Gamma} gv$$



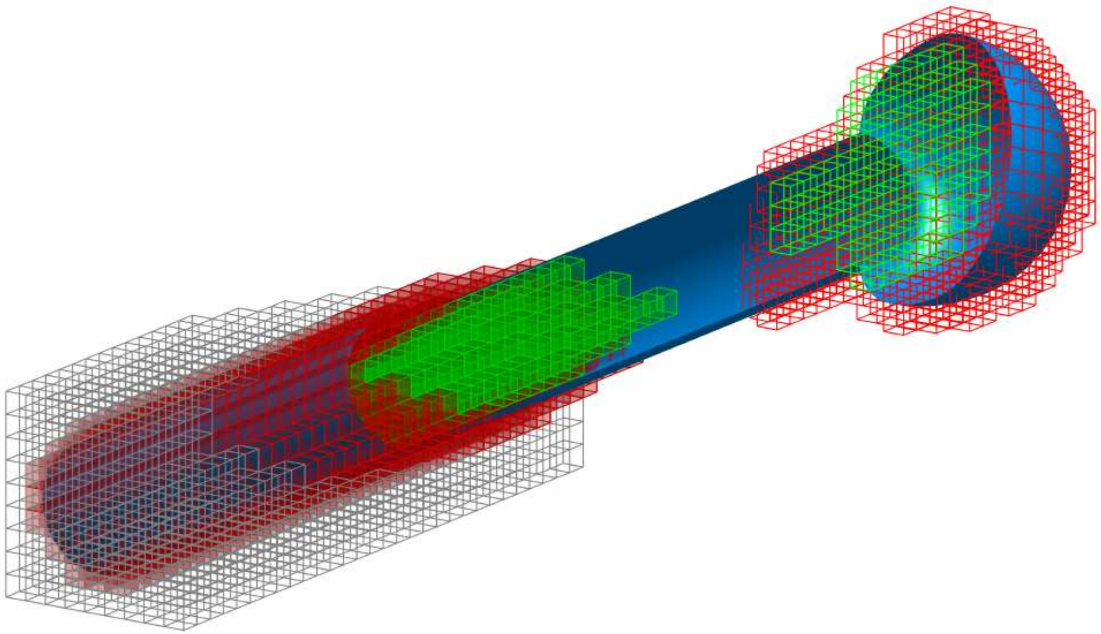
Flagstaff

displacement (30x) and principle stress (color)



rod length:	4.0m	head diameter:	0.4m
material:	aluminum	λ:	$5.71 \cdot 10^8 \text{N}/\text{dm}^2$
		μ:	$2.69 \cdot 10^8 \text{N}/\text{dm}^2$
max displacement:	9mm	max stress:	$1.7 \text{kN}/\text{cm}^2$

Flagstaff

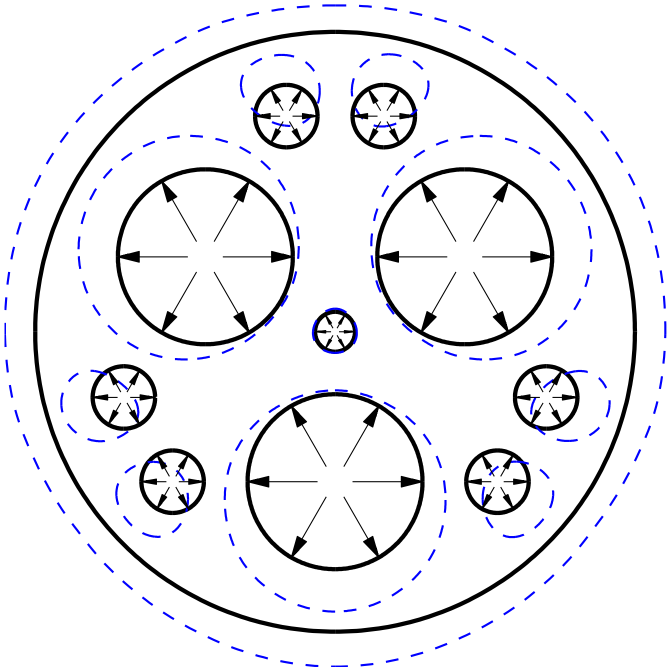


grid cell classification

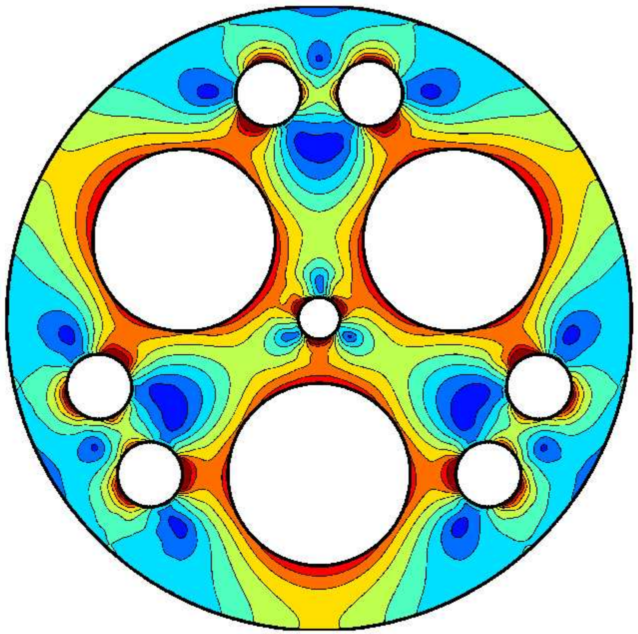
grid width	0.9	0.45	0.225	0.1125
inner	56	1036	10506	101606
boundary	450	1794	7242	31862
ratio	0.889	0.634	0.408	0.239

Plane Strain

$$\varepsilon_{3,\ell} = \varepsilon_{\ell,3} = 0$$



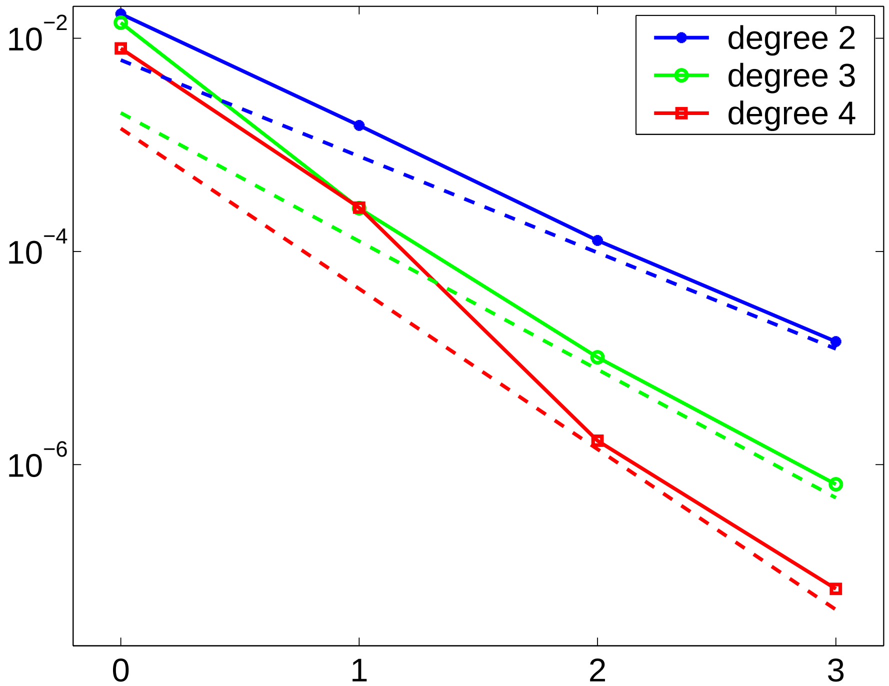
displacement



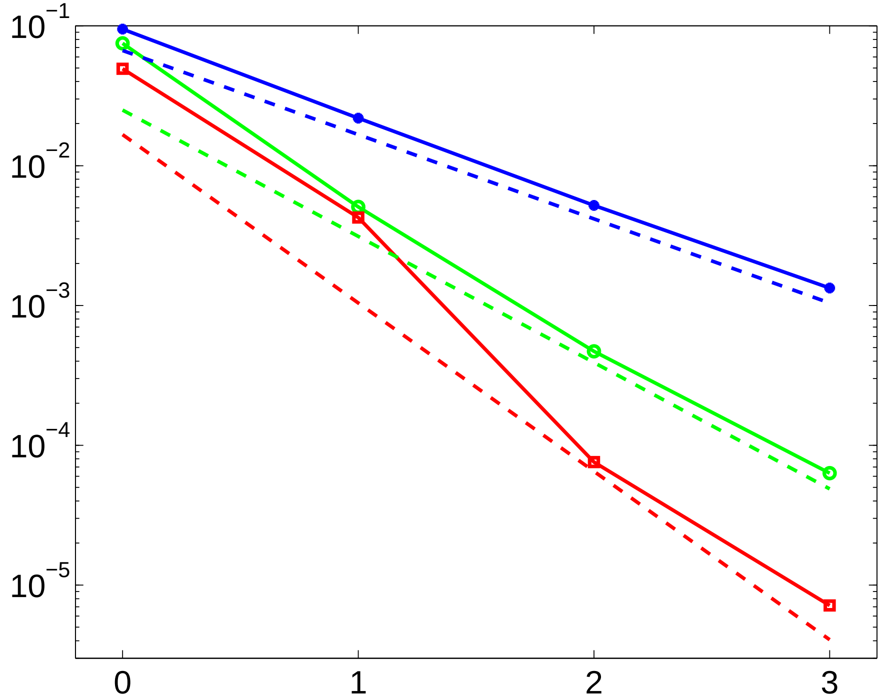
principal stress

same pressure (40 kp/cm^2) in all pipes, no force on outer boundary
material steel: $\lambda = 1.15e7$ N/cm^2, $\mu = 7.7e6$ N/cm^2
outer circle: $r_o = 120$cm

L^2- and H^1-Error



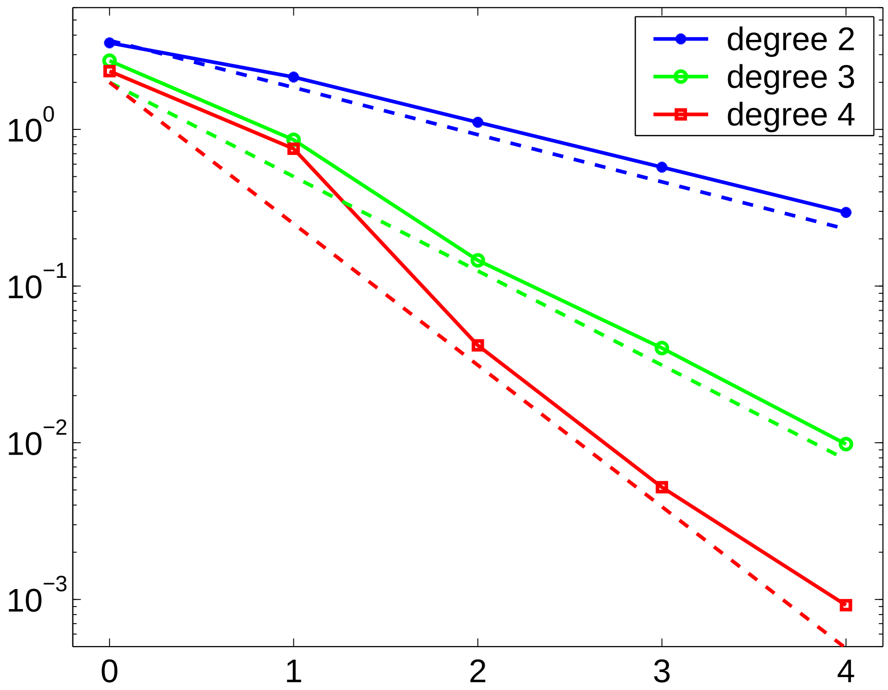
$$\|u - u_h\|_0 \preceq h^{n+1}$$

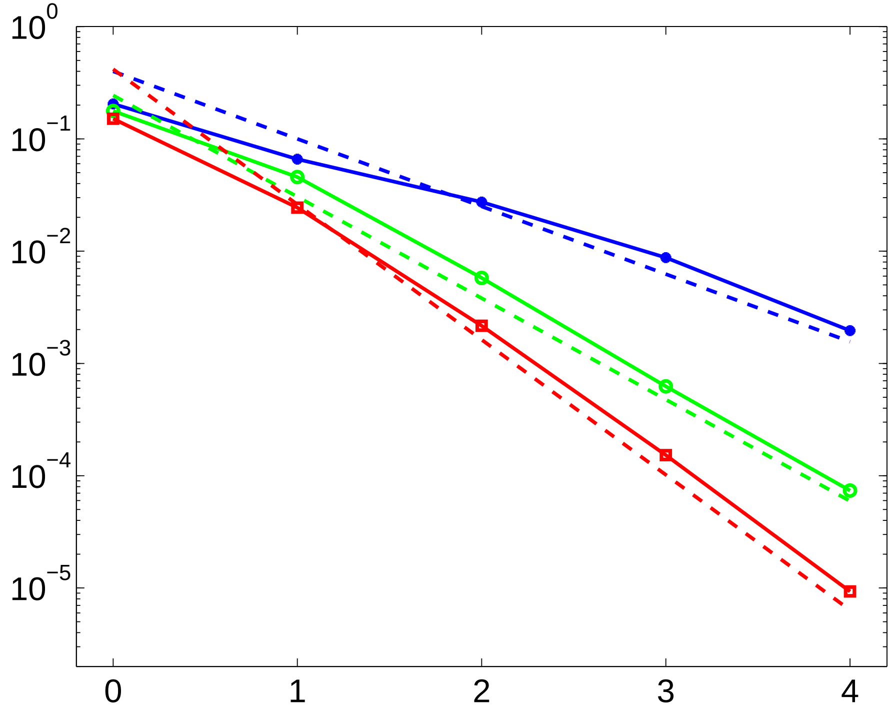


$$\|u - u_h\|_1 \preceq h^n$$

$h = 2^{-k} h_0$, 84 ... 4752 basis functions

Residuum and Boundary Error

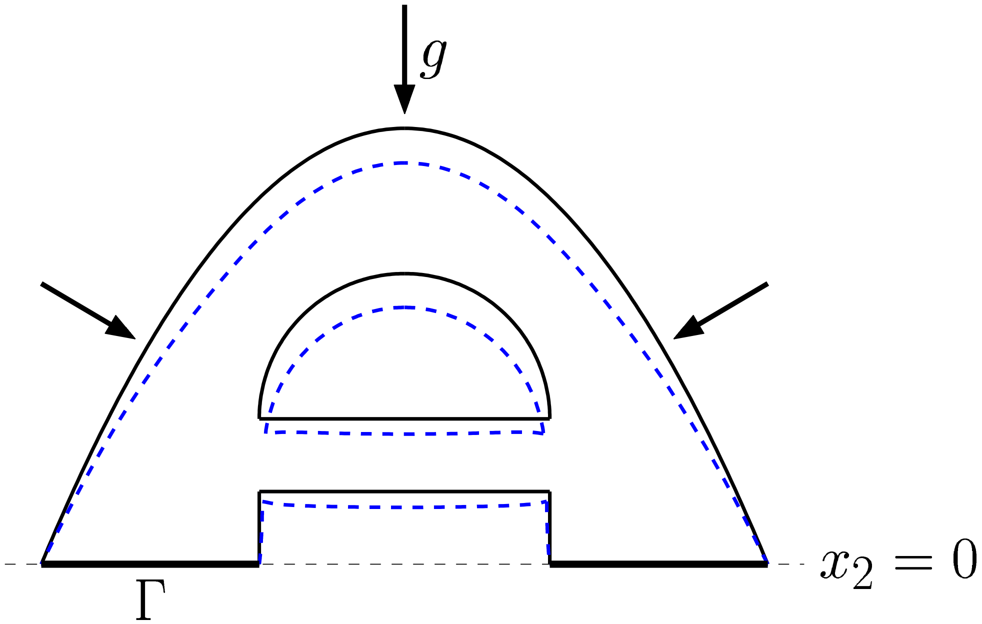




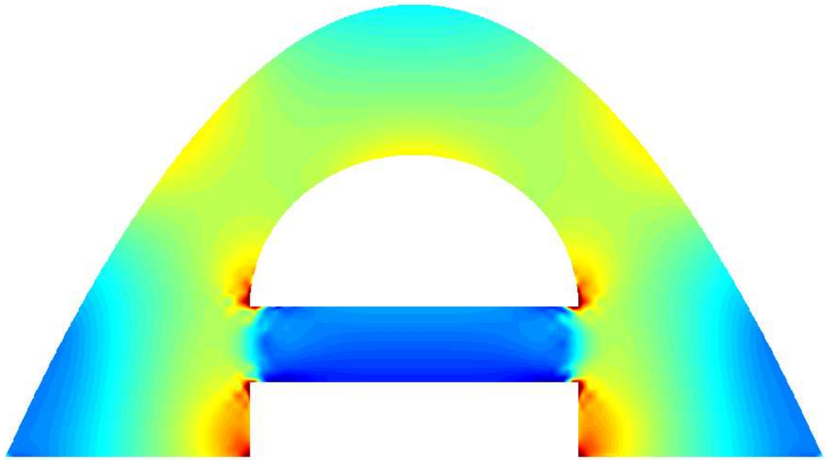
$$\|\mathrm{div}\sigma + f\|_0 \preceq h^{n-1}$$

$$\max_{\partial\mathcal{D}\setminus\Gamma} |\sigma\xi - g| \preceq h^n$$

Singular Solution



displacement

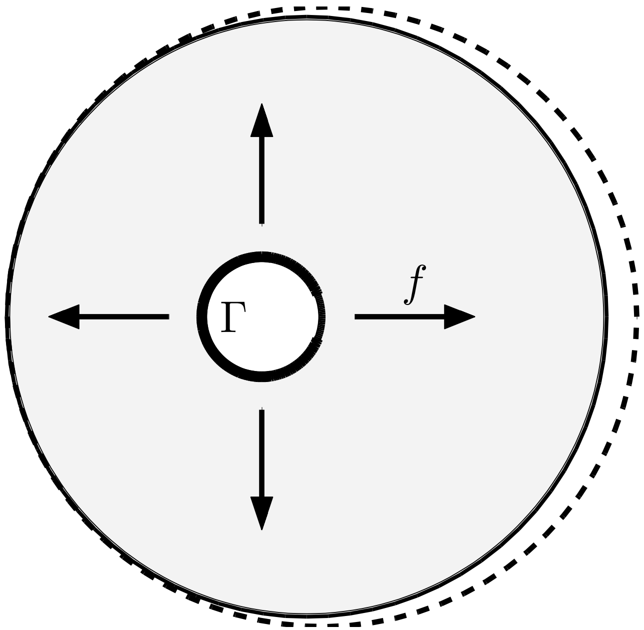


principal stress

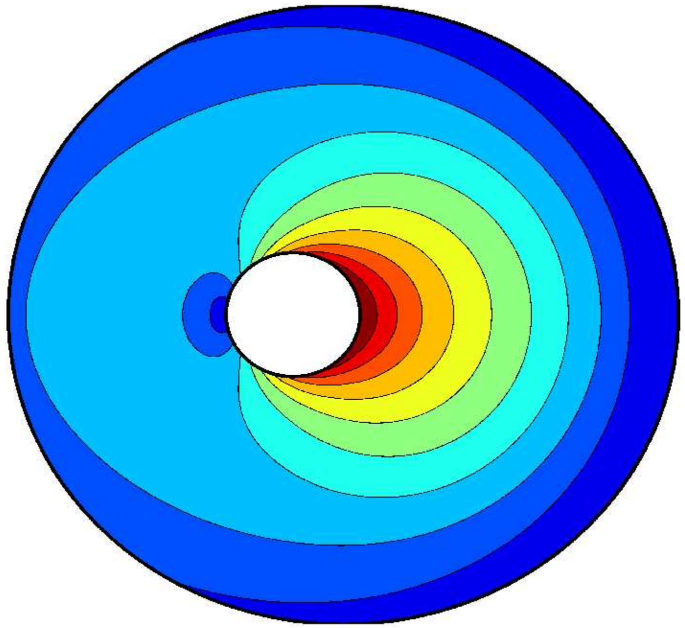
material concrete: $\lambda = 8.3e9$ N/m^2, $\mu = 1.25e10$ N/m^2

Plane Stress

$$\sigma_{3,\ell} = \sigma_{\ell,3} = 0$$



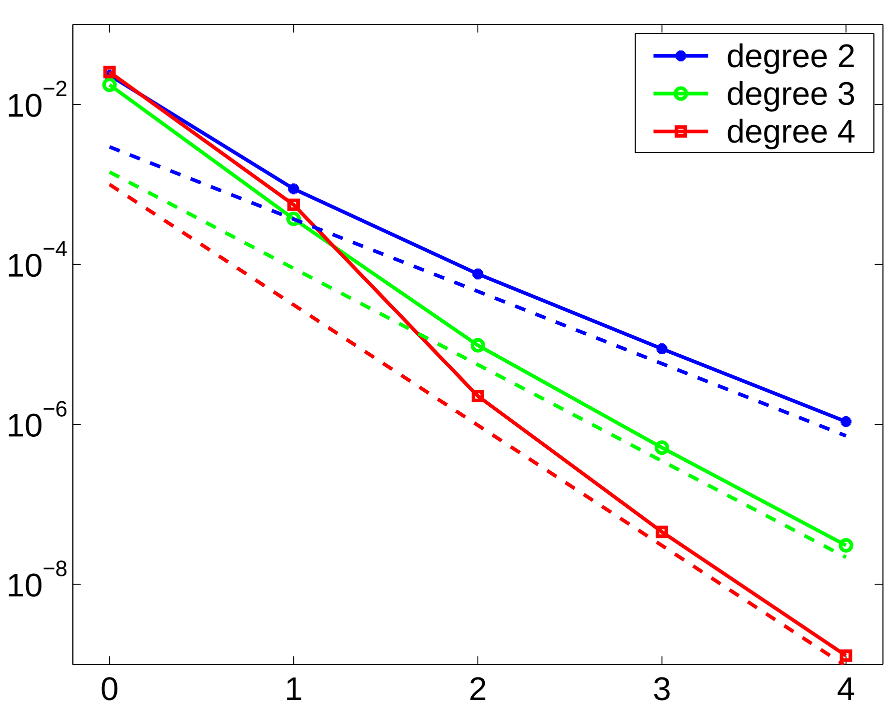
displacement



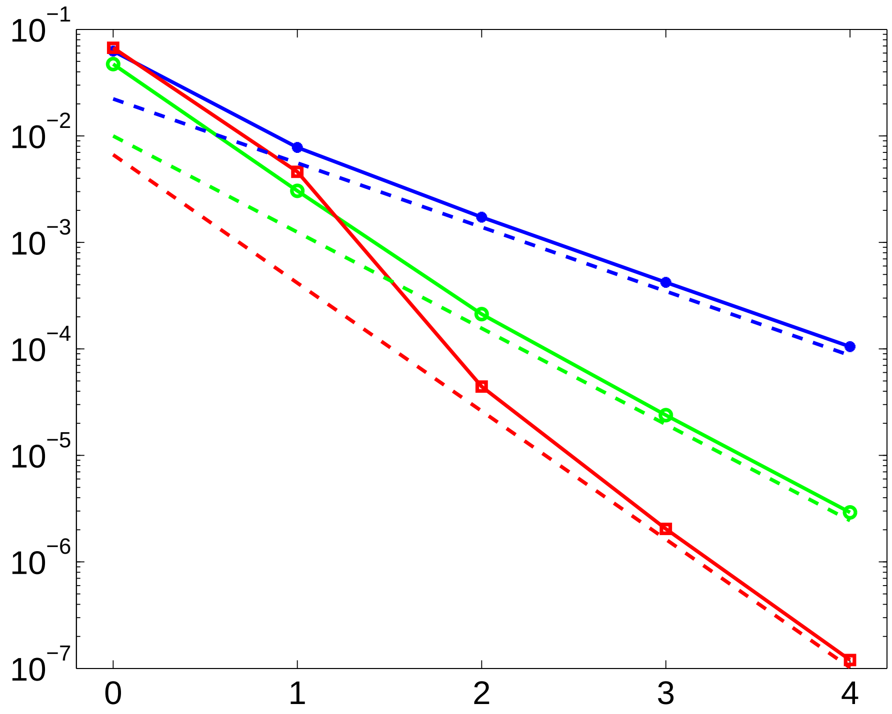
principal stress

material steel: $\lambda = 1.05\text{e}7\ \text{N/cm}^2$, $\mu = 8.28\text{e}6\ \text{N/cm}^2$
volume force $f = 10^4(x,y)^t\ \text{N/cm}^3$
outer circle: $r_o = 5\text{cm}$, $M_o = (3/4, 0)$
inner circle: $r_i = 1\text{cm}$, $M_i = (0,0)$

L^2- and H^1-Error



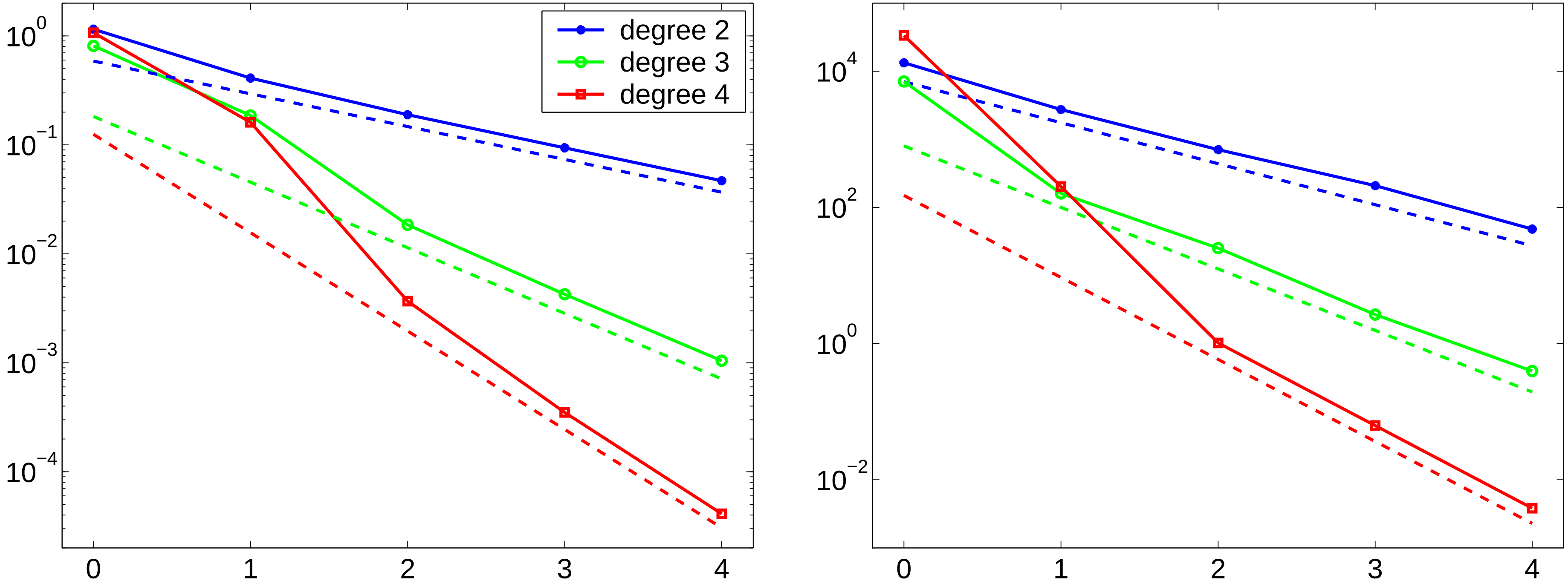
$\|u - u_h\|_0 \preceq h^{n+1}$



$\|u - u_h\|_1 \preceq h^{n}$

$h = 2^{-k} h_0, \qquad$ 196 . . . 40896 basis functions

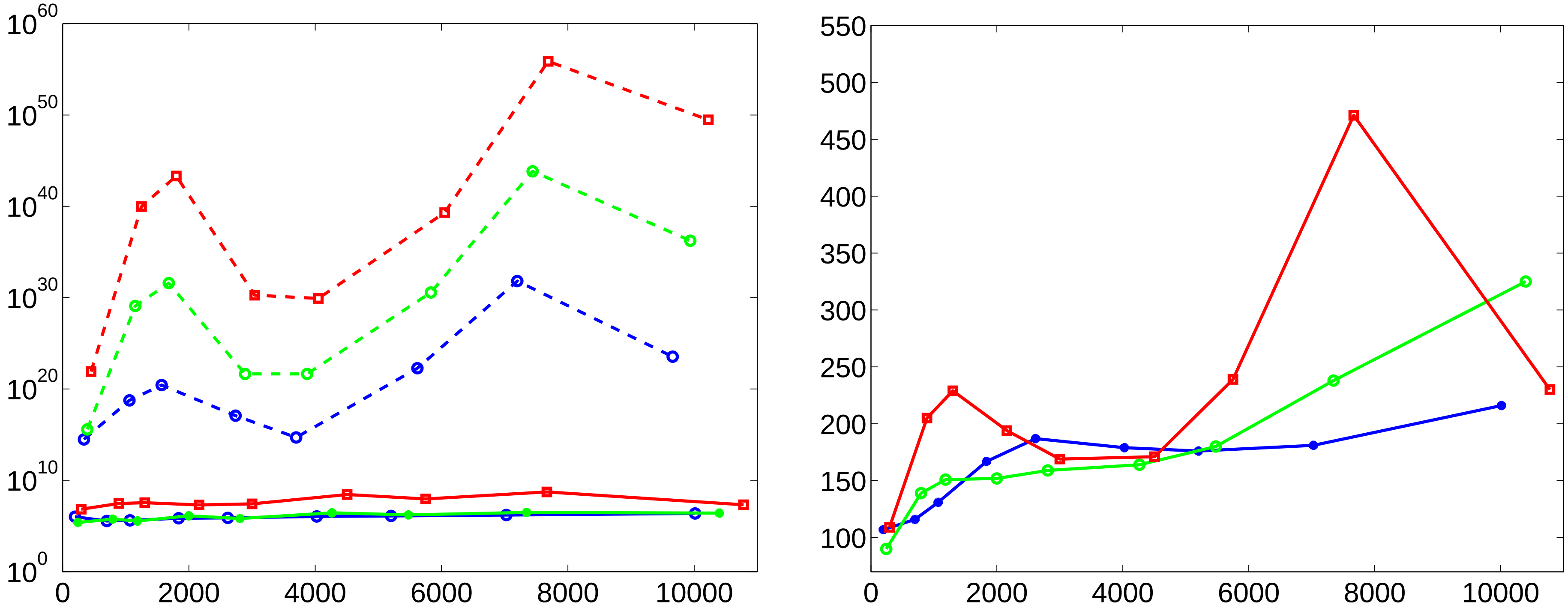
Residuum and Boundary Error



$$\|\mathrm{div}\sigma + f\|_0 \preceq h^{n-1}$$

$$\max_{\partial\mathcal{D}\setminus\Gamma} |\sigma\xi - g| \preceq h^n$$

Condition Number and Iteration Count



$$\mathrm{cond}(G_h) \sim h^{-2} \sim \mathrm{dim}^{-1}$$

Advantages of web-Splines

- no mesh generation
- simple basis functions
- multigrid techniques
- high accuracy with few parameters
- arbitrary smoothness and approximation order
- hierarchical bases