Subdivision Schemes in Geometric Modelling

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Contents

1 Introduction

The first work on a subdivision scheme was by de Rahm [20]. He showed that the scheme he presented produces limit functions with a first derivative everywhere and a second derivative nowhere. The pioneer work of Chaikin [5] introduced subdivision as a practical algorithm for curve design. His algorithm served as a starting point for generalizations into subdivision algorithms generating any spline functions. The importance of subdivision to applications in computer aided geometric design became clear with the generalizations of the tensor product spline rules to control nets of arbitrary topology. This important step has been introduced in two papers by Doo and Sabin [27] and by Catmull and Clark [3]. The surfaces generated by their subdivision schemes are no longer restricted to representing bivariate functions, and they can easily represent surfaces of arbitrary topology.

In recent years the subject of subdivision gained popularity due to many new applications, such as 3D computer graphics, and due to the close relation of subdivision analysis to wavelets analysis. Subdivision algorithms are ideally suited for computer applications; they are simple to apprehend, easy to implement, highly flexible and very attractive to users and to researchers. In free form surface design applications, such as in the 3D animation industry, subdivision methods are already in extensive use, and the next venture is to introduce these methods to the more conservative, and more demanding world of geometric modeling in the industry.

Important steps in subdivision analysis has been made in the last two decades, and the subject expanded into new directions due to various generalizations and applications. This review does not claim to cover all the knowledge about subdivision schemes, their analysis and their application. It is rather a personal view of the authors on the subject. For example, the convergence analysis is not presented in its most generality and is restricted to uniform convergence which is relevant to geometric modeling. On the other hand, the review deals with the analysis and applications of non-stationary subdivision scheme, which the authors view as important for future developments. While most of the review deals with convergence, regularity and approximation order, it also relates the results to practical issues such as attaining the optimal approximation and computing limit values.

The presentation starts with the basic notions of non-stationary subdivision, definitions of limit functions and basic limit functions and the refinement relations they satisfy. Different forms of representation of subdivision schemes and the basic convolution property of subdivision schemes are also presented in Section 2. These are later used throughout the review for presenting and proving the main results. Next we present a gallery of examples of different types of subdivision schemes, interpolatory and non-interpolatory, linear and non-linear, stationary and non-stationary, matrix subdivision and Hermite-type subdivision and bivariate subdivision on regular and non-regular nets. In the same section we overview some extensions of subdivision schemes which are not studied in this review. The material in sections §2 and §3 is intended to provide a broad map of the subdivision area for tourists and for new potential users.

In §4, the analysis of convergence of a univariate and bivariate subdivision processes and the smoothness of the limit functions are presented via the related difference schemes for stationary schemes. It is presented also for non-stationary schemes, relating the results to the analysis of stationary subdivision and using smoothing factors and convolutions as main tools. The central results are presented some with full proofs and some with only sketches of proofs. The special analysis of convergence and smoothness at extraordinary points, for subdivision schemes for nets of general topology, is reviewed in §6. In §7 we discuss two practical issues in the practical application of subdivision schemes. One is the computation of exact limit values of the function (surface) and the limit derivatives at diadic points. The other is the approximation power of subdivision and how to attain the theoretical approximation power.

2 Basic notions

This review presents subdivision schemes mainly as a tool for geometric modeling, starting from the general point of view of non-stationary schemes.

2.1 Non-stationary schemes

A subdivision scheme is defined as a set of refinement rules relative to a set of nested meshes of isolated points (nets),

$$\{N_k \ : \ k \in \mathbb{Z}_+\} \ , \ N_k \subseteq N_{k+1} \ , \ N_k \in \mathbb{R}^s \ .$$

Each refinement rule maps real values defined on a level k net, N_k, to real values defined on a refined net N_{k+1}. The subdivision scheme is the repeated refinement of initial data defined on N_0, by the set of refinement rules.

Let us first consider the regular grid case, namely the net $N_0 = \mathbb{Z}^s$ for $s \in \mathbb{Z}_+ \backslash 0$ and its binary refinements, namely the refined nets $N_k = 2^{-k}\mathbb{Z}^s$, $k \in \mathbb{Z}_+\backslash 0$. Let $\mathbf{f}^k$ be the values attached to the net $N_k = 2^{-k}\mathbb{Z}^s$,

$$\mathbf{f}^k = \{f^k_\alpha : \alpha \in \mathbb{Z}^s\} \tag{2.1}$$

with f^k_α attached to $2^{-k}\alpha$.

The refinement rule at refinement level k is of the form

$$f^{k+1}_\alpha = \sum_{\beta\in\mathbb{Z}^s} a^k_{\alpha-2\beta} f^k_\beta \ , \ \alpha \in \mathbb{Z}^s \ , \tag{2.2}$$

which we write formally as

$$\mathbf{f}^{k+1} = R_{\mathbf{a}^k}\mathbf{f}^k \ . \tag{2.3}$$

The set of coefficients $\mathbf{a}^k = \{a^k_\alpha : \alpha \in \mathbb{Z}^s\}$ determines the refinement rule at level k and is termed the k'th level mask. Let $\sigma(\mathbf{a}^k) = \{\gamma \mid a^k_\gamma \neq 0\}$ be the support of the mask $\mathbf{a}^k$. Here we restrict the discussion to the case that the origin is in the convex hull of $\sigma(\mathbf{a}^k)$, and that $\sigma(\mathbf{a}^k)$ are finite sets, for $k \in \mathbb{Z}_+$. A more general form of refinement, corresponding to a dilation matrix M, is

$$f^{k+1}_\alpha = \sum_{\beta\in\mathbb{Z}^s} a^k_{\alpha-M\beta} f^k_\beta \ , \tag{2.4}$$

where M is an $s \times s$ matrix of integers with $|\det(M)| > 1$ (see e.g. [13], [54]). In this case the refined nets are $M^{-k}\mathbb{Z}^s$, $k \in \mathbb{Z}_+$. We restrict our discussion to binary refinements corresponding to $M = 2I$, with I the $s \times s$ identity matrix, namely to (2.2).

If the masks $\{\mathbf{a}^k\}$ are independent of the refinement level, namely if $\mathbf{a}^k = \mathbf{a}$, $k \in \mathbb{Z}_+$, the subdivision scheme is termed stationary, and is denoted by $S_\mathbf{a}$. In the non-stationary case, the subdivision scheme is determined by $\{\mathbf{a}^k : k \in \mathbb{Z}_+\}$, and is denoted as a collection of refinement rules $\{R_{\mathbf{a}^k}\}$, or by the shortened notation $S_{\{\mathbf{a}^k\}}$.

2.2 Notions of convergence

A continuous function $f \in C(\mathbb{R}^s)$ is termed the limit function of the subdivision scheme $S_{\{\mathbf{a}^k\}}$, from the initial data $\mathbf{f}^0$, and is denoted by $S^\infty_{\{\mathbf{a}^k\}}\mathbf{f}^0$, if

$$\lim_{k\to\infty} \max_{\alpha\in\mathbb{Z}^s\cap K} |f^k_a - f(2^{-k}\alpha)| = 0 \ , \tag{2.5}$$

where $\mathbf{f}^k$ is defined recursively by (2.2), and K is any compact set in $\mathbb{R}^s$.

This is equivalent [4] to f being the uniform limit on compact sets of $\mathbb{R}^s$ of the sequence $\{F_k : k \in \mathbb{Z}_+\}$ of s-linear spline functions interpolating the data at each refinement level, namely

$$F_k(2^{-k}\alpha) = f^k_\alpha \ , \quad F_k\big|_{2^{-k}(\alpha+[0,1]^s)} \in \pi^T_1 \ , \quad \alpha \in \mathbb{Z}^s \ , \tag{2.6}$$

where π^T_1 is the tensor product space of the spaces of linear polynomials in each of the variables.

From this equivalence we get,

$$\lim_{k\to\infty} \|f(t) - F_k(t)\|_{\infty,K} = 0 \ . \tag{2.7}$$

If we do not insist on the continuity of f in (2.5) or on the L_∞-norm in (2.7), we get weaker notions of convergence, e.g. L_p-convergence, by requiring the existence of $f \in L_p(\mathbb{R}^s)$ satisfing $\lim_{k\to\infty} \|f(t)-F_k(t)\|_p = 0$ [97],[57]. The case $p = 2$ is important in the theory of wavelets [14]. In this paper we consider mainly the notion of uniform convergent, corresponding to (2.7), which is relevant to geometric modeling. We also mention here the weakest notion of convergence [22], termed "weak convergence" or "distributional convergence". A subdivision scheme $S_{\{\mathbf{a}^k\}}$, generating the values $f^k = R_{\mathbf{a}} f^{k-1}$, for $k \in \mathbb{Z}, \ \ k > 0$, converges weakly to an integrable function f, if for any $g \in C^\infty_0$ (infinitely smooth and of compact support),

$$\lim_{k\to\infty} 2^{-k} \sum_{\alpha\in\mathbb{Z}^s} g(2^{-k}\alpha) f^k_\alpha = \int^s_{\mathbb{R}} f(x)g(x)dx \ .$$

Definition 2.1 *A subdivision scheme is termed uniformly convergent, if for any initial data there exists a limit function in the sense of* (2.7)*, (or equivalently, if for any initial data there exists a continuous limit function in the sense of* (2.5)*) and if the limit function is non-trivial for at least one initial data. A uniformly convergent subdivision scheme is termed C^m, or C^m-convergent, if for any initial data the limit function has continuous derivatives up to order m.*

In the following we use the term convergence for uniform convergence, since this notion of convergence is central to this review.

An important initial data is $\mathbf{f}^0 = \boldsymbol{\delta} = \{f^0_\alpha = \delta_{\alpha,0} \ : \ \alpha \in \mathbb{Z}^s\}$. If $S_{\{\mathbf{a}^k\}}$ is convergent, then there exists a non-trivial limit function starting from this initial data,

$$\phi_{\{\mathbf{a}^k\}} = S^\infty_{\{\mathbf{a}^k\}} \boldsymbol{\delta} \ .$$

By the uniformity of the refinement rules, (each refinement rule operates in the same way at all locations) and by their linearity,

$$S^\infty_{\{\mathbf{a}^k\}} \mathbf{f}^0 = \sum_{\alpha\in\mathbb{Z}^s} f^0_\alpha \phi_{\{\mathbf{a}^k\}}(\cdot - \alpha) \ , \tag{2.8}$$

for any initial data $\mathbf{f}^0$. Thus if $\phi_{\mathbf{a}^k} \in C^m(\mathbb{R}^s)$ for some $m \geq 0$, so is any limit function generated by $S_{\mathbf{a}^k}$, and the scheme is C^m.

When the initial data consists of a sequence of vectors

$$\mathbf{P}^0 = \{P^0_\alpha \in \mathbb{R}^d \ : \ \alpha \in \mathbb{Z}^s\} \in (\ell_\infty(\mathbb{Z}^s))^d \ ,$$

the limit of the subdivision, given by (2.8) with $\mathbf{f}^0$ replaced by $\mathbf{P}^0$, is a parametric representation of a manifold in $\mathbb{R}^d$. In geometric modeling $s = 1$ corresponds to curves in $\mathbb{R}^d$ for $d = 2, 3$ and $s = 2, d = 3$ to surfaces in $\mathbb{R}^3$. The set of refined points $\mathbf{P}^k$, for $k \in \mathbb{Z}_+$, is termed "the control points at level k".

2.3 The refinement equations

The function $\phi_{\{\mathbf{a}^k\}} = S^\infty_{\{\mathbf{a}^k\}}\boldsymbol{\delta}$, termed the "basic limit function" of the subdivision scheme $S_{\{\mathbf{a}^k\}}$, is the first in the family of functions $\{\phi_\ell \ : \ \ell \in \mathbb{Z}_+\}$, defined by

$$\phi_\ell = S^\infty_\ell \boldsymbol{\delta} \ , \tag{2.9}$$

where $S_\ell = \{R_{\mathbf{a}^k} \ : \ k \geq \ell, \ k \in \mathbb{Z}_+\}$. Each function in this family is a basic limit function of a subdivision scheme defined in terms of a subset of the masks $\{\mathbf{a}^k\}$. If $S_0 = S_{\{\mathbf{a}^k\}}$ is convergent so is any S_ℓ for $\ell \in \mathbb{Z}_+$ [41] (see §4.1). Thus all the functions $\{\phi_\ell \ : \ \ell \in \mathbb{Z}_+\}$ are well defined, if S_0 is convergent. Moreover, by (2.9)

$$S^\infty_\ell \mathbf{f}^0 = \sum_{\alpha\in\mathbb{Z}^s} f^0_\alpha \phi_\ell(\cdot - \alpha) \ . \tag{2.10}$$

The support of ϕ_ℓ can be determined by the the supports of the masks $\{\mathbf{a}^k\}$. Recalling that $\sigma(\mathbf{a}^k)$ denotes the support of the mask $\mathbf{a}^k$, which is a finite set of points in $\mathbb{Z}^s$, then by the refinement rules (2.2) and by (2.9), the support $\sigma(\phi_\ell)$ of ϕ_ℓ is given by

$$\sigma(\phi_\ell) = \overline{\sum_{k=\ell}^{\infty} 2^{\ell-k-1}\sigma(\mathbf{a}^k)} \ , \tag{2.11}$$

where the sum above is the Minkowski sum of sets. In the stationary case and in the univariate case (2.11) can be further elaborated.

In the univariate case, $s = 1$, let $[\ell^k, u^k] =< \sigma(\mathbf{a}^k) >$, be the convex hull of $\sigma(\mathbf{a}^k)$, and let

$$\ell_k = \sum_{j=k}^{\infty} 2^{k-j-1}\ell^j \ , \quad u_k = \sum_{j=k}^{\infty} 2^{k-j-1}u^j \ .$$

Then

$$\sigma(\phi_k) \subseteq [\ell_k, u_k] \ . \tag{2.12}$$

In the stationary case [4], (2.11) yields

$$\sigma(\phi_{\mathbf{a}}) \subseteq < \sigma(\mathbf{a}) > \ . \tag{2.13}$$

The functions $\{\phi_k \ : \ k \in \mathbb{Z}_+\}$ are related by a system of functional equations, termed refinement equations. To see this, observe that $(R_{\mathbf{a}^k}\boldsymbol{\delta})_\alpha = a^k_\alpha, \ \alpha \in \mathbb{Z}^s$, and by the linearity of the refinement rules,

$$\phi_k = \sum_\alpha a^k_\alpha \phi_{k+1}(2\cdot - \alpha) \ , \ k \in \mathbb{Z}_+ \ . \tag{2.14}$$

In the stationary case, namely when $\mathbf{a}^k = \mathbf{a}, \ k \in \mathbb{Z}_+$, this system of equations reduces to a single functional equation

$$\phi_{\mathbf{a}} = \sum_\alpha a_\alpha \phi_{\mathbf{a}}(2\cdot - \alpha) \ . \tag{2.15}$$

with $\mathbf{a} = \{a_\alpha \ : \ \alpha \in \mathbb{Z}^s\}$, and $\phi_{\mathbf{a}} = S^\infty_{\mathbf{a}}\boldsymbol{\delta}$.

The refinement equation (2.15) is the key to the generation of multiresolution analysis and wavelets [14, 76]. In case the scheme $S_{\mathbf{a}}$ converges, the unique compactly supported solution of the refinement equation (2.15) coincides with $S^\infty_{\mathbf{a}}\boldsymbol{\delta}$. The refinement equation (2.15) suggests another way to compute its unique compactly supported solution. This method is termed the "cascade algorithm", see e.g. [16]. It involves the repeated use of the operator

$$T_{\mathbf{a}} g = \sum_\alpha a_\alpha g(2\cdot - \alpha) \ .$$

defined on continuous compactly supported functions. The cascade algorithm:

1. Choose a continuous compactly supported function, ψ_0, as a "good" initial guess (e.g. H as in (2.20)).
2. iterate $\psi_{k+1} = T_{\mathbf{a}}\psi_k$.

It is easy to verify that the operator $T_{\mathbf{a}}$ is the adjoint of the refinement rule $R_{\mathbf{a}}$, in the following sense; for any ψ continuous and of compact support, [4]

$$\sum_\alpha (R_{\mathbf{a}} f)_\alpha \psi(2\cdot - \alpha) = \sum_\alpha f_\alpha (T_{\mathbf{a}}\psi)(\cdot - \alpha) \tag{2.16}$$

Note that while the refinement rule $R_{\mathbf{a}}$ is defined on sequences, the operator $T_{\mathbf{a}}$ is defined on functions. A similar operator to $T_{\mathbf{a}}$, defined on sequences is

$$(\widetilde{T}_{\mathbf{a}}\mathbf{f})_\alpha = \sum_\beta a_\beta f_{2\alpha-\beta} = \sum_\gamma a_{2\alpha-\gamma} f_\gamma \ . \tag{2.17}$$

This operator is the adjoint of the operator $R_{\mathbf{a}}$ on the space of sequences defined on $\mathbb{Z}^s$. The operator $\widetilde{T}_{\mathbf{a}}$ in (2.17) is termed the "transfer operator" [14], and plays a major role in the analysis of the solutions of refinement equations of the form (2.15) (see e.g. [58, 53, 54, 60].

2.4 Representations of subdivision schemes

The notions introduced above regard a subdivision scheme $S_{\{\mathbf{a}^k\}} = \{R_{\mathbf{a}^k}\}$ as a set of operators defined on sequences in $\ell_\infty(\mathbb{Z}^s)$. Each refinement rule can be represented as a bi-infinite matrix with each element indexed by two index vectors from $\mathbb{Z}^s$,

$$f_\alpha^{k+1} = \sum_{\beta\in\mathbb{Z}^s} A_{\alpha,\beta}^k f_\beta^k \ , \quad \alpha \in \mathbb{Z}^s \ , \tag{2.18}$$

where the bi-infinite matrix A^k has elements

$$A_{\alpha,\beta}^k = a_{\alpha-2\beta}^k \ . \tag{2.19}$$

Finite sections of these matrices are used in the analysis of the subdivision scheme $S_{\{\mathbf{a}^k\}}$ (see §5).

One may also regard a subdivision scheme as a set of operators $\{R_k \ : \ k \in \mathbb{Z}_+\}$ defined on a function space [41], if one considers the functions $\{F_k\}$ introduced in (2.6). The set of operators $\{R_k\}$ has the property that R_k maps F_k into F_{k+1}. More specifically, let H be defined by

$$H(\alpha) = \boldsymbol{\delta}_{0,\alpha} \ , \quad H\big|_{(\alpha+[0,1]^s)} \in \pi_1^T \ , \quad \alpha \in \mathbb{Z}^s \ . \tag{2.20}$$

Define the operators $\{R_k\}$ on $C(\mathbb{R}^s)$ as

$$R_k g = \sum_{\alpha\in\mathbb{Z}^s} H(2^{k+1}\cdot - \alpha) \sum_{\beta\in\mathbb{Z}^s} a_{\alpha-2\beta}^k g(2^{-k}\beta) \ , \ k \in \mathbb{Z}_+ \ , \tag{2.21}$$

for any $g \in C(\mathbb{R}^s)$. Then the subdivision scheme $S_{\{\mathbf{a}^k\}}$ is related to the set of operators $\{R_k\}$, in several ways, e.g

$$(R_k g)\big|_{2^{-k-1}\mathbb{Z}^s} = R_{\mathbf{a}^k}(g\big|_{2^{-k}\mathbb{Z}^s}) \ ,$$

and the more significant relation

$$S_{\{\mathbf{a}^k\}}^\infty \mathbf{f}^0 = \lim_{k\to\infty} R_k R_{k-1} \cdots R_0 g \ , \tag{2.22}$$

where $g \in C(\mathbb{R}^s)$ is any interpolant to $\mathbf{f}^0$ on $\mathbb{Z}^s$, namely

$$g(\alpha) = f_\alpha^0 \ , \ \alpha \in \mathbb{Z}^s \ .$$

In particular g can be

$$g = \sum_{\alpha\in\mathbb{Z}^s} H(\cdot - \alpha) f_\alpha^0 \ .$$

Another important relation is

$$\|R_k\| = \|R_{\mathbf{a}^k}\|_\infty = \max_{\alpha\in E_s}\{\sum_{\beta\in\mathbb{Z}^s} |a_{\alpha-2\beta}^k|\} \ , \tag{2.23}$$

where E_s is the set of extreme points of $[0,1]^s$. The representation of subdivision schemes in terms of sequences of operators on $C(\mathbb{R}^s)$, facilitates the application of standard operator-theory tools to the analysis of subdivision schemes, e.g. to deduce convergence properties of non-stationary schemes from those of related stationary ones [41] (see §4.1).

A representation of the refinement rule (2.2), which is a central tool in the convergence and smoothness analysis of stationary schemes, is in terms of z-transforms (Laurent series). Let the symbol of the mask $\mathbf{a}^k$ be defined as the Laurent polynomial

$$a^k(z) = \sum_{\alpha \in \mathbb{Z}^s} a^k_\alpha z^\alpha \ . \tag{2.24}$$

Here we use the multi index notation $z^n = z_1^{n_1} \cdots z_s^{n_s}$, for $z \in \mathbb{R}^s$, $n \in \mathbb{Z}^s$, and $z^n = z_1^n \cdots z_s^n$, for $z \in \mathbb{R}^s$, $n \in \mathbb{Z}$. Obviously, a subdivision scheme $S_{\{\mathbf{a}^k\}}$ is identified with the set of its symbols $\{a^k(z)\}$. In our notations we exchange freely between the mask and its symbol, e.g. $\phi_{\{a^k(z)\}}$ denotes the basic limit function of $S_{\{a^k(z)\}} = S_{\{\mathbf{a}^k\}}$.

Let the z-transform of the sequence $\mathbf{f} = \{f_\alpha \ : \ \alpha \in \mathbb{Z}^s\}$ be denoted by $L(\mathbf{f}; z)$, namely

$$L(\mathbf{f}; z) = \sum_{\alpha \in \mathbb{Z}^s} f_\alpha z^\alpha \ .$$

Then the refinement rule (2.2) can be written in the form

$$L(\mathbf{f}^{k+1}; z) = a^k(z) L(\mathbf{f}^k; z^2) \ , \tag{2.25}$$

with the formal meaning of the equality above being that corresponding powers of z on both sides of the equality have equal coefficients. Iterating the relation (2.25), we obtain

$$L(\mathbf{f}^{k+\ell}; z) = a^{k+\ell-1}(z) a^{k+\ell-2}(z^2) \cdots a^k(z^{2^{\ell-1}}) L(\mathbf{f}^k; z^{2^\ell}) \ . \tag{2.26}$$

Thus, the ℓ-iterated symbol from level k to level $k+\ell$ is

$$a^{[k;\ell]}(z) = \sum_{\alpha \in \mathbb{Z}^s} a^{[k;\ell]}_\alpha z^\alpha = \prod_{j=1}^{\ell} a^{k+\ell-j}(z^{2^{j-1}}) \ . \tag{2.27}$$

In the stationary case we denote the ℓ-iterated symbol by $a^{[\ell]}$

$$a^{[\ell]}(z) = \prod_{j=1}^{\ell} a(z^{2^{j-1}}) \tag{2.28}$$

2.5 The convolution property.

Here we present an important property of schemes, which is easily expressed in terms of the Laurent polynomial representation. This property is presented in three different forms, depending on the notion of convergence used.

1. Let $S_{\{\mathbf{a}^k\}}$ and $S_{\{\mathbf{b}^k\}}$ be two (uniformly) convergent schemes with corresponding basic limit functions $\phi_{\{\mathbf{a}^k\}}$ and $\phi_{\{\mathbf{b}^k\}}$. Then, the scheme $S_{\{\mathbf{c}^k\}}$ defined by the symbols

$$c^k(z) = 2^{-s} a^k(z) b^k(z) \ , \tag{2.29}$$

is also convergent and its basic limit function is

$$\phi_{\{\mathbf{c}^\mathbf{k}\}} = \phi_{\{\mathbf{a}^k\}} * \phi_{\{\mathbf{b}^\mathbf{k}\}} \ . \tag{2.30}$$

Here the symbol $*$ stands for the s-dimensional convolution [4], [41].

The convolution property which is repeatedly used in this paper for $s > 1$, is of a different form:

2. Let $S_{\{\mathbf{b}^\mathbf{k}\}}$ be a convergent s-variate subdivision scheme, and let $S_{\{\mathbf{a}^k\}}$ be a univariate scheme, which is convergent in the sense of (2.5) to integrable limit functions. Then the symbols

$$c^k(z) = 2^{-1} a^k(z^\lambda) b^k(z) \ , \tag{2.31}$$

with $\lambda \in \mathbb{Z}^s$, define a convergent scheme $S_{\{\mathbf{c}^\mathbf{k}\}}$. Morover

$$\phi_{\{\mathbf{c}^k\}}(x) = \phi_{\{\mathbf{a}^k\}} *_\lambda \phi_{\{\mathbf{b}^k\}}(x) \equiv \int_{\mathbb{R}} \phi_{\{\mathbf{a}^k\}}(x - \lambda t) \phi_{\{\mathbf{b}^k\}}(t) dt \ . \tag{2.32}$$

The convolution property is also valid in case of weak convergence of $S_\mathbf{a}$. This property is used only for one example in the paper.

3. Let $S_{\{\mathbf{b}^k\}}$ be an s-variate subdivision scheme convergent in the sense of (2.5), with $\phi_{\{\mathbf{b}^k\}}$ continuous in its support, and let $S_{\{\mathbf{a}^k\}}$ be a weakly convergent s-variate scheme, with $\phi_{\{\mathbf{a}^k\}}$ continuous in its support. Then the scheme $S_{\{\mathbf{c}^\mathbf{k}\}}$ defined by the symbols in (2.29) is convergent, and $\phi_{\{\mathbf{c}^\mathbf{k}\}}$ is given by (2.30).

Here we indicate how to verify the convolution property 2 ((2.31), (2.32)). The verification of the convolution property in its other two forms is based on the same reasoning. Observe that for $\mathbf{f}^k = R_{\mathbf{a}^{k-1}} \cdots R_{\mathbf{a}^0} \boldsymbol{\delta}$, we have $L(\mathbf{f}^k; z) = a^{[0,\ell]}(z)$, and that in polynomial multiplication the coefficients are computed by convolutions of the coefficients of the factors. Thus, the relations (2.31) and (2.27) yield

$$c^{[0,\ell]}(z) = 2^{-\ell} a^{[0,\ell]}(z^\lambda) b^{[0,\ell]}(z) \ ,$$

or equivalently

$$L(\mathbf{g}^\ell; z) = 2^{-\ell} L(\mathbf{f}^\ell; z^\lambda) L(\mathbf{h}^\ell; z) \ , \tag{2.33}$$

with $\mathbf{g}^\ell = R_{\mathbf{c}^{\ell-1}} \cdots R_{\mathbf{c}^0} \boldsymbol{\delta}$, and $\mathbf{h}^\ell = R_{\mathbf{b}^{\ell-1}} \cdots R_{\mathbf{b}^0} \boldsymbol{\delta}$.

Now, (2.32) can be concluded, by equating coefficients of equal powers of z on both sides of (2.33), taking into account the convergence of $\{\mathbf{f}^k\}_{k \in \mathbb{Z}_+}$ and of $\{\mathbf{h}^k\}_{k \in \mathbb{Z}_+}$ to the compactly supported limit functions $\phi_{\{\mathbf{a}^k\}}$, and $\phi_{\{\mathbf{b}^k\}}$ respectively.

3 The variety of subdivision schemes

Subdivision schemes have been first known as a tool for generating spline functions [5, 91, 11]. The renewed interest in this subject in geometric modeling has evolved as subdivision processes were extended to general topologies [3, 27]. In recent years interesting applications emerged, such as wavelets theory, and some very challenging theoretical issues arose. In the following we overview the major different types of subdivision schemes, most of them relevant to geometric modeling.

- B-splines and Box-splines schemes.
- The up-function.
- Exponential Splines and Box-splines schemes.
- Interpolatory schemes.
- Shape preserving schemes.
- General matrix schemes.
- Hermite-type and moment interpolatory schemes.
- Tensor product schemes.
- Different topologies for surface subdivision.

While assessing the various types we incorporate the notions of local support and support size, smoothness and approximation order. These issues will be further developed and investigated in the next sections. Here we take the liberty of using these properties in a heuristic manner.

3.1 Elementary schemes and their convolutions

The simplest elementary univariate uniform stationary scheme is the scheme with the symbol

$$a^k(z) = a(z) = 1 + z \ . \tag{3.1}$$

The corresponding basic limit function is the characteristic function of $[0, 1]$,

$$\phi_{\mathbf{1+z}} = B_0(\cdot) = \chi_{[0,1]} \ . \tag{3.2}$$

By the convolution property 1,

$$\phi_{2^{-m}(1+z)^{m+1}} = B_0(\cdot) * B_0(\cdot) * ... * B_0(\cdot) = B_m(\cdot) \ . \tag{3.3}$$

Thus, the scheme with symbol $a(z) = 2^{-m}(1 + z)^{m+1}$ has as a basic limit function the uniform mth-degree B-spline function with integer knots, supported in $[0, m + 1]$,

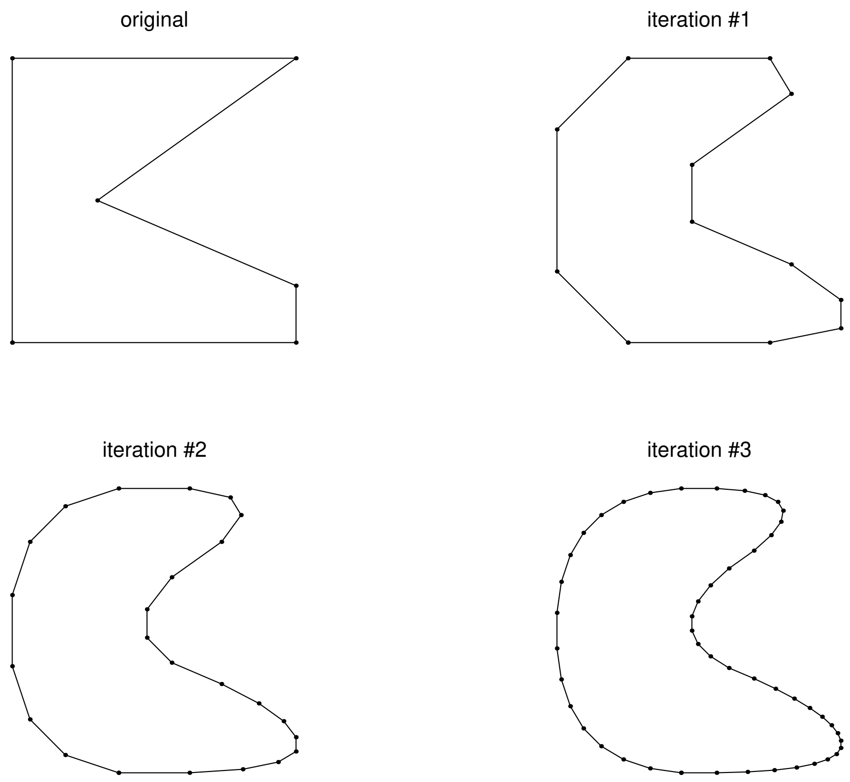

Figure 1: The Chaikin algorithm

which is in $C^{m-1}(\mathbb{R})$. As shown in §4.2.4, the symbol of a C^m univariate uniform stationary binary scheme, under an additional mild condition, must contain the factor $(1+z)^{m+1}$. The earliest example of a spline subdivision is the Chaikin's algorithm [5]

$$f_{2i}^{k+1} = \frac{3}{4}f_i^k + \frac{1}{4}f_{i+1}^k\ , \qquad f_{2i+1}^{k+1} = \frac{1}{4}f_i^k + \frac{3}{4}f_{i+1}^k\ , \tag{3.4}$$

which converges to a quadratic B-spline curve $\sum f_i^0 B_2(\cdot - i)$. The Chaikin's algorithm is also the basic example of a 'corner cutting' algorithm, which served as a starting point to various generalizations, e.g., in [18, 49]. The application of three iterations on a simple control polygon (the polygonal line joining the control points) is presented in figure 1.

Another interesting scheme which is constructed by convolutions of the elementary scheme is defined by:

$$a^k(z) = 2^{-k+1}(1+z)^k\ . \tag{3.5}$$

The corresponding basic limit function is the Rvachev's up-function [94, 22] which is in $C^\infty(\mathbb{R})$ and is supported in $[0, 2]$ (see Example 4.18). The spaces $V_k = span\{\phi_k(2^k\cdot -\alpha)\ :\ \alpha \in \mathbb{Z}^s\}$, $k \in \mathbb{Z}_+$ with $\{\phi_k\}$ defined as (2.9) with respect to the symbols at (3.5) provide spectral approximation order [47].

Products of the elementary univariate factors in directions in $\mathbb{Z}^s$ generate Box-splines in $\mathbb{R}^s$ as basis limit functions. Let $\Lambda = \{\lambda_1,\ ..., \lambda_\ell\} \subset \mathbb{Z}^s$, and define the stationary scheme with the symbol

$$a(z) = 2^{s-\ell} \prod_{j=1}^{\ell} (1 + z^{\lambda_j}) \ . \tag{3.6}$$

This scheme is related to the box-spline with directions Λ [19], [12]. Convergence is guaranteed if there is a subset of s directions $\{\lambda_{i_1}\lambda_{i_2}, ..., \lambda_{i_s}\} \in \Lambda$ such that $det(\lambda_{i_1}\lambda_{i_2} \cdot \cdot \cdot \lambda_{i_s}) = 1$. Furthermore, if any subset of $\ell - m - 1$ directions span $\mathbb{R}^s$, then $\phi_{\mathbf{a}}$ is in C^m [19].

An important example here is the scheme generating the C^2 quartic three-directional box-spline, namely, the scheme with the symbol

$$a(z) = 2^{-4}(1 + z^{(1,0)})^2(1 + z^{(0,1)})^2(1 + z^{(1,1)})^2 \ . \tag{3.7}$$

It is easy to check that the above conditions are satisfied with $m = 2$, and thus the basic limit function is a box-spline in C^2.

The uniform non-stationary elementary scheme is again a scheme define by symbols which are linear polynomials in z, namely,

$$a^k(z) = 1 + r_k z \ . \tag{3.8}$$

The parameters $\{r_k\}_{k\in\mathbb{Z}_+}$ are free parameters which determine the convergence of the subdivision process and the regularity of the limit function [41]. To examine these issues we write the scheme explicitly as

$$f_{2i}^{k+1} = f_i^k \ , \quad f_{2i+1}^{k+1} = r_k f_i^k \ , \quad , \quad i \in \mathbb{Z} \ . \tag{3.9}$$

Starting the subdivision with initial data $\mathbf{f}^0 = \boldsymbol{\delta}$, the limit at a diadic point $x = \sum_{i=1}^k d_i 2^{-i} \in [0,1)$, $d_i \in \{0,1\}$, is determined at level k of the subdivision. It is easy to verify that the value of the basic limit function ϕ at such x is

$$\phi(x) = \prod_{i=1}^{k} r_{i-1}^{d_i} \ . \tag{3.10}$$

Let us define $\phi(x)$ at non-diadic points by

$$\phi(x) = \prod_{i=1}^{\infty} r_{i-1}^{d_i} \ , \quad x = \sum_{i=1}^{\infty} d_i 2^{-i} \in [0,1) \ , \tag{3.11}$$

and $\phi(x) = 0$ for all $x \notin [0,1)$. If we assume that the parameters $\{r_k\}$ satisfy $\sum_{k\in\mathbb{Z}_+} |1 - r_k| < \infty$, then all the above infinite products converge, and we find out that ϕ is continuous at all non-diadic points. At diadic points in $[0,1)$ ϕ is right-continuous, hence, it is integrable. As proved in [41], ϕ is also left-continuous at all diadic points in $(0,1)$ if and only if $r_k = e^{c2^{-k}}$ for some constant c.

Exponential B-splines. The univariate elementary non-stationary scheme defined by

$$a^k(z) = 1 + e^{c2^{-k-1}} z \ , \quad k \in \mathbb{Z}_+ \ , \tag{3.12}$$

generates the exponential B-spline

$$\phi_{\{\mathbf{a^k}\}}(x) = e^{cx}\chi_{[0,1]}(x) \ . \tag{3.13}$$

Consequently, by the convolution property 1, the scheme generating the mth-order exponential B-spline with exponents $c_1, ..., c_m$ is

$$a^k(z) = 2^{-m+1}\prod_{j=1}^{m}(1 + e^{c_j 2^{-k-1}}z) \ . \tag{3.14}$$

Similarly, one can derive symbols of schemes generating exponential box-splines and exponential up-function [41].

Generating circumscribed circle. A special example of a scheme which is obtained by convolution of elementary schemes is given by the symbol

$$a^k(z) = \frac{1}{2(1+cos(\alpha_k))}(1+z)(1+e^{i\alpha_k}z)(1+e^{-i\alpha_k}z) \ , \quad \alpha_k = 2^{-k-1}\alpha_0 \ , \quad k \in \mathbb{Z}_+. \tag{3.15}$$

This is a C^1 "corner cutting" scheme which reproduces constants and also $sin(\alpha_0 x)$ and $cos(\alpha_0 x)$. If the initial control polygon is a regular n-gon and $\alpha_0 = \frac{2\pi}{n}$, then the limit curve is the circle circumscribed in the n-gon. The tensor product of the above scheme with any other stationary scheme generates surfaces of revolution [79]. Note that a circle cannot be generated by a stationary scheme.

3.2 Interpolatory schemes

A class of subdivision schemes with many specific features is that of "interpolatory subdivision schemes" [40]. The schemes in this class generate the refined values by retaining the values at the vertices of the current net, and defining new values at the new vertices of the refined net.

Among the B-spline schemes, only those generating B_0 and B_1 are interpolatory schemes, namely satisfying,

$$f^{k+1}_{2j} = f^k_j \ , \quad j \in \mathbb{Z} \quad , \ k \in \mathbb{Z}_+ \ , \tag{3.16}$$

together with insertion rules for new points $\{f^{k+1}_{2j+1}\}_{j\in\mathbb{Z}}$. The interpolatory refinement rules on $N_k = 2^{-k}\mathbb{Z}^s$ have the form

$$f^{k+1}_{2\alpha} = f^k_\alpha \ , \quad f^{k+1}_{\gamma+2\alpha} = \sum_{\beta\in\mathbb{Z}^s} a^k_{\gamma+2\beta} f^k_{\alpha-\beta} \ , \quad \gamma \in E_s\backslash 0 \ , \quad \alpha \in \mathbb{Z}^s \ . \tag{3.17}$$

The masks corresponding to an interpolatory subdivision scheme have the feature

$$a^k_{2\alpha} = \delta_{\alpha,0} \ , \quad \alpha \in \mathbb{Z}^s \ , \quad k \in \mathbb{Z}_+ \ .$$

It is easy to realize that in case of a convergent scheme, all the points

$$(2^{-k}\alpha, f^k_\alpha) \ , \quad \alpha \in \mathbb{Z}^s \ , \quad k \in \mathbb{Z}_+ \ ,$$

are on the graph of the limit function. In this setting there is (uniform) convergence if the values generated at the diadic points $\{f^k_\alpha \ : \ \alpha \in \mathbb{Z}^s, \ \ k \in \mathbb{Z}_+\}$, are continuous.

The basic limit functions $\{\phi_k \ : \ k \in \mathbb{Z}^s\}$ satisfy,

$$\phi_k(\alpha) = \delta_{\alpha,0} \ , \quad \alpha \in \mathbb{Z}^s \ , \quad k \in \mathbb{Z}_+ \ ,$$

thus their integer shifts $\{\phi_k(\cdot - \alpha) \ : \ \alpha \in \mathbb{Z}^s\}$, are linearly independent for any $k \in \mathbb{Z}_+$.

The 4-point scheme. In the class of stationary interpolatory schemes one looks for maximal smoothness and minimal support. The first attempts in this direction were the 4-point schemes presented in [28] and [33]. The 4-point scheme is the univariate scheme defined by (3.16) and the insertion rule

$$f^{k+1}_{2j+1} = -wf^k_{j-1} + (\frac{1}{2} + w)f^k_j + (\frac{1}{2} + w)f^k_{j+1} - wf^k_{j+2} \ , \tag{3.18}$$

for $j \in \mathbb{Z}$, and $k \in \mathbb{Z}_+$, where w is a shape parameter of the scheme. The symbol of the 4-point scheme is

$$a_w(z) = \frac{1}{2z}(z+1)^2(1 + wb(z)) \tag{3.19}$$

where

$$b(z) = -2z^{-2}(z-1)^2(z^2+1) \ . \tag{3.20}$$

For $w = 0$ the limit is the piecewise linear interpolant to the data. As w increases the limit function is less tight. The symbol contains the elementary factor $(z+1)^2$ necessary for C^1 convergence, and the challenge in [33] was to determine the range of values of the shape parameter w for which the scheme is C^1. The particular value $w = \frac{1}{16}$ is analyzed also in [28]. In this case the symbol contains the factor $(z+1)^4$, which means that the scheme reproduces cubic polynomials (see §4.2.3). Yet, the limit function is not even C^2. It is shown in [35] that the 4-point scheme is C^1 for any $w \in (0, \frac{\sqrt{5}-1}{8})$, and in [24] that for $w = \frac{1}{16}$ the first derivative is Hölder continuous for any Hölder exponent $0 < \nu < 1$, yet the second derivative does not exist at diadic points [33].

Deslauriers-Dubuc interpolatory schemes The 4-point scheme of [28] has been generalized to symmetric $2n$-point interpolatory schemes by Deslauriers and Dubuc in [24]. The insertion rule for f^{k+1}_{2j+1} is defined by polynomial interpolation of degree $2n - 1$ at $2^{-k-1}(2j+1)$ interpolating the $2n$ values $f^k_{j-n+1}, ..., f^k_{j+n}$. Let us denote the resulting symbol by $d_{(2n)}(z)$. These schemes are studied in [24] by Fourier analysis, and their convergenceis proved. The smoothness of $S_{\mathbf{d}_{(2n)}}$ grows linearly but slowly with n [14]. Generalizations to multidimensional interpolatory schemes is presented in [34, 90].

In analogy to the up-function, it is possible to get C^∞_0 interpolatory basic functions using the symbols of Deslauriers-Dubuc interpolatory schemes. This is achieved in [8] by defining the non-stationary subdivision symbols as $a^k(z) = d_{(2k)}(z)$.

Non-linear, stationary, shape preserving 4-point schemes. A significant drawback of linear interpolatory schemes is the lack of shape preservation properties. If one is interested in both interpolation and shape preservation, then linearity has to be given up. A beautiful example of a non-linear, stationary, shape preserving interpolatory scheme is the following 4-point C^1 convexity preserving scheme due to Kuijt and van Damme [65], where the rule replacing (3.18) is:

$$f_{2j+1}^{k+1} = \frac{1}{2}(f_j^k + f_{j+1}^k) - \frac{1}{4}\frac{1}{\frac{1}{d_j^k} + \frac{1}{d_{j+1}^k}} \; ; \qquad d_j^k = f_{j+1}^k - 2f_j^k + f_{j-1}^k \; . \tag{3.21}$$

Starting with a strictly convex initial functional data, it is shown in [65] that the limit function is a strictly convex C^1 function. Kuijt and van Damme have also developed non-linear schemes preserving monotonicity [66]. It is also possible to use the linear 4-point scheme and to generate a convex limit function from an initial strictly convex data, by choosing $w \in (0, w^*)$, where w^* depends on the initial data [39].

3.3 Matrix schemes and Hermite-type schemes

While interpolatory schemes preserve the function data at the points of the previous level, it is sometimes desirable to preserve other quantities. Two related families of schemes of this kind are the Hermite-type schemes and the moment interpolating schemes. We may view interpolatory schemes as schemes generating limit functions with specified values at the integers. Hermite-type schemes generate limit functions with specified function values and certain derivatives' values at the integers. Moment interpolating schemes produce limit functions with specified moments on the intervals $[i, i+1]$, $i \in \mathbb{Z}$. In both cases, the data attached to the vertices of the nets is a vector of values, and the subdivision operator is defined by a mask with matrix elements.

A univariate uniform stationary matrix subdivision scheme, operating on sequences of vectors in $\mathbb{R}^n$, is defined by a set of real $n \times n$ matrix coefficients $\{A_j : j \in \mathbb{Z}\}$, with a finite number of non-zero A_j's, generating sequences of control points in $\mathbb{R}^n$, $v^k = \{v_j^k \in \mathbb{R}^n \ : j \in \mathbb{Z}\}$, $k \geq 0$, recursively by

$$v_i^{k+1} = \sum_{j \in \mathbb{Z}} A_{i-2j} v_j^k \ , \ i \in \mathbb{Z} \ . \tag{3.22}$$

As an example of such a scheme, we consider the scheme generating the double-knot cubic spline. The matrix mask is defined by its matrix symbol,

$$A(z) = \frac{1}{16}\begin{pmatrix} 2 + 6z + z^2 & 2z + 5z^2 \\ 5 + 2z & 1 + 6z + 2z^2 \end{pmatrix} = \sum_{i \in \mathbb{Z}} A_i z^i \ . \tag{3.23}$$

Here there are two basic sets of initial data, namely, $\mathbf{v}^{1,0} = (1,0)^t \boldsymbol{\delta}$ and $\mathbf{v}^{2,0} = (0,1)^t \boldsymbol{\delta}$. The two basic limit vector functions are

$$S_{\mathbf{A}}^{\infty} \mathbf{v}^{1,0} = (\phi_1, \phi_1)^t \ , \quad S_{\mathbf{A}}^{\infty} \mathbf{v}^{2,0} = (\phi_2, \phi_2)^t \ , \tag{3.24}$$

where ϕ_1 and ϕ_2 are the two different cubic B-splines spanning the space of cubic splines with double knots at the integers [85].

Let us now return to the Hermite-type and moment interpolating schemes. In the Hermite case we start with Hermite-type data, $\{v_j^0 = (f_j^0, g_j^0)^t\}_{j\in\mathbb{Z}}$ where the values $\{g_j^0\}$ represent derivative data. We now consider the scheme

$$v_{2i}^{k+1} = v_v^k \ , \ v_{2i+1}^{k+1} = \sum A_{1-2j}^{(k)} v_{i+j}^k \ , \quad k \geq 0 \ , \tag{3.25}$$

or, equivalently,

$$v_i^{k+1} = \sum_j A_{i-2j}^{(k)} v_j^k \ , \quad k \geq 0 \ , \tag{3.26}$$

where $\{A_i^{(k)}\}$ are 2×2 matrices, possibly depending upon the refinement level k, and $A_{2j}^{(k)} = \delta_{j,0} I_{2\times 2}$. The Hermite-type scheme recursively defines values $\{v_j^k = (f_j^k, g_j^k)^t\}_{j\in\mathbb{Z}}$ attached respectively to the diadic points $\{j2^{-k}\}_{j\in\mathbb{Z}}$. We say that the scheme is C^r if there exists a function $f \in C^r(\mathbb{R})$ such that

$$v_j^k = (f_j^k, g_j^k)^t = (f(j2^{-k}), f'(j2^{-k}))^t \ , j \in \mathbb{Z} \ , \ k \in \mathbb{Z}_+ \ . \tag{3.27}$$

The first interesting example, presented in [77], is an extension of the interpolatory Hermite-cubic rule. The non-zero matrices of its mask are

$$A_1^{(k)} = \begin{pmatrix} \frac{1}{2} & \alpha 2^{-k} \\ -\beta 2^k & \frac{1-\beta}{2} \end{pmatrix} \ , \quad A_{-1}^{(k)} = \begin{pmatrix} \frac{1}{2} & -\alpha 2^{-k} \\ \beta 2^k & \frac{1-\beta}{2} \end{pmatrix} \ . \tag{3.28}$$

This scheme with $\alpha = 1/8$ and $\beta = 3/2$ produces the piecewise Hermite-cubic interpolant to the given initial data, and thus it is a C^1 scheme. We note that the matrices depend upon k, and they are even unbounded as $k \to \infty$. However, as shown in [42], if we consider in this case the scheme for transforming the vector of values $u_j^k = (g_j^k, df_j^k)^t$, with $df_j^k = 2^k(f_j^k - f_{j-1}^k)$, this scheme becomes stationary, i.e., with a constant matrix mask. Here, if the original scheme is C^1, then both elements of $\{u_j^k\}$ should converge to the same limit function f'.

The moment interpolation problem for m moments is defined as follows: Let $b^\ell(x) = \frac{(m-1)!}{\ell!(m-1-\ell)!} x^l (1-x)^{m-1-l} \cdot \chi_{[0,1]}$ denotes the ℓ-th Bernstein polynomial of degree $m-1$ for the interval $[0,1]$, truncated to $[0,1]$. Define

$$b_j^\ell(x) = b^\ell(x-j),$$

the translate of b^ℓ that "lives" on $[j, j+1]$ and has L_1-norm 1.

Given the local moments of a function f,

$$\beta_j^\ell = \langle f, b_j^\ell \rangle \ , \qquad j \in \mathbb{Z} \ , \quad 0 \leq \ell < m, \tag{3.29}$$

the problem is to construct a "smooth" function $\tilde{f}$ matching those moments. A solution of this problem by a subdivision process is presented in [25]. Also shown there is the close relation between the moment interpolating subdivision schemes and the Hermite interpolatory subdivision schemes. In the sections on the analysis of subdivision schemes, we consider only schemes with scalar masks. The analysis of schemes with a matrix mask is not reviewed here. The interested reader may consult [85, 7, 9, 78, 43].

3.4 Tensor product schemes and related ones

The simplest subdivision schemes on $\mathbb{Z}^2$ are the stationary tensor product schemes, obtained by applying one stationary univariate scheme in the x-direction and then a second (or the same) stationary univariate scheme in the y-direction. Let us denote the symbols of the stationary univariate schemes by $x(z)$ and $y(z)$ respectively, then the symbol of the tensor product scheme S_t is $t(z_1, z_2) = x(z_1)y(z_2)$. Obviously, the tensor product subdivision scheme inherits the convergence and smoothness properties of the univariate schemes. Tensor products of univariate spline schemes are special cases of box-splines, using only two directions in (3.6). For example, the mask generating the biquadratic and the bicubic B-spline functions are respectively defined by the symbols

$$a(z_1, z_2) = 2^{-4}(1 + z_1)^3(1 + z_2)^3 \ . \tag{3.30}$$

$$a(z_1, z_2) = 2^{-6}(1 + z_1)^4(1 + z_2)^4 \ . \tag{3.31}$$

Yet, tensor product schemes are not ideal with respect to the size of the support of the mask per given smoothness. In the case of splines, the same smoothness may be achieved by using more directions in (3.6), and less linear factors (see §4.3.1).

Considering the case of interpolatory schemes, the tensor product of two 4-point schemes (3.18), has the mask $t_w(z_1.z_2) = a_w(z_1)a_w(z_2)$, with support size of $4 \times 4 = 16$ points. Yet, as shown in [38], a schemes of a smaller support size (12 points), interpolatory, and with the same polynomial precision and smoothness exists. The suggested scheme is obtained by removing all the w^2 terms in t_w. The resulting symbol is

$$c_w(z_1, z_2) = \frac{1}{4}(1 + z_1)^2(1 + z_2)^2 z_1^{-1} z_2^{-1}(1 - w[b(z_1) + b(z_2)]) \ , \tag{3.32}$$

where b is given in (3.20).
The stencils (see §3.5) of the insertion rule of this truncated tensor-product scheme are shown in Figure 2.

The scheme S_{c_w} reproduces cubic polynomials for $w = \frac{1}{16}$, and it reduces to the 4-point scheme in one direction, when the data is constant along the other direction [38]. An interpolatory subdivision on quadrilateral nets (see §3.5) with arbitrary topology based on the 4-point scheme, is proposed by Kobbelt in [62].

3.5 Subdivision on nets

A control net for surfaces, consists of control points in $\mathbb{R}^3$ with topological relations between them. The refinement rules are defined with respect to a control net, and generate a refined control net with new control points. The topological relations in the refined net are determined by the type of net, while the control points are determined by the subdivision scheme as weighted averages of topologically neighboring control points.

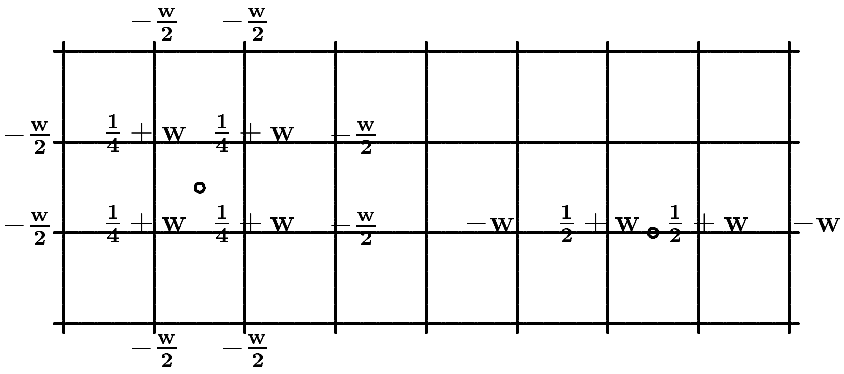

Figure 2: The two stencils of the truncated tensor-product scheme S_{c_w}

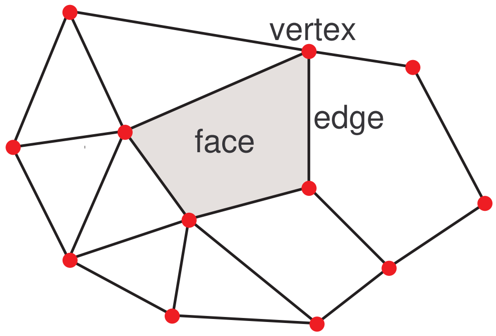

Figure 3: A net.

In this section we present subdivision schemes that are defined over nets of arbitrary topology in 3D space. Such nets are valuable for the task of free surface design. The surfaces generated by subdivision schemes on such nets are no longer restricted to representing bivariate functions, and they can represent surfaces of arbitrary topology. We describe three types of nets - triangular, Catmull-Clark type (primal type) and Doo-Sabin type (dual type), which are the most commonly used.

In addition to the above types of nets, there are hexagonal nets. Very few subdivision schemes with respect to hexagonal nets are available, see e.g. [44], [46], and they are not considered here.

3.5.1 Nets of general topology

A net $N(V, E, F)$, as shown in figure 3. is a configuration of a finite set V of points in $\mathbb{R}^3$ called *vertices*, with two sets of topological relations between them E and F, called *edges* and *faces* (A similar description of nets can be found in [64]).

An edge denotes a connection between two vertices. A face is a cyclic list of vertices where every pair of consecutive vertices share an edge. The *valency* of a vertex or a face is the number of edges that share that vertex or that face. While

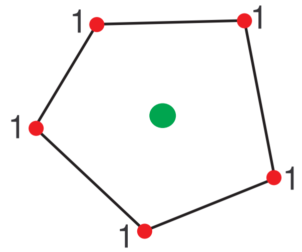

Figure 4: A stencil.

edges can always be represented by straight line segments, the vertices of a face are not necessarily co-planar, therefore a face is not associated with any geometric shape (in contrast to the faces of a polyhedron, which are planar pieces).

An edge e is called a *boundary edge* of $N(V, E, F)$ if it is not shared by two faces. A vertex v is called a *boundary vertex* if it belongs to a boundary edge.

We restrict our attention to nets $N(V, E, F)$ that satisfy the following properties:

1. Every pair of vertices share at most one edge.

2. The valency of each vertex is at least 2.

3. The valency of each face is at least 3.

4. Every boundary edge belongs to exactly one face.

5. Three boundary edges cannot share a vertex.

$N(V, E, F)$ is said to be *closed* if it has no boundary edges. Otherwise, $N(V, E, F)$ is an *open* net. A triangular net is a net whose faces all have valency 3. A closed triangular net is termed regular, or a regular triangulation, if the valency of each vertex is 6. A regular triangular net is locally topologically equivalent to a portion of the three directional grid, namely the grid $\mathbb{Z}^2$ with edges connecting (i, j) with $(i \pm 1, j)$, $(i, j \pm 1)$ and $(i \pm 1, j \pm 1)$, for $(i, j) \in \mathbb{Z}^2$. A quad-mesh is a net whose faces all have valency 4. A quad-mesh (quadrilateral net) is termed regular, if it is topologically equivalent to $\mathbb{Z}^2$, namely, the valency of each vertex is 4.

The subdivision process transforms the net $N(V, E, F)$ into a refined net $N(V', E', F')$, where each new vertex in V' is associated with an element or a configuration c of elements from $N(V, E, F)$. The method for calculating a new vertex $v' \in V'$ can be described as a weighted average (with possibly negative weights) of vertices of V. The weight given to every vertex $v \in V$ depends only on its topological relation to c. The set of weights, together with their topological location in V relative to c, constitute the *stencil* which is determined by the subdivision scheme. There are different stencils for different topological configurations.

For example, suppose that a vertex v' is associated with a face $f \in F$ that has valency 5. The stencil in figure 4 represents the rule: v' is the average of the vertices of f. The set of vertices with non-zero weights, the support of the stencil, is topologically

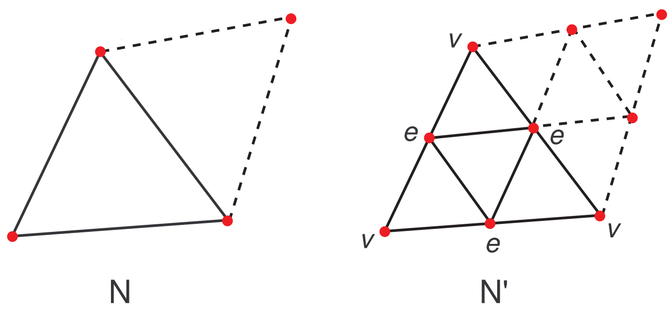

Figure 5: Triangular subdivision

related to c, but not necessarily coincide with c as it does in the last example. Together with the definition of V', there is a proper definition of the new edges E' and faces F', and these are described later for the different types of nets.

Let S denote a subdivision operator for nets. Let $N_0 = N(V, E, F)$ be a given initial net. A sequence of finer nets $N_k = N\left(V^k, E^k, F^k\right)$ is defined for $k = 1, 2, \ldots$ by

$$N_{k+1} = SN_k, \quad k = 0, 1, \ldots . \tag{3.33}$$

Ideally, the convergence of the sequence of nets $\{N_k \ : \ k \in \mathbb{Z}_+\}$ to a limit surface X should be defined independently of any parametrization of the surface. In the following definition, a surface X is considered as a closed subset of $\mathbb{R}^3$. We say that X is the limit surface of the subdivision scheme (3.33) if

$$\lim_{k\to\infty} dist\left(V^k, X\right) = 0. \tag{3.34}$$

where $dist(X, Y)$ denoted the Euclidean Hausdorff distance between two closed subsets $X, Y \subset \mathbb{R}^3$. In case a limit surface X exists we denote it by $S^\infty N_0 = X$. In practice, however, the convergence is studied with respect to appropriate local parametrizations of the limit surface.

3.5.2 Triangular subdivision

Triangular subdivision schemes are defined over triangular nets, i.e. nets whose faces all have valency 3 and therefore can be regarded as planar triangles. The new vertices are divided to v-vertices, and e-vertices. Each v-vertex in V' is associated with a vertex in V. Every e-vertex in V' is associated with an edge in E. For each type of vertex there is a different stencil. The new edges E' are defined between a new v-vertex and all the e-vertices such that their "parents" in E share the parent of the v-vertex in V, and between any two e-vertices such that their parent edges share a face in F. Thus every triangle in the original net $N(V, E, F)$ is replaced by four triangles in the new net $N(V', E', F')$. The topology of the new triangular net is shown in figure 5.

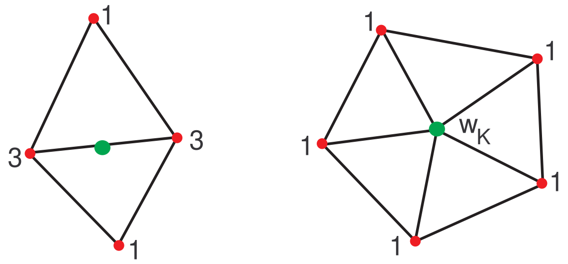

Figure 6: Loop scheme
Stencils for e-vertex (left) and for v vertex (rigth)

A regular vertex in a triangular net, is a vertex with valency 6. In a closed net, every new e-vertex has valency 6, and every new v-vertex inherits the valency of its parent vertex. Therefore, the number of irregular vertices in a net remains constant, and most of the net is a regular triangular net.

One of the commonly used triangular subdivision schemes is the Loop subdivision scheme [74] defined for closed triangular nets. The stencils for the new e-vertices and v-vertices are depicted in figure 6.

The weight w_n given to the original vertex, in the stencil for its corresponding new v-vertex depends on the valency K of that vertex. It is given by the following formula:

$$w_K = \frac{64K}{40 - \left(3 + 2\cos\left(\frac{2\pi}{K}\right)\right)^2} - K, \quad K = 3, 4, \dots . \tag{3.35}$$

Loop scheme generalizes the three-directional box-spline scheme (3.7), in the sense that it coincides with it in the regular parts of the net. This implies that the limit surface is C^2 almost everywhere, and this is achieved with stencils of very small support. Near irregular vertices of the original net, the surface is C^1 [74]. Another important property for geometric modeling of this scheme is shape preservation which is due to the positivity of the weights in the stencils of Loop scheme.

An interpolatory triangular subdivision scheme with stencil of small support is the Butterfly scheme [34]. This scheme is defined over closed triangular nets. It has improved stencils in the vicinity of irregular vertices [101] which produce better looking and smoother surfaces in the presence of irregular vertices.

As an interpolatory scheme, the new v-vertices inherit their location from their parent vertices. Figure 7 shows the stencils for new e-vertices. The Butterfly stencil is used to calculate new e-vertices whose parent edge is "regular", namely, has two regular vertices. A different stencil is used when the parent edge is "iregular, namely, has one vertex which is regular and one which has valency $K \neq 6$. The weights $\{s_j\}_{j=0,\dots,K-1}$ depend on the valency of the irregular vertex, and are given by

$$s_j = \frac{1}{K}\left(\frac{1}{4} + \cos\left(\frac{2\pi j}{K}\right) + \frac{1}{2}\cos\left(\frac{4\pi j}{K}\right)\right), \quad j = 0, \dots, K-1.$$

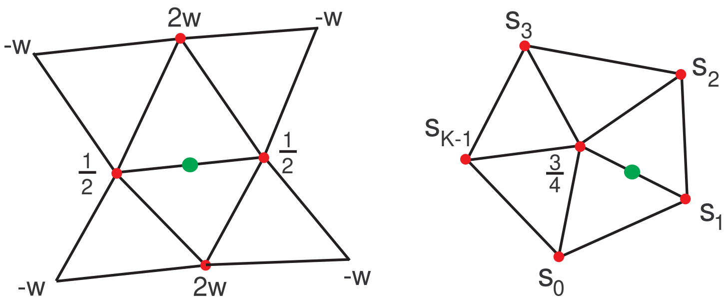

Figure 7: Modified Butterfly scheme
Stencils corresponding to a "regular" edge (left) and an "irregular' edge (right)

The case where both of the vertices of the parent edge are irregular can occur only in the initial net. In such a case, in the first refinement step the calculation of the new e-vertex may be done in any reasonable way. The limit surfaces generated by the butterfly scheme are C^1 continuous everywhere, a property valuable for computer graphics applications [101]. An extended butterfly interpolatory subdivision scheme for the generation of C^2 surfaces (except for extraordinary vertices) is presented in [68].

3.5.3 Subdivision on an arbitrary net

The two types of refinements of nets of arbitrary topological structure are the Catmull-Clark type, also called 'primal', and the Doo-Sabin type, also called 'dual'.

In the **primal type refinement**, every face of valency n in the original net $N(V, E, F)$ is replaced by n quadrilateral faces in the new net $N'(V', E', F')$, as shown in figure 8.

The new vertices are divided to v-vertices, e-vertices and f-vertices. Each v-vertex in V' is associated with a vertex in V. Each e-vertex in V' is associated with an edge in E. Each f-vertex in V' is associated with a face in F.

Figure 8 indicates the topological relations in $N(V', E', F')$, with the points v, e, and f indicating v-vertices, e-vertices and f-vertices respectively. The new edges are marked by line segments and the faces by the quadrilaterals formed.

A regular vertex in this setting is a vertex with valency 4, and a regular face is also of valency 4, namely, a quadrilateral face. Vertices or faces with valency $\neq 4$ are termed "irregular" or "extraordinary". In a closed net, every new e-vertex has valency 4. Every new v-vertex inherits the valency of its parent vertex, and every new f-vertex inherits the valency of its parent face. Therefore, the number of irregularities in a net remains constant throughout the subdivision process. Note that after one subdivision iteration, all the faces are quadrilateral. The actual locations in $\mathbb{R}^3$ of the vertices V' are determined by the stencils of the subdivision scheme.

Catmull-Clark scheme. The first example of a primal type scheme is the Catmull-Clark scheme [3, 27], defined as an extension of the bicubic B-spline scheme

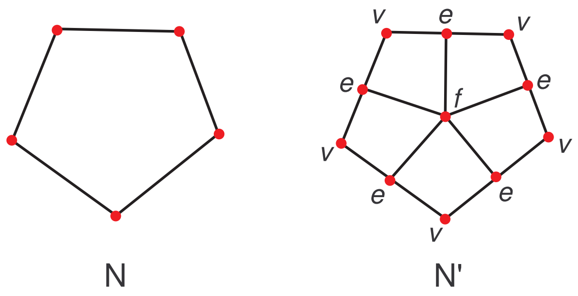

Figure 8: Primal type refinement

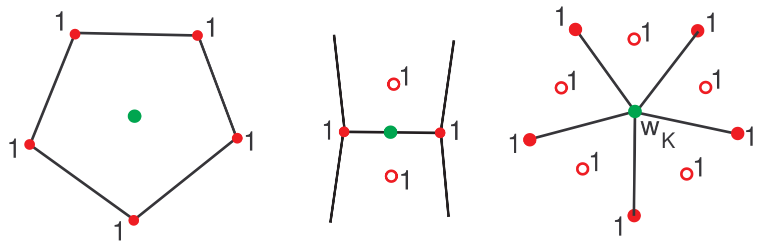

Figure 9: Catmull-Clark scheme
f-stencil (left), e-stencil (middle) and v-stencil (right)

(3.31) to closed nets of arbitrary topology. Its stencils are depicted in figure 9.

The stencils for the new e-vertices and v-vertices involve the neighboring new f-vertices (depicted as empty circles). The weight W_K in the stencil for the new v-vertex depends on the valency K of that vertex. Different formulae for W_K produce different behavior of the limit surface near irregular vertices. A commonly used formula for W_K is

$$W_K = K(K-2), \quad K = 3, 4, \ldots .$$

As long as $W_4 = 8$, the limit surfaces of this scheme are C^2 away from irregular points. Different variants of this scheme were investigated by Ball and Storry [2, 1]. It is observed there that for every choice of W_K, the surface curvature near an irregular point either tends to zero, or is unbounded. Applications of Catmull-Clark scheme can be found in [23, 51].

Here we present an example (see figure 10) of two surfaces generated from an initial triangulation, one by Loop scheme and the other by Catmull-Clark scheme which regards the triangulation as a general net. Note that in the later case, most of the initial control points are irregular.

The **dual type refinement** is depicted in figure 11. Every new vertex in $v' \in V'$ corresponds to a pair $(v \in V, f \in F)$ such that v is a vertex of f in the original net N. It is considered a *dual* scheme, since vertices and edges in the original net $N = N(V, E, F)$ correspond to faces in the new net $N' = N(V', E', F')$. A regular

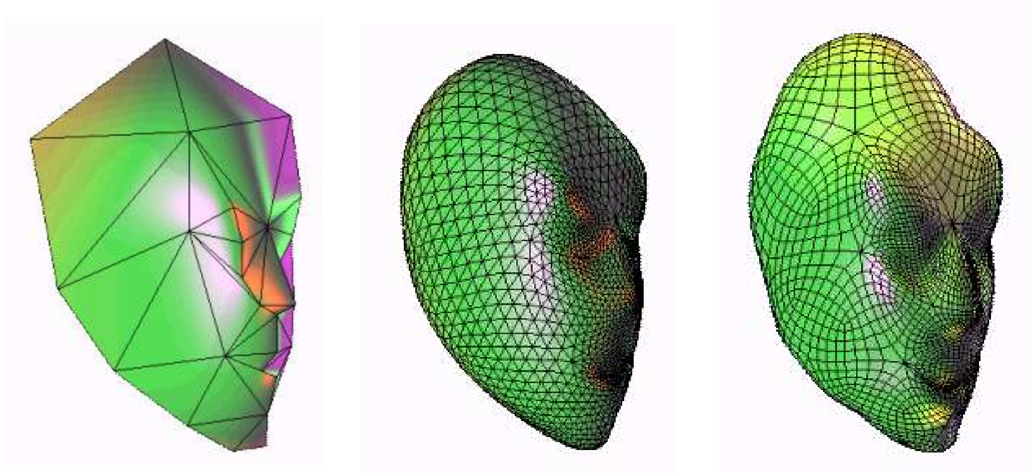

Figure 10: Head - initial contol net (left),
two iterations with Loop scheme (middle) and with Catmull-Clark scheme (right)

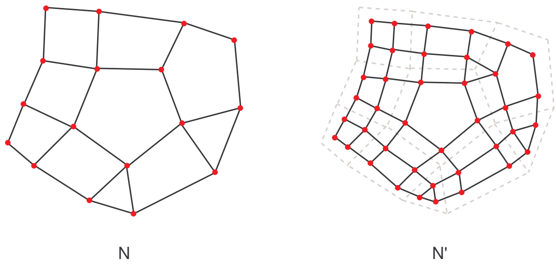

Figure 11: Dual type refinement

vertex in this setting is a vertex with valency 4, and a regular face is a quadrilateral face.

Doo-Sabin scheme The dual scheme due to Doo and Sabin generalizes the biquadratic B-spline scheme to subdivision of closed nets of arbitrary topological type.

The vertex v' is calculated by a weighted average of the vertices of f, with the stencils shown in figure 12. The weights $\{S_j\}_{j=0,\dots,K-1}$ depend on the valency K of the face in the original net, and are given by

$$s_0 = \frac{K+5}{4K}, \qquad s_j = \frac{3+2\cos\left(\frac{2\pi j}{K}\right)}{4K}, \quad j = 1, \dots, K-1.$$

Almost everywhere the new nets are regular quadrilateral nets and the scheme reduces to the scheme defined by the symbol (3.30) giving the C^1 biquadratic spline surface.

In both examples, of the Catmull-Clark scheme and of the Doo-Sabin scheme, the mask parameters near an extraordinary vertex are so chosen to achieve an overall C^1 limit surface. In §6 we describe the main results on the analysis of smoothness of stationary subdivision schemes near irregular vertices. Another dual type subdivision

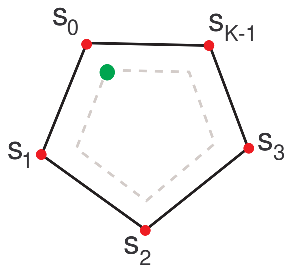

Figure 12: Doo-Sabin scheme - stencil for an f vertex

scheme is 'the simplest scheme for smoothing polyhedra' presented in [83]. In this scheme, given a polyhedron, a new polyhedron is constructed by connecting every edge-midpoint to its four neighboring edge-midpoints. The limit surface is piecewise quadratic C^1 surface except at some extraordinary vertices. For additional material about subdivision schemes on general nets and their applications in computer graphics see [56, 101, 102, 103].

3.6 Further extensions

The inspiring iterative refinement idea which is the basic concept in subdivision and in wavelets, motivated many new research directions. In this section we briefly mention several extensions and generalizations of the uniform binary subdivision which are not discussed in this review. These include extensions to

- Non-uniform schemes.
- Quasi-uniform and Combined subdivision.
- Lie group valued subdivision.
- Set-valued subdivision.
- Polyscale subdivision.
- Variational subdivision.
- Quasi-linear subdivision.

Non-uniform schemes. In many applications the data may be given on an irregular mesh and a scheme for iterative refinement of such data should be different from the standard uniform subdivision schemes. Also, convergence and smoothness analysis cannot be performed using the standard tools such as z-transform or Fourier transform. The tools that are being used for subdivision schemes over irregular grids are generalizations of the local matrix analysis (§5) and of the divided defference schemes (§4.2). See, e.g., [98], [50] and [15]. Another type of non-uniform schemes is still on uniform grids, but the subdivision refinement rules may differ from one point to the other. Here again it seems that the divided difference tools are the only way to analyse convergence and smoothness, as is done by Gregory and Qu for general corner cutting schemes [49]. A systematic method for deriving the difference schemes, using a variation of the z-transform method, is presented in [73]. A general analysis of shape preserving schemes for non-uniform data is done in [67].

Quasi-uniform and Combined subdivision. The analysis presented in this review is restricted to the case of closed nets, i.e., there are no boundary edges. In real applications, there are boundaries of surface patches and boundaries may occur inside a patch if the patch should pass through a curve or a system of curves. For a subdivision scheme, a boundary treatment requires the definition of special rules in the vicinity of the boundary, and consequently, a special smoothness analysis. A subdivision scheme, together with special boundary rules, is termed in [71, 70] a combined subdivision scheme. In these works, analysis tools for combined subdivision schemes are developed, and combined schemes, based on some of the most "popular" bivariate schemes, are designed. The problem of matching boundary conditions or curve interpolation by subdivision surfaces is also treated in [82, 81, 72]. A boundary may also be the border between two regions, or two patches, where in each patch a different uniform subdivision scheme is applied. This is termed quasi-uniform or piecewise uniform, and here also a special smoothness analysis is required, as presented in [36] for the univariate case and in [69, 71, 100] for surfaces.

Lie group valued subdivision. In some applications the data is restricted to a manifold W in $\mathbb{R}^d$, and the limit function is also expected to be a function from $\mathbb{R}^s$ into W. The usual subdivision schemes are defined via linear averaging refinement rules that not necessarily give points in W. In a recent work [26] the general case of Lie group valued data is considered. The main approach is based on the fact that each Lie group has its associated Lie algebra, related through the exponential map, and the subdivision operation are performed in the Lie algebra and are translated back to the group by the exponential map.

Set-valued subdivision. For these schemes the initial data and the refined data generated by the scheme, are sequences of sets in $\mathbb{R}^d$, and the limit function is a set-valued function. This is motivated by the problem of the reconstruction of 3D objects from their 2D cross sections. The given data is a sequence of cross sections and the set-valued function describes a 3D object. Subdivision schemes for set-valued data require the definition of operations on sets and the study of notions of convergence and smoothness of set-valued functions. These issues, for convex sets using Minkowski averages, and for general compact sets using the "metric average", are studied in [29], [30], [31].

Polyscale subdivision. A subdivision scheme is a two-scale process, using data at one refinement level to compute the values at the next refinement level. In [21] polyscale subdivision schemes are introduced. Such schemes compute the next refinement level from several previous levels, using several masks. This new idea is related also to the notion of poly-scale refinable functions, and opens up new theoretical convergence and smoothness issues. These issues, several interesting examples, and the relation of poly-scale subdivision schemes to matrix subdivision schemes, are presented in [21].

Variational subdivision. A variational approach to interpolatory subdivision is presented in [63]. The resulting schemes are global, i.e., every new point depends on all the points of the control polygon to be refined. The refinement is defined by minimizing a quadratic "energy" functional, resulting in a highly smooth limit surface.

Quasi-linear subdivision. Quasi-linear schemes are non-linear binary interpolatory schemes defined on a regular grid, with linear insertion rules which are data dependent. In [10] a specific class based upon the ENO and weighted-ENO interpolation techniques is analysed.

4 Convergence and smoothness analysis on regular grids

In this section, analysis of the (uniform) convergence of subdivision schemes on regular grids is presented, together with analysis of the smoothness of the limit functions.

First we present a method which relates the convergence and smoothness of non-stationary schemes to the convergence and smoothness of related stationary schemes [41], then we present a method for the analysis of stationary schemes, based on difference schemes (see [32] and references therein). This method is also applied directly to certain non-stationary schemes.

The main other approaches to the convergence and smoothness analysis are in terms of Fourier transforms, and in terms of the joint spectral radius of a finite set of finite dimentional matrices. The later approach is briefly reviewed in §5.2. The Fourier analysis approach is not surveyed here. The interested reader may consult [6, 24, 14, 16].

4.1 Analysis of non-stationary schemes via relations to stationary schemes

The analysis of the convergence of non-stationary schemes presented here, relies on the representation of a subdivision scheme $S_{\{\mathbf{a}^k\}}$ in terms of a sequence of operators $\{R_k \ : \ k \in \mathbb{Z}_+\}$ as in(2.22), where each R_k is defined by (2.21).

The main results are based on several observations on sequences of bounded linear operators in a Banach space. From now on all operators considered are bounded and linear. A sequence of operators $\{A_k \ : \ k \in \mathbb{Z}_+\}$ in a Banach space $\{X, \|\cdot\|\}$ defines the iterated process $x_{k+1} = A_k x_k, \ k \in \mathbb{Z}_+$, with $x_0 \in X$. Such a sequence is termed convergent if for any $m \in \mathbb{Z}_+$ and any $x \in X$, $\lim_{k\to\infty} x_{m,k}$ exists, where $x_{m,k} = A_{m+k} \cdots A_{m+1} A_m x$. The sequence $\{A_k\}$ is termed stable if

$$\|A_{m+k} \cdots A_{m+1} A_m\| \le M < \infty, \quad \forall m, k \in \mathbb{Z}_+ \ . \tag{4.1}$$

Two sequences of bounded operators $\{A_k\}$ and $\{B_k\}$ are called asymptotically equivalent if there exists $L \in \mathbb{Z}$, such that

$$\sum_{k=\max\{0,-L\}}^{\infty} \|A_{k+L} - B_k\| < \infty \ . \tag{4.2}$$

Proposition 4.1 *Let $\{A_k\}$ and $\{B_k\}$ be asymptotically equivalent. Then $\{A_k\}$ is stable if and only if $\{B_k\}$ is stable.*

The proof of this proposition [41] introduces the $\{A_k\}$-norms

$$\|x\|_m = \sup_k \|A_{m+k}, \cdots, A_m x\|, \quad m \in \mathbb{Z}_+ \ ,$$

which are equivalent to the norm of the Banach space, when $\{A_k\}$ is stable. It also introduces the Banach spaces $X_m = \{X, \|\cdot\|_m\}$. The key observation is that A_m as an operator from X_m to X_{m+1}, is bounded in norm by 1. From this observation follows Proposition 4.1. By a similar reasoning one gets

Proposition 4.2 *Let $\{A_k\}$ and $\{B_k\}$ be asymptotically equivalent. Then $\{A_k\}$ is stable and convergent if and only if so is $\{B_k\}$.*

This analysis of sequences of operators in a Banach space, leads to the important notion "asymptotic equivalence" between two subdivision schemes. Here we use the representation of subdivision schemes as operators on $X = C(\mathbb{R}^s)$, with the maximum norm. Two schemes $S_{\{\mathbf{a}^k\}}, S_{\{\mathbf{b}^k\}}$ are defined to be "asymptotically equivalent" if for some fixed $L \in \mathbb{Z}$,

$$\sum_{k=\max\{0,-L\}}^{\infty} \|\mathbf{a}^{k+L} - \mathbf{b}^k\|_\infty < \infty \ , \tag{4.3}$$

where $\|\mathbf{a}^k - \mathbf{b}^j\|_\infty = \max_{\alpha \in E^s} \sum_{\beta \in \mathbb{Z}^s} |a^k_{\alpha-2\beta} - b^j_{\alpha-2\beta}|$.

A scheme $S_{\{\mathbf{a}^k\}}$ is termed "stable" if there exists $M > 0$, such that for all $k, j \in \mathbb{Z}_+$

$$\|R_{j+k} \cdots R_{j+1} R_j\|_\infty < M \ , \tag{4.4}$$

with $\{R_j\}$ the operators corresponding to $S_{\{\mathbf{a}^k\}}$ as in (2.21). It is easy to conclude from (2.10) that a convergent scheme $S_{\{\mathbf{a}^k\}}$ is stable, iff the functions $\Phi_k = \sum_{\alpha \in \mathbb{Z}^s} |\phi_k(\cdot - \alpha)|$ are uniformly bounded for $k \in \mathbb{Z}_+$.

Two stable asymptotically equivalent schemes have similar convergence properties. This is easily concluded from Proposition 4.2.

Theorem 4.3 *Let $S_{\{a^k\}}$ and $S_{\{b^k\}}$ be asymptotically equivalent. Then $S_{\{a^k\}}$ is stable and convergent iff $S_{\{b^k\}}$ is stable and convergent.*

If $S_{\{\mathbf{b}^k\}} = S_{\mathbf{b}}$ is stationary, namely $\mathbf{b}^k = \mathbf{b}$ for $k \in \mathbb{Z}_+$, and $S_{\mathbf{b}}$ is convergent then by (2.8) $S_{\mathbf{b}}$ is stable. Thus

Corollary 4.4 *Let $S_{\{\mathbf{a}^k\}}$ and $S_{\mathbf{b}}$ be asymptotically equivalent. If $S_{\mathbf{b}}$ is convergent then $S_{\{\mathbf{a}^k\}}$ is stable and convergent.*

Example 4.5 As an example of convergence implied by Corollary 4.4, we consider the non-stationary subdivision scheme with symbols

$$a^k(z) = 2\prod_{i=1}^{m} \frac{1}{2}\left(1 + e^{\eta_i 2^{-k}} z\right) , \quad k \in \mathbb{Z}_+ , \tag{4.5}$$

with $\eta_1, \ldots , \eta_m$ distinct complex constants.

It is easy to verify that $S_{\{\mathbf{a}^k\}}$ is asymptotically equivalent to $S_{\mathbf{b}}$ with symbol

$$b(z) = 2^{-m+1}(1 + z)^m . \tag{4.6}$$

$S_{\mathbf{b}}$ is a convergent stationary subdivision scheme with a basic limit function the polynomial B-spline of order m (degree $m - 1$) with integer knots and support $[0, m]$ (see §3.1).

Thus the non-stationary scheme (4.5) is convergent. In fact its basic limit function is the exponential B-spline in span $\{e^{\frac{\eta_i}{2}x} : 1 \le i \le m\}$ with integer knots and support $[0, m]$. (For more about exponential B-splines, see e.g. [95]).

One way to analyze the smoothness of the basic limit function of a non-stationary scheme $S_{\{\mathbf{a}^k\}}$ (and therefore all limit functions generated by $S_{\{a^k\}}$, as implied by (2.8)), is in terms of smoothing factors [41].

Theorem 4.6 *Let the symbols of $S_{\{\mathbf{a}^k\}}$ be of the form*

$$a^k(z) = \frac{1}{2}(1 + r_k z^\lambda)b^k(z), \quad k \ge K \in \mathbb{Z}_+ , \tag{4.7}$$

with $\lambda \in \mathbb{Z}^s$, where $S_{\{\mathbf{b}^k\}}$ is a stable and convergent subdivision scheme with $\phi_{\{\mathbf{b}^k\}}$ of compact support and in $C^m(\mathbb{R}^s)$. If

$$r_k = e^{\eta 2^{-k}}(1 + \varepsilon_k), \quad \sum_{k=K}^{\infty} |\varepsilon_k| 2^k < \infty , \tag{4.8}$$

then $\phi_{\{\mathbf{a}^k\}}$ and $\partial_\lambda \phi_{\{\mathbf{a}^k\}}$ are in $C^m(\mathbb{R}^s)$

The factors $\frac{1}{2}(1 + r_k z^\lambda)$ in (4.7) are termed "smoothing factors", and for $\varepsilon_k = 0$ in (4.8), are related to the univariate elementary non-stationary scheme of (3.12),

Sketch of the proof: The key to the proof is the convolution property 2, which in this case has the form

$$\phi_{\{\mathbf{a}^k\}} = \int_{\mathbb{R}} \phi_{\{\mathbf{b}^k\}}(\cdot - \lambda t)\phi_{\{1+r_k z\}}(t)dt .$$

Since $\phi_{\{1+r_k z\}}$ is supported on $[0, 1]$ and is integrable (as discussed in §3.1), $\phi_{\{\mathbf{a}^k\}} \in C^m$. The result $\partial_\lambda \phi_{\{a^k\}} \in C^m$ follows from the general observation that for a univariate integrable function h with $\sigma(h) = [0, 1]$, and for a bounded continuous function $g \in C(\mathbb{R})$,

$$(g * h)(x) = \int_{x-1}^{x} g(t)h(x - t)dt \in C^1(\mathbb{R}) .$$

Since for a multivariate function the conditions $\partial_\lambda f \in C^m$ for $\lambda \in \Lambda$, with Λ a set of s linearly independent directions in $\mathbb{R}^s$, imply that $f \in C^{m+1}(\mathbb{R}^s)$, we conclude from Theorem 4.6 and the convolution property 2,

Corollary 4.7 *Let*

$$a^k(z) = \prod_{i=1}^{s} \frac{1}{2}(1 + r_{i,k} z^{\lambda_i}) b^k(z), \quad k \geq K \in \mathbb{Z}_+$$

where $S_{\{\mathbf{b}^k\}}$ satisfies the conditions of Theorem 4.6.

If for $i = 1, \ldots, m$,

$$r_{i,k} = e^{\eta_i 2^{-k}}(1 + \varepsilon_{i,k}), \quad \sum_{k=K}^{\infty} |\varepsilon_{i,k}| 2^k < \infty \ ,$$

and if $\lambda_1, \ldots, \lambda_s \in \mathbb{Z}^s$ are linearly independent, then $\phi_{\{a^k\}} \in C^{m+1}$.

A good example where smoothness is concluded with the help of Theorem 4.6, is provided by the non-stationary, univariate interpolatory schemes which reproduce finite dimensional spaces of exponential polynomials [45].

Example 4.8 Consider finite dimensional spaces of univariate exponential polynomials of the form

$$V_{\boldsymbol{\gamma},\boldsymbol{\mu}} = \mathrm{span}\{x^j e^{\gamma_\ell t}, j = 0, \ldots, \mu_{\ell-1}, \ \ell = 1, \ldots, \nu\}$$

where $\boldsymbol{\gamma} = \{\gamma_1, \ldots, \gamma_\nu\}$ are the roots with multiplicities $\boldsymbol{\mu} = \{\mu_1, \ldots, \mu_\nu\}$ of a real polynomial of degree $\sum_{i=1}^{\nu} \mu_i$.

A scheme $S_{\{\mathbf{a}^k\}}$ is termed a reproducing scheme of $V_{\boldsymbol{\gamma},\boldsymbol{\mu}}$, if for any $k \in \mathbb{Z}_+$ and $\mathbf{f}^k = \{f_j^k = f(2^{-k} j) : j \in \mathbb{Z}\}$, with $f \in V_{\boldsymbol{\gamma},\boldsymbol{\mu}}$,

$$S_{\mathbf{a}^k} \mathbf{f}^k = \mathbf{f}^{k+1}$$

It is proved in [45] that an interpolatory scheme $S_{\{a^k\}}$ with support $\sigma(a^k)$ fixed for $k \in \mathbb{Z}_+$ which reproduces $V_{\boldsymbol{\gamma},\boldsymbol{\mu}}$ and does not reproduce any bigger space of exponential polynomials containing $V_{\boldsymbol{\gamma},\boldsymbol{\mu}}$, has the property that its symbols $\{a^k(z) : k \in \mathbb{Z}_+\}$ are Laurent polynomials of degree $2n$ satisfying

$$\frac{d^r}{dz^r} a^k(z_n^k) = 2\delta_{0,r}, \quad \frac{d^r}{dz^r} a^k(-z_n^k) = 0, \quad r = 0, 1, \ldots, \mu_n - 1, \quad n = 1, \ldots, \nu \ , \qquad (4.9)$$

where $z_n^k = \exp(2^{-(k+1)} \gamma_n), \ \ n = 1, \ldots, \nu, \ \ k \in \mathbb{Z}_+$.

For the case $n = 2\ell$, it can be concluded from (4.9) that the masks $\{\mathbf{a}^k : k \in \mathbb{Z}_+\}$ with $\sigma(\mathbf{a}^k) = [-n, n]$, tend as $k \to \infty$ to the mask $\mathbf{a}$ with $\sigma(\mathbf{a}) = [-n, n]$ of the interpolatory scheme, introduced in [24], which reproduces the space π_{n-1} of all polynomials of degree not exceeding $n - 1$ (see §3.2). More specifically

$$\|\mathbf{a}^k - \mathbf{a}\|_\infty < 2^{-k} B, \quad 0 < B < \infty \ ,$$

and $a(z)$ is divisible by $(1 + z)^n$ as follows from (4.9). Thus $S_{\{\mathbf{a}^k\}}$ is asymptotically equivalent to $S_{\mathbf{a}}$, and since $S_{\mathbf{a}}$ is convergent [24] so is $S_{\{\mathbf{a}^k\}}$. To conclude the smoothness of $\phi_0 = \phi_{\{\mathbf{a}^k\}}$ from the smoothness of $\phi_{\mathbf{a}}$ Theorem 4.6 is invoked. Assume $\phi_{\mathbf{a}} \in C^m$. Then by the theory of smoothness of stationary schemes (see §4.2.4) $m \leq n$. Consider for each $k \in \mathbb{Z}_+$, the m linear factors of $a^k(z)$, $\prod_{i=1}^m (1 + (z_{n_i}^k)^{-1} z)$, where $n_1, \ldots n_m$ are fixed integers in $\{1, \ldots, \nu\}$, such that $\#\{n_i : n_i = j\} \leq \mu_j$. The existence of these factors is guaranteed by (4.9). Each of the m factors divided by 2 is a smoothing factor. Now, the symbols $\{c^k(z) : k \in \mathbb{Z}_+\}$ given by

$$c^k(z) = \frac{a^k(z) 2^m}{\prod_{i=1}^m (1 + (z_{n_i}^k)^{-1} z)}, \quad k \in \mathbb{Z}_+ ,$$

define a scheme $S_{\{\mathbf{c}^k\}}$ which is asymptotically equivalent to the scheme $S_{\mathbf{c}}$ with symbol

$$c(z) = \frac{a(z) 2^m}{(1+z)^m} .$$

Since $\phi_{\mathbf{a}} \in C^m$, it follows from the analysis of stationary schemes (see §4.2.4) that $S_{\mathbf{c}}$ is convergent. Thus $S_{\{\mathbf{c}^k\}}$ is convergent, and by Theorem 4.6 and Corollary 4.7, $\phi_{\{\mathbf{a}^k\}} \in C^m$.

Next we consider a similar example but in the multivariate setting, with general smoothing factors.

Example 4.9 Given are the symbols

$$a^k(z) = 2^{s-\ell} \prod_{j=1}^{\ell} \left(1 + r_k^{(j)} z^{\lambda^{(j)}}\right), \quad k \in \mathbb{Z}_+ ,$$

with directions $\Lambda = \{\lambda^{(1)}, \cdots, \lambda^{(\ell)}\} \subset \mathbb{Z}^s$. If $r_k^{(j)}$ for $j = 1, \ldots, \ell$ satisfy (4.8) and if the set Λ contains a subset of s directions with determinant ± 1, and any subset of $\ell - m - 1$ directions span $\mathbb{R}^s$, then $\phi_{\{\mathbf{a}^k\}}$ is in C^m.

To see this, observe that under the conditions of the example, $S_{\{\mathbf{a}^k\}}$ is asymptotically equivalent to $S_{\mathbf{a}}$ with

$$a(z) = 2^s \prod_{\lambda^{(j)} \in \Lambda} (1 + z^{\lambda^{(j)}})/2 .$$

By the conditions on Λ, $S_{\mathbf{a}}$ is convergent and $\phi_{\mathbf{a}}$ is the polynomial box-spline with directions Λ, which is C^m (see the previous Section). Let $\Lambda_0 \subset \Lambda$ be the smallest subset of Λ for which $S_{\mathbf{b}}$ with $b(z) = 2^s \prod_{\lambda^{(j)} \in \Lambda_0} (1 + z^{\lambda^{(j)}})/2$ is C^0. The scheme $S_{\{\mathbf{b}^k\}}$ with $b^k(z) = 2^s \prod_{\lambda^{(j)} \in \Lambda_0} (1 + r_k^{(j)} z^{\lambda^{(j)}})/2$ is asymptotically equivalent to $S_{\mathbf{b}}$. Hence, by Corrolary 4.4, it follows that $\phi_{\{\mathbf{b}^k\}} \in C(\mathbb{R})$. The maximal m for which $S_{\mathbf{a}}$ is C^m is determined by repeated convolutions with respect to appropriate directions in $\Lambda \setminus \Lambda_0$. The same procedure of adding directions, in view of Theorem 4.6, proves that $S_{\{\mathbf{a}^k\}}$ is also C^m.

4.2 Analysis of univariate schemes via difference schemes

The case $s = 1$ is the simpler to analyze, and the theory for the stationary case is almost complete. This theory provides a method of analysis, based on necessary and sufficient conditions for convergence, and in the most interesting cases also for smoothness.

The method presented here is general in the sense that it also applies to non-stationary schemes with symbols that are all divisible by the elementary factor $(1+z)$ and its powers, as in the stationary case. Yet, in the stationary case this divisibility is necessary and sufficient, while in the non-stationary case it is only sufficient.

A necessary condition for convergence (for any $s \in \mathbb{Z}_+\backslash 0$) [4, 32], which is the key to this analysis in the univariate case, is easily derived from the stationary refinement step

$$f_\alpha^{k+1} = \sum_{\beta\in\mathbb{Z}^s} a_{\alpha-2\beta} f_\beta^k, \quad \alpha \in \mathbb{Z}^s .$$

Considering large k such that $|f_\alpha^j - (S_{\mathbf{a}}^\infty f^0)(2^{-j}\alpha)| < \varepsilon$, $j = k, k+1$ for ε small enough, and taking into account that $\sigma(\mathbf{a})$ is finite, so that $2^{-k}\beta$ in the above sum is close to $2^{-k-1}\alpha$, we conclude

Theorem 4.10 *If $S_{\mathbf{a}}$ is (uniformly) convergent then*

$$\sum_{\beta\in\mathbb{Z}^s} a_{\alpha+2\beta} = 1 , \quad \alpha \in E^s , \tag{4.10}$$

where E^s are the extreme points of $[0,1]^s$.

4.2.1 Analysis of univariate stationary schemes

In the univariate case ($s = 1$) conditions (4.10) imply that $a(-1) = 0$, $a(1) = 2$. Thus $a(z)$ is divisible by $(1+z)$, the elementary univariate factor of (3.1). As will become clear in the sequel $(1+z)/2$ is the stationary univariate smoothing factor.

Let the mask $\mathbf{a}$ satisfy (4.10). Then, $a(z) = (1+z)b(z)$, with $S_{\mathbf{b}}$ a scheme related to $S_{\mathbf{a}}$ by

$$S_{\mathbf{b}}\Delta\mathbf{f} = \Delta(S_{\mathbf{a}}\mathbf{f}) , \tag{4.11}$$

where $\Delta\mathbf{f} = \{(\Delta\mathbf{f})_j = f_j - f_{j-1} : j \in \mathbb{Z}\}$. The verification of (4.11) is easily done in terms of the z-transform representation of subdivision schemes (2.25). Since

$$L(\Delta\mathbf{f}; z) = \sum_{j\in\mathbb{Z}} (\Delta\mathbf{f})_j \ z^j = (1-z)L(\mathbf{f}; z)$$

it follows from (2.25) and from the factorization of $a(z)$, that

$$L(\Delta\mathbf{f}^{k+1}; z) = (1-z)a(z)L(\mathbf{f}^k; z^2) = b(z)(1-z^2)L(\mathbf{f}^k; z) = b(z)L(\Delta\mathbf{f}^k; z^2)$$

which proves (4.11).

From now on we consider only masks that satisfy (4.10). It is clear that if $S_{\mathbf{a}}$ is convergent, then $\lim_{k\to\infty}\sup_{j\in\mathbb{Z}}|\Delta f_j^k| = 0$ with $\mathbf{f}^k = S_{\mathbf{a}}^k\mathbf{f}^0$, or $\Delta\mathbf{f}^k = S_{\mathbf{b}}^k\Delta\mathbf{f}^0$. Thus if $S_{\mathbf{a}}$ is convergent, $S_{\mathbf{b}}$ maps any initial data to zero, or shortly, is contractive. The inverse of this relation also holds.

Theorem 4.11 *Let $a(z) = (1+z)b(z)$. $S_{\mathbf{a}}$ is convergent if and only if $S_{\mathbf{b}}$ is contractive.*

Proof: It remains to prove that if $S_{\mathbf{b}}$ is contractive then $S_{\mathbf{a}}$ is convergent. Consider the sequence $\{F_k(t)\}_{k\in\mathbb{Z}_+}$ defined by (2.6). To show convergence of $S_{\mathbf{a}}$ it is sufficient to show that $\{F_k(t)\}_{k\in\mathbb{Z}_+}$ is a Cauchy sequence with respect to the sup-norm. Now by definition, and by the observation that a piecewise linear function attains its extreme values at its breakpoints

$$\sup_{t\in\mathbb{R}}|F_{k+1}(t) - F_k(t)| = \max\left\{|\sup_{i\in\mathbb{Z}}|f_{2i}^{k+1} - g_{2i}^{k+1}|, \sup_{i\in\mathbb{Z}}|f_{2i+1}^{k+1} - g_{2i+1}^{k+1}|\right\}, \tag{4.12}$$

where

$$g_{2i}^{k+1} = f_i^k \quad \text{and} \quad g_{2i+1}^{k+1} = \frac{1}{2}(f_i^k + f_{i+1}^k) . \tag{4.13}$$

It is easy to verify that (4.13) is represented in terms of the z-transform, by

$$L(\mathbf{g}^{k+1}; z) = \frac{(1+z)^2}{2z}L(\mathbf{f}^k; z^2) .$$

Thus

$$\begin{aligned} L(\mathbf{f}^{k+1}; z) - L(\mathbf{g}^{k+1}; z) &= \Big(a(z) - \frac{(1+z)^2}{2z}\Big)L(\mathbf{f}^k; z^2) \\ &= (1+z)\Big(b(z) - \frac{1+z}{2z}\Big)L(\mathbf{f}^k; z^2) = (1+z)d(z)L(\mathbf{f}^k; z^2) \end{aligned}$$

with $d(z) = b(z) - \frac{1+z}{2z}$. Since by (4.10) $a(1) = 2$, $d(1) = b(1) - 1 = 0$ and hence $d(z) = (1-z)e(z)$.

This leads finally to

$$L(\mathbf{f}^{k+1} - \mathbf{g}^{k+1}; z) = e(z)(1-z^2)L(\mathbf{f}^k; z^2) = e(z)L(\Delta\mathbf{f}^k; z^2) . \tag{4.14}$$

Recalling that by (4.12) $\|F_{k+1} - F_k\|_\infty = \sup_{j\in\mathbb{Z}}|f_j^{k+1} - g_j^{k+1}| = \|\mathbf{f}^{k+1} - \mathbf{g}^{k+1}\|_\infty$, and that by (4.14) and (4.11)

$$\mathbf{f}^{k+1} - \mathbf{g}^{k+1} = S_{\mathbf{e}}\Delta\mathbf{f}^k = S_{\mathbf{e}}S_{\mathbf{b}}^k\Delta\mathbf{f}^0 ,$$

we finally get

$$\|F_{k+1} - F_k\|_\infty = \|\mathbf{f}^{k+1} - \mathbf{g}^{k+1}\|_\infty \le \|S_{\mathbf{e}}\|_\infty\|S_{\mathbf{b}}^k\Delta\mathbf{f}^0\|_\infty . \tag{4.15}$$

Now, if $S_{\mathbf{b}}$ is contractive, namely if $S_{\mathbf{b}}^k\mathbf{f}$ tends to zero for all $\mathbf{f}$, then there exists $M \in \mathbb{Z}_+\backslash 0$ such that $\|S_b^M\|_\infty = \mu < 1$. Thus (4.15) leads to

$$\|F_{k+1} - F_k\|_\infty = \|\mathbf{f}^{k+1} - \mathbf{g}^{k+1}\|_\infty \leq \|S_{\mathbf{e}}\|_\infty \mu^{[\frac{k}{M}]} \max_{0\leq j<M} \|\Delta \mathbf{f}^j\| \leq C\eta^k \ , \tag{4.16}$$

where $\eta = (\mu)^{\frac{1}{M}} < 1$ and C is a generic constant. Thus $\{F_k : k \in \mathbb{Z}_+\}$ is uniformly convergent.

With the analysis presented, we can design an algorithm for checking the convergence of $S_{\mathbf{a}}$ given the mask $\mathbf{a}$. Consider the iterated scheme $S_{\mathbf{b}}^\ell$, transforming data at level k to data at level $\ell + k$. Recall that the symbol of $S_{\mathbf{b}}^\ell$ can be computed by (2.28) as $b^{[\ell]}(z) = \prod_{j=1}^{\ell} b(z^{2^{j-1}})$, and thus to check the contractivity of $S_{\mathbf{b}}$ the norms of $S_{\mathbf{b}}^\ell$, $\ell = 1, 2, \dots$, have to be evaluated in terms of $b^{[\ell]}(z) = \sum_{j\in\mathbb{Z}} b_j^{[\ell]} z^k$, according to

$$\|S_b^\ell\|_\infty = \max\left\{ \sum_{j\in\mathbb{Z}} |b_{i-2^\ell j}^{[\ell]}| : 0 \leq i < 2^\ell \right\} . \tag{4.17}$$

The norm in (4.17) reflects the fact that there are 2^ℓ different rules in the iterated scheme $S_{\mathbf{b}}^\ell$:

$$\mathbf{g}^{k+\ell} = S_{\mathbf{b}}^\ell \mathbf{g}^k \Leftrightarrow g_i^{k+\ell} = \sum_{j\in\mathbb{Z}} b_{i-2^\ell j}^{[\ell]} g_j^k, \quad i \in \mathbb{Z} \ .$$

Schemes for which $S_{\mathbf{b}}$ is contractive, but $\|S_{\mathbf{b}}^\ell\|_\infty \geq 1$ for large ℓ ($\ell > 5$), are of no practical value, since a large number of iterations is required to observe convergence (small $\|\Delta \mathbf{f}^k\|_\infty$). Thus the algorithm has an input parameter M_0, such that if $\|S_{\mathbf{b}}^{M_0}\|_\infty \geq 1$, the scheme is declared to be practically "not convergent". A reasonable choice of M_0 is in the range $5 < M_0 \leq 10$.

An algorithm for verifying convergence, given the symbol $a(z)$ of the scheme.

1. If $a(-1) \neq 0$, or $a(1) \neq 2$, the scheme does not converge. Stop!
2. Compute $b^{[1]}(z) = a(z)/(1+z) = \sum_j b_j^{[1]} z^j$.
3. For $\ell = 1, \dots, M_0$.
4. Compute $N_\ell = \max_{0\leq i<2^\ell} \sum_{j\in\mathbb{Z}} |b_{i-2^\ell j}^{[\ell]}|$.
5. If $N_\ell < 1$, the scheme is convergent. Stop!
6. If $N_\ell \geq 1$ compute $b^{[\ell+1]}(z) = b^{[1]}(z) b^{[\ell]}(z^2) = \sum_{j\in\mathbb{Z}} b_j^{[\ell+1]} z^j$.
7. End loop.
8. $S_{\mathbf{b}}$ is not contractive after M_0 iterations. Stop!

The parameters μ, M from the proof of Theorem 4.11 corresponding to a mask $\mathbf{a}$, determine also the Hölder exponent of $\phi_{\mathbf{a}}$ (or any $S_{\mathbf{a}}^\infty f^0$), and the rate of convergence of the subdivision scheme.

Theorem 4.12 *Let* $\mathbf{a}, \mu, M, \eta$, *be as in the proof of Theorem 4.11, and define* $\nu = -(\log_2 \mu)/M$. *Then*

$$|\phi_{\mathbf{a}}(y) - \phi_{\mathbf{a}}(x)| \leq C|x-y|^{\nu} \ .$$

Moreover, the rate of convergence of the sequence $\{F_k(t)\}_{k \in \mathbb{Z}_+}$ *defined in* (2.6) *is*

$$\|F_k(t) - S_{\mathbf{a}}^{\infty}\mathbf{f}^0\|_{\infty} \leq C\eta^k \ .$$

Here C *is a generic constant.*

Proof: Both claims of the theorem follow from (4.16). The second directly with the aid of the observation

$$|(S_{\mathbf{a}}^{\infty}\mathbf{f}^0 - F_k)(x)| = \lim_{\ell \to \infty} |F_\ell(x) - F_k(x)| \leq \sum_{j=k}^{\infty} |F_{j+1}(x) - F_j(x)| \ .$$

To verify the first claim, we use the second claim in the bound

$$|\phi_{\mathbf{a}}(x) - \phi_{\mathbf{a}}(y)| \leq |\phi_{\mathbf{a}}(x) - F_k(x)| + |\phi_a(y) - F_k(y)| + |F_k(x) - F_k(y)| \ ,$$

and the obvious bound

$$|F_k(x) - F_k(y)| \leq 2\|\Delta\mathbf{f}^k\|_{\infty} \ ,$$

both holding for any k. The first claim now follows by estimating $\Delta f^k = S_{\mathbf{b}}^k \Delta\boldsymbol{\delta}$ in terms of $\|S_{\mathbf{b}}^M\|_{\infty} < \mu$, and by the observation that for $2^{-k} \leq |x-y| \leq 2^{-k+1}$,

$$\mu^{[\frac{k}{M}]} \leq C\mu^{\frac{k}{M}} = C2^{-k\nu} \leq C|x-y|^{\nu} \ .$$

The tools for the analysis of smoothness are similar to the tools for convergence analysis. The analysis of smoothness is based on the observation that in the stationary case $(1+z)/2$ is a smoothing factor.

Theorem 4.13 *Let* $a(z) = \frac{1+z}{2}q(z)$. *If* $S_{\mathbf{q}}$ *is convergent and* C^{ℓ}, *then* $S_{\mathbf{a}}$ *is convergent and* $C^{\ell+1}$.

Sketch of the proof: By the convolution property 2 and by (3.2), $S_{\mathbf{a}}$ is convergent, and

$$\phi_{\mathbf{a}}(x) = \int_{x-1}^{x} \phi_{\mathbf{q}}(t)dt \ . \tag{4.18}$$

Thus

$$\phi_{\mathbf{a}}'(x) = \phi_{\mathbf{q}}(x) - \phi_{\mathbf{q}}(x-1) \ . \tag{4.19}$$

Theorem 4.13 supplies a sufficient condition for smoothness. A repeated use of Theorem 4.13 together with Theorem 4.11, leads to

Corollary 4.14 *Let* $a(z) = \frac{(1+z)^{m+1}}{2^m} b(z)$ *with* $S_{\mathbf{b}}$ *contractive. Then* $\phi_{\mathbf{a}} \in C^m(\mathbb{R})$*. Moreover*

$$\phi_{\mathbf{a}}^{(\ell)} = S^{\infty}_{a(z)(1+z)^{-\ell}2^{\ell}} \Delta^{\ell} \boldsymbol{\delta} \ , \quad \ell = 0, 1, \dots, m$$

where $\Delta^{\ell} = \Delta\Delta^{\ell-1}$ *is defined recursively.*

$S_{a(z)2^{\ell-1}(1+z)^{-\ell}}$ is contractive, it is suggested in [92] for a scheme with symbol $a(z) = 2^{-m}(1+z)^{m+1}b(z)$, to compute the numbers $\|S_{\mathbf{b}}^{\ell}\|_{\infty} = \mu_{\ell}$ and $\nu_{\ell} = -(\log_2 \mu_{\ell})/\ell$. If $m - \nu_{\ell} > 0$, then $\phi_{\mathbf{a}} \in C^{[m-\nu_{\ell}]}$. Defining $\nu = \sup_{\ell \geq 1} \nu_{\ell}$, if $m - \nu > 0$, then $\phi_{\mathbf{a}} \in C^{[m-\nu]}$, and $\phi_{\mathbf{a}}^{([m-\nu])}$ has Hölder exponent $n - \epsilon$ for any $\epsilon > 0$ with $n = m - \nu - [m - \nu]$.

Example 4.15 *Consider the stationary interpolatory 4-point scheme with symbol* (3.19)

$$a_w(z) = \frac{1}{2z}(1+z)^2[1 - 2wz^{-2}(1-z)^2(z^2+1)] \ .$$

By Theorem 4.11, the range of w *for which* $S_{\mathbf{a_w}}$ *is convergent is the range for which* $S_{\mathbf{b_w}}$ *with symbol*

$$b_w(z) = \frac{1}{2z}(1+z)[1 - 2wz^{-2}(1-z)^2(z^2+1)] \ ,$$

is contractive. The condition $\|S_{\mathbf{b_w}}\|_{\infty} < 1$ *yields the range* $-\frac{3}{8} < w < \frac{-1+\sqrt{13}}{8}$*, while the condition* $\|S^2_{\mathbf{b_w}}\|_{\infty} < 1$ *yields the range* $-\frac{1}{4} < w < \frac{-1+\sqrt{17}}{8}$*. Thus a range of* w *for which* $S_{\mathbf{a_w}}$ *is convergent is* $-\frac{3}{8} < w < \frac{-1+\sqrt{17}}{8} \cong 0.39$ *[35].*

By Corollary 4.14, it is sufficient to show that $S_{\mathbf{c_w}}$ *with symbol*

$$c_w(z) = \frac{1}{z}[1 - 2wz^{-2}(z-1)^2(z^2+1)] \ ,$$

is contractive, in order to prove that $S_{\mathbf{a_w}}$ *is* C^1*. Now,* $\|R_{\mathbf{c_w}}\|_{\infty} \geq 1$*, while* $\|R^2_{\mathbf{c_w}}\|_{\infty} < 1$ *for* $0 < w < \frac{\sqrt{5}-1}{8}$*, as is shown in [35].*

The fact that $\phi_{\mathbf{a_w}} \notin C^2(\mathbb{R})$*, can be deduced from necessary conditions that are violated (see subsection 5.2). In [16] it is shown, by methods as in subsection 5.2, that* $\phi'_{\mathbf{a_w}}$ *is differentiable except at all the diadic points in its support.*

After deriving similar results to the above for a class of non-stationary schemes, we return to the stationary case, and show that if $\phi_{\mathbf{a}} \in C^m(\mathbb{R})$ then necessarily the symbol $a(z)$ is divisible by $(1+z)^{m+1}$ in most interesting cases. In this sense the form of $a(z)$ in Corollary 4.14 is necessary for $S_{\mathbf{a}}$ with C^m limit functions. This result holds if $\phi_{\mathbf{a}}$ is L_{∞}-stable, namely if for any bounded bi-infinite sequence $\mathbf{f} = \{f_i : i \in \mathbb{Z}\}$

$$C_2 \sup_{i \in \mathbb{Z}} |f_i| \leq \Big\| \sum_{i \in \mathbb{Z}} f_i \phi(x-i) \Big\|_{\infty} \leq C_1 \sup_{i \in \mathbb{Z}} |f_i| \tag{4.20}$$

with $0 < C_2 \leq C_1 < \infty$. For most interesting schemes the basic limit function is L_{∞}-stable, e.g. for interpolatory schemes and for spline schemes. We also study the related property, that for $S_{\mathbf{a}}$ with $\phi_{\mathbf{a}} \in C^m$ and L_{∞}-stable, the space of limit functions of $S_{\mathbf{a}}$, contains π_m – the space of polynomials of degree $\leq m$.

4.2.2 Analysis of non-stationary schemes with symbols devisible by stationary smoothing factors

In this section the tools of analysis of §4.2.1 are extended to a class of non-stationary schemes. Theorem 4.13 holds also in case of a non-stationary scheme with symbols

$$a^k(z) = \frac{(1+z)}{2} q^k(z), \quad k \in \mathbb{Z}_+, \quad k \geq K$$

with K some positive integer, and such that $S_{\{\mathbf{q}^k\}}$ is convergent. A version of Theorem 4.11 holds also in the non-statinary case. It supplies only a sufficient condition for convergence.

Theorem 4.16 *Let a non-stationary scheme be given by the symbols*

$$a^k(z) = (1+z)b^k(z), \quad k \in \mathbb{Z}_+, \quad k \geq K \in \mathbb{Z}_+ \ .$$

If $S_{\{\mathbf{b}^k\}}$ is contractive then $S_{\{\mathbf{a}^k\}}$ is convergent.

This theorem holds since $R_{\mathbf{a}^k}$ and $R_{\mathbf{b}^k}$ defined by (2.2) and (2.3) are related by

$$\Delta R_{\mathbf{a}^k}\mathbf{f} = R_{\mathbf{b}^k}\Delta\mathbf{f}, \quad k \in \mathbb{Z}_+, \quad k \geq K \ , \tag{4.21}$$

and therefore by the same arguments, as in the stationary case, the contractivity of $S_{\{\mathbf{b}^k\}}$ implies the convergence of $S_{\{\mathbf{a}^k\}}$. A simple sufficient condition for the contractivity of $S_{\{\mathbf{b}^k\}}$ is

$$\|R_{\mathbf{b}^k}\|_\infty = \max\Big(\sum_{j\in\mathbb{Z}} |b^k_{i-2j}| \ \ : \ \ i \in \{0,1\}\Big) \leq \mu < 1, \quad k \in \mathbb{Z}_+, \quad k \geq K \ , \tag{4.22}$$

since then for $\mathbf{g}^k = R_{\mathbf{b}^{k-1}}R_{\mathbf{b}^{k-2}}\cdots R_{\mathbf{b}^0}\mathbf{g}^0$ we have $\|\mathbf{g}^k\|_\infty \leq \mu^k\|\mathbf{g}^0\|_\infty$.

From Theorem4.16 and the remark above it, we conclude

Corollary 4.17 *Let a non-stationary scheme be given by the symbols*

$$a^k(z) = \frac{(1+z)^{m+1}}{2^m} b^k(z), \quad k \in \mathbb{Z}_+, \quad k \geq K \in \mathbb{Z}_+ \ .$$

If $S_{\{\mathbf{b}^k\}}$ is contractive then $S_{\{\mathbf{a}^k\}}$ is C^m.

Example 4.18 *In this example we study properties of the up-function introduced in §3.1, by applying the analysis tools of this section.*

Let a non-stationary scheme be given by the symbols as in (3.5)

$$a^k(z) = \frac{(1+z)^k}{2^{k-1}}, \quad k \in \mathbb{Z}_+ \ .$$

To show that $\phi_{\{\mathbf{a}^k\}} \in C^\infty(\mathbb{R})$, we show that for any $m \in \mathbb{Z}_+$, $\phi_{\{\mathbf{a}^k\}} \in C^m(\mathbb{R})$.

Now for $k \geq m+2$

$$a^k(z) = \frac{(1+z)^{m+1}}{2^m} \quad \cdot \frac{(1+z)^{k-m-1}}{2^{k-m-1}} \ ,$$

and by Corollary 4.17, $\phi_{\{\mathbf{a}^k\}} \in C^m$ *if* $S_{\{\mathbf{b}^k\}}$ *is contractive, with*

$$b^k(z) = \frac{(1+z)^{k-m-1}}{2^{k-m-1}}, \quad k \in \mathbb{Z}_+, \quad k \geq m+2 \ .$$

But $\|R_{\mathbf{b}^k}\|_\infty = \frac{1}{2}$ *for* $k \in \mathbb{Z}_+$, $k \geq m+2$, *which proves that* $S_{\{\mathbf{b}^k\}}$ *is contractive.*

Next we show that $\sigma(\phi_{\{\mathbf{a}^k\}}) = [0,2]$. *Using* (2.11) *we get from* (3.5)

$$\sigma(\phi_{\{\mathbf{a}^k\}}) = \sum_{j=0}^{\infty} 2^{-j-1}\sigma(\mathbf{a}^j) = \sum_{j=0}^{\infty} 2^{-j-1}[0, j+1] = [0,2] \ .$$

4.2.3 Polynomials generated by univariate stationary schemes

For stationary interpolatory schemes in $\mathbb{R}^s$ it is easy to show [40] that $\phi_{\mathbf{a}} \in C^m$ implies that π_m is reproduced by the scheme, namely

$$\left.\begin{array}{l} R_{\mathbf{a}} p|_{\mathbb{Z}^s} = p(\frac{\cdot}{2})|_{\mathbb{Z}^s}, \\ S_{\mathbf{a}}^\infty p|_{\mathbb{Z}^s} = p \ . \end{array}\right. \qquad p \in \pi_m(\mathbb{R}^s) \tag{4.23}$$

For a subdivision scheme with a stable basic limit function, the proof is more involved. It was first proved in [4]. Here we present a proof for $s = 1$, which is extendable to univariate matrix subdivison schemes [43] and to multivariate schemes.

The proof is based on the important observation in [98]:

Theorem 4.19 *Let* $S_{\mathbf{a}}$ *be a* C^m *convergent univariate, stationary subdivision scheme. Let* $\mathbb{B}$ *denote the set of bi-infinite sequences, and let* $\mathbf{v} = \{v_j : j \in \mathbb{Z}\} \in \mathbb{B}$ *be an eigenvector of* $R_{\mathbf{a}}$ *with eigenvalue* λ

$$R_{\mathbf{a}}\mathbf{v} = \lambda\mathbf{v} \ . \tag{4.24}$$

Then

1. *If* $|\lambda| \geq 2^{-m}$ *either* $S_{\mathbf{a}}^\infty \mathbf{v} \equiv 0$ *or* $S_{\mathbf{a}}^\infty \mathbf{v} = x^i$ *for some* $0 \leq i \leq m$, *and* $\lambda = 2^{-i}$. *Also* $\lambda = 2^{-i}$, $0 \leq i \leq m$, *cannot have a generalized eigenvector* $\mathbf{u} \in \mathbb{B}$, *satisfying*

$$R_{\mathbf{a}}\mathbf{u} = \lambda\mathbf{u} + \mathbf{v} \ . \tag{4.25}$$

2. *If* $|\lambda| < 2^{-m}$ *then* $(S_{\mathbf{a}}^\infty \mathbf{v})^{(\ell)}(0) = 0$, $\ell = 0, \dots, m$.

3. *If* $\lambda \neq 2^{-i}$, $0 \leq i \leq m$, *and* $\mathbf{u}$ *is a corresponding generalized eigenvector satisfying* (4.25), *then* $(S_{\mathbf{a}}^\infty \mathbf{u})^{(\ell)}(0) = 0$, $\ell = 0, \dots, m$.

The proof of Theorem 4.19 is based on the relations

$$(S_{\mathbf{a}}^\infty \mathbf{v})(x) = \lambda(S_{\mathbf{a}}^\infty \mathbf{v})(2x), \quad (S_{\mathbf{a}}^\infty \mathbf{u})(x) = \lambda(S_{\mathbf{a}}^\infty \mathbf{u})(2x) + (S_{\mathbf{a}}^\infty \mathbf{v})(2x)$$

for $\mathbf{v}, \mathbf{u}$ satisfying (4.24) and (4.25) respectively, and on the continuity at $x = 0$ of the derivatives of order up to m of $S_{\mathbf{a}}^\infty \mathbf{u}$, $S_{\mathbf{a}}^\infty \mathbf{v}$.

A direct consequence of Theorem 4.19 deals with polynomials generated by a univariate stationary subdivision scheme with smooth limit functions [37].

Theorem 4.20 *Let $S_{\mathbf{a}}$ be a C^m subdivision scheme. Then there exist $\mathbf{v}^{[i]} \in \mathbb{B}$, $i = 0, \dots, m$, such that*

$$R_{\mathbf{a}}\mathbf{v}^{[i]} = 2^i \mathbf{v}^{[i]}, \quad S_{\mathbf{a}}^{\infty}\mathbf{v}^{[i]} = x^i, \quad i = 0, \dots, m \; . \tag{4.26}$$

The argument leading to (4.26) is that 2^{-i} must be an eigenvalue of $R_{\mathbf{a}}$ for $i = 0, \dots, m$ otherwise there exists $\ell \in \{0, 1, \dots, m\}$ such that $2^{-\ell}$ is not an eigenvalue of $R_{\mathbf{a}}$, implying that $\phi_{\mathbf{a}}^{(\ell)} \equiv 0$, in view of Theorem 4.19. But $\phi_{\mathbf{a}}$ is of compact support, $\phi_a \not\equiv 0$, which contradicts $\phi_{\mathbf{a}}^{(\ell)} \equiv 0$. Next we show that $\mathbf{v}^{[i]}$ in Theorem 4.20 is of the form $\mathbf{v}^{[i]} = x^i|_{\mathbb{Z}} + p_i|_{\mathbb{Z}}$ with $p_i \in \pi_{i-1}$, $i = 0, \dots, m$ (here $p_0 \equiv 0$). For this proof the L_∞-stability of $\phi_{\mathbf{a}}$ is needed. We term a scheme L_∞-stable if its basic limit function is L_∞-stable.

Theorem 4.21 *Let $S_{\mathbf{a}}$ be C^m and L_∞-stable. Then there exist polynomials $p_i \in \pi_{i-1}$, $i = 0, \dots, m$, with $p_0 \equiv 0$, such that*

$$S_{\mathbf{a}}^{\infty}(x^i + p_i)|_{\mathbb{Z}} = x^i, \quad i = 0, \dots, m \; . \tag{4.27}$$

Sketch of the proof: The case $i = 0$ follows directly from (4.10), because $R_{\mathbf{a}}$ maps the constant sequence $\mathbf{1} = \mathbf{u} = \{u_j = 1 : j \in \mathbb{Z}\}$ on itself.

In the following we indicate the proof for $i = 1$. For $i = 2, \dots, m$, the proof is similar. Let $\mathbf{v} = \mathbf{v}^{[1]}$ satisfy $S_{\mathbf{a}}^{\infty}\mathbf{v} = x$, and for $r \in \mathbb{Z}_+ \backslash 0$ let $\Delta^{(r)}\mathbf{v} = \{v_{j+r} - v_j : j \in \mathbb{Z}\}$. Then the linearity and uniformity of $S_{\mathbf{a}}$ leads to $S_{\mathbf{a}}^{\infty}\Delta^{(1)}\mathbf{v} = x + 1 - x = 1$ or

$$S_{\mathbf{a}}^{\infty}(\Delta^{(1)}\mathbf{v} - \mathbf{1}) \equiv 0 \; . \tag{4.28}$$

If $\Delta^{(1)}\mathbf{v} - \mathbf{1} \in \mathbb{B}$ is bounded, then by the L_∞-stability of $\phi_{\mathbf{a}}$, $\Delta^{(1)}\mathbf{v} = \mathbf{1}$, which is equivalent to $\mathbf{v} = x|_{\mathbb{Z}} + c\mathbf{1}$ for some $c \in \mathbb{R}$. Thus the claim of the theorem for $i = 1$ follows. To show the boundedness of $\Delta^{(1)}\mathbf{v} - \mathbf{1}$ we consider (4.28) at the integers, which in view of (2.8) has the form

$$\sum_{j \in \mathbb{Z}} \big((\Delta^{(1)}\mathbf{v})_j - 1\big)\phi_{\mathbf{a}}(n - j) = 0, \quad n \in \mathbb{Z} \; . \tag{4.29}$$

Equation (4.29) can be regarded as a finite difference equation for $\Delta_1\mathbf{v} - \mathbf{1}$, since $\phi_{\mathbf{a}}|_{\mathbb{Z}}$ is finitely supported, and is not identically equal to zero (otherwise $\phi_{\mathbf{a}} \equiv 0$ by (2.15)). As a solution of (4.29) $\Delta^{(1)}\mathbf{v} - \mathbf{1}$ either vanishes or grows at least polynomially but not linearly, as $j \to \infty$ or $j \to -\infty$. But the later possibility is eliminated, since

$$(R_{\mathbf{a}}\Delta^{(r)}\mathbf{v})_\alpha = \sum_{j \in \mathbb{Z}} a_{\alpha - 2j}(v_{j+r} - v_r) = \frac{1}{2}v_{\alpha+2r} - \frac{1}{2}v_\alpha = \frac{1}{2}(\Delta^{(2r)}\mathbf{v})_\alpha \; ,$$

from which it is concluded, in view of (4.28), that $S_{\mathbf{a}}^{\infty}\Delta^{(1)}\mathbf{v} = \lim_{\ell\to\infty} 2^{-\ell}\Delta^{(2^\ell)}\mathbf{v} = 1$, or that $v_{\pm 2^\ell} = v_0 \pm 2^\ell + o(1)$, which is in contradiction to faster than linear growth.

As a direct consequence of Theorem 4.21 we get,

Corollary 4.22 *Let $S_{\mathbf{a}}$ be C^m and L_∞-stable. Then $\pi_m|_{\mathbb{Z}}$ is invariant under $R_{\mathbf{a}}$, and for $p \in \pi_i$, $0 \le i \le m$*

$$R_{\mathbf{a}} p|_{\mathbb{Z}} = q\left(\frac{\cdot}{2}\right)|_{\mathbb{Z}} \ ,$$

with $q \in \pi_i$ and $p - q \in \pi_{i-1}$, while $p = q$ for $i = 0$.

In the following subsection we derive the factorization of the symbol of a scheme satisfying the requirements of Corollary 4.22.

4.2.4 Factorization of symbols of stationary, smooth, L_∞-stable schemes, and related necessary conditions

First we show that if $S_{\mathbf{a}}$ is C^m and L_∞-stable, then its symbol has the factor $(1+z)^{m+1}$. Latter we show that necessarily $S_{2^m a(z)(1+z)^{-m-1}}$ is contractive. A similar result holds for L_p-stability and convergence in the L_p-norm, $1 \le p \le \infty$ [57]. Thess results are important in the analysis of smoothness of univariate stationary schemes (see section 4.2.1).

Theorem 4.23 *Let $S_{\mathbf{a}}$ be C^m and L_∞-stable. Then*

$$a(z) = (1+z)^{m+1} b(z) \tag{4.30}$$

with $b(z)$ a Laurent polynomial.

Proof: The proof is by a recursive construction of "divided difference" schemes with symbols

$$a^{[i]}(z) = 2^i (z+1)^{-i} a(z), \quad i = 0, 1, \ldots, m+1 \ .$$

If $a^{[i]}(z)$ is a Laurent polynomial, then in view of (4.11), $S_{\mathbf{a}^{[i]}}$ is related to $S_{\mathbf{a}}$ by

$$S_{\mathbf{a}^{[i]}} d_k^i \mathbf{f} = d_{k+1}^i S_{\mathbf{a}} \mathbf{f}, \quad \mathbf{f} \in \mathbb{B} \ ,$$

where $d_k^i \mathbf{f} = (2^k)^i \Delta^i f$, is the sequence of divided differences of order i on refinement level k. Since by Corollary 4.22 $R_{\mathbf{a}}$ maps $\mathbf{1} \in \mathbb{B}$ to itself, $\sum_{i\in\mathbb{Z}} a_{2i} = \sum_{i\in\mathbb{Z}} a_{2i+1} = 1$ and $a(z)$ is divisible by $(1+z)$. This guarantees that $a^{[1]}$ exists. Now, $R_{\mathbf{a}}$ maps $\mathbf{v} = x|_{\mathbb{Z}}$, to $R_{\mathbf{a}}\mathbf{v} = \frac{1}{2}x|_{\mathbb{Z}} + c\mathbf{1}$ for some $c \in \mathbb{R}$, so $R_{\mathbf{a}^{[1]}}$ maps $\mathbf{1} \in \mathbb{B}$ into itself. Thus $a^{[1]}(z)$ is divisible by $(1+z)$, and $a^{[2]}$ exists. The general argument is similar.

By applying $(2^k)^i \Delta^i \mathbf{f}$ to $\mathbf{f} = x^i|_{\mathbb{Z}}$ we get a constant sequence. This sequence is mapped by $R_{a^{[i]}}$ to $(2^{k+1})^i \Delta^i R_{\mathbf{a}} \mathbf{f}$ which is the same constant sequence. This is the case, since $R_{\mathbf{a}}\mathbf{f} = (\frac{\cdot}{2})^i + q(\frac{\cdot}{2})$ with $q \in \pi_{i-1}$ by Corollary 4.22, and $\Delta^i q(\frac{\cdot}{2}) = 0$. Again, if $R_{a^{[i]}}$ maps the constant sequence on itself then $a^{[i]}(z)$ is divisible by $(1+z)$.

Using this argument for $i = 0, 1, \ldots, m$ we conclude that $a^{[i]}$ exists for $i = 1, \ldots, m+1$, and thus (4.30) holds.

Example 4.24 Consider the symbol

$$a(z) = \frac{1}{4}(1+z)(1+z^2)^2 = \frac{1}{4}(1 + z + 2z^2 + 2z^3 + z^4 + z^5) \ . \tag{4.31}$$

It is easy to see that (4.10) holds, since $a(1) = 2$, $a(-1) = 0$. To verify that $S_{\mathbf{a}}$ is convergent, we show that $S_{\mathbf{b}}$ with $b(z) = \frac{1}{4}(1+z^2)^2$ is contractive. Now, $b(z) = \frac{1}{4}(1+2z^2+z^4)$ and therefore $\|S_{\mathbf{b}}\|_\infty = 1$. Yet from (2.28) and (4.17), we get

$$b^{[2]}(z) = \frac{1}{16}(1+z^4)^2(1+z^2)^2 = \frac{1}{16}(1+2z^2+3z^4+4z^6+3z^8+2z^{10}+z^{12})$$

and therefore $\|S_{\mathbf{b}}^2\|_\infty = \frac{1}{2}$.

Since the symbol $c(z) = 1+z^2$ satisfies $c(1) = 2$, $S_{\mathbf{c}}$ converges weakly [22]. It is easy to verify that $S_{\mathbf{c}}^\infty \boldsymbol{\delta} = \frac{1}{2}\chi_{[0,1]}$ in the sense of weak convergence. By the convolution property 3

$$\phi_{\mathbf{a}} = \tfrac{1}{4}\chi_{[0,1]} * \chi_{[0,1]} * \chi_{[0,1]} \ .$$

Thus $\phi_{\mathbf{a}} \in C^1$, while $a(z)$ is not divisible by $(1+z)^2$. This indicates, in view of Theorem 4.23, that $\phi_{\mathbf{a}}$ is not L_∞-stable. Indeed, consider the sequence $\mathbf{u} = \{u_i = (-1)^i : i \in \mathbb{Z}\}$. Clearly $\mathbf{u}$ is bounded. Now in view of (4.31) $|R_{\mathbf{a}}\mathbf{u} = \mathbf{0} \in \mathbb{B}$, and therefore $S_{\mathbf{a}}^\infty \mathbf{u} = \sum_{i\in\mathbb{Z}}(-1)^i \phi_{\mathbf{a}}(\cdot - i) \equiv 0$, and $\phi_{\mathbf{a}}$ is not L_∞-stable.

Once we have the factorization of the symbol of a stationary C^m, L_∞-stable scheme,

$$a(z) = (1+z)^{m+1}b(z)$$

we can show that $\frac{2^m a(z)}{(1+z)^{m+1}}$ is the symbol of a contractive scheme. For that we need two results, which are of importance beyond their current use.

Theorem 4.25 *Let ϕ be a solution of the functional equation*

$$\phi(x) = \sum_{\alpha\in\mathbb{Z}} a_\alpha \phi(2x-\alpha) \tag{4.32}$$

with a mask $\mathbf{a}$ *satisfying* (4.10). *If ϕ is compactly supported, continuous and L_∞-stable, then $S_{\mathbf{a}}$ is convergent.*

This theorem was first proved in [4]. Here we give a sketch of a different proof [43].

Sketch of the proof: Recalling the relation in (2.16), we observe that since $\phi = T_{\mathbf{a}}\phi$, and $\mathbf{a} = R_{\mathbf{a}}\boldsymbol{\delta}$,

$$\phi(x) = \sum_{\alpha\in\mathbb{Z}^s}(R_{\mathbf{a}}\delta)\phi(2x-\alpha) = \sum_{\alpha\in\mathbb{Z}^s}(R_{\mathbf{a}}^k\delta)\phi(2^kx-\alpha) \ , \tag{4.33}$$

and that for all $k \in \mathbb{Z}_+$

$$\sum_{\alpha\in\mathbb{Z}}\phi(x-\alpha) = \sum_{\alpha\in\mathbb{Z}}(R_{\mathbf{a}}^k\mathbf{1})_\alpha\phi(2^kx-\alpha) = \sum_{\alpha\in\mathbb{Z}}\phi(2^kx-\alpha) \ . \tag{4.34}$$

The continuity and L_∞-stability of ϕ together with (4.34) leads, after proper normalization, to

$$\sum_{\alpha\in\mathbb{Z}}\phi(\cdot-\alpha) \equiv 1 \ . \tag{4.35}$$

Combining (4.33) and (4.35) we get

$$0 = \sum_{\alpha\in\mathbb{Z}} \phi(2^k x - \alpha)\big[(R_{\mathbf{a}}^k \delta)_\alpha - \phi(x)\big]$$

which together with the continuity, compact support and L_∞-stability of ϕ yields

$$\lim_{k\to\infty} \sup_{\alpha\in\mathbb{Z}\cap K} |(R_{\mathbf{a}}^k \boldsymbol{\delta})_\alpha - \phi(2^{-k}\alpha)| = 0 \ ,$$

for any compact set $K \subset \mathbb{R}$. This is the convergence of $S_{\mathbf{a}}$ in the sense of (2.5), to a continuous limit function ϕ, hence uniform convergence.

The second theorem is taken from [43], where it is proved for matrix masks.

Theorem 4.26 *Let $a(z) = \frac{1+z}{2} q(z)$, with $S_{\mathbf{a}}$, L_∞-stable and C^1. Then*

$$\varphi = \sum_{\alpha\in\mathbb{Z}} \phi'_{\mathbf{a}}(\cdot - \alpha)$$

is a continuous, L_∞-stable solution of

$$\varphi(x) = T_{\mathbf{q}}\varphi(x) = \sum_{\alpha\in\mathbb{Z}} q_\alpha \varphi(2x - \alpha) \ . \tag{4.36}$$

Sketch of the proof: The function φ is well defined, continuous and of compact support. It is related to $\phi_{\mathbf{a}}$ by

$$\phi_{\mathbf{a}}(x) = \int_{x-1}^{x} \varphi(t)dt = \varphi * \chi_{[0,1]} \ . \tag{4.37}$$

Suppose φ is not L_∞-stable, then there exists a bounded non-zero sequence $\mathbf{u} \in \mathbb{B}$ such that $\sum_{\alpha\in\mathbb{Z}} u_\alpha \varphi(\cdot - \alpha) \equiv 0$. By integrating this relation from $x-1$ to x we obtain $\sum_{\alpha\in\mathbb{Z}} \mathbf{u}_\alpha \phi_{\mathbf{a}}(x-\alpha) \equiv 0$. This last relation contradicts the L_∞-stability of $\phi_{\mathbf{a}}$. Thus φ is also L_∞-stable. To verify that $\varphi = T_{\mathbf{q}}\varphi$, we observe that $\phi_{\mathbf{a}} = T_{\mathbf{a}}\phi_{\mathbf{a}}$, after taking Fourier transform, is equivalent to

$$\hat{\phi}_{\mathbf{a}}(w) = \frac{1}{2}\hat{a}\Big(\frac{w}{2}\Big)\hat{\phi}_{\mathbf{a}}\Big(\frac{w}{2}\Big) \tag{4.38}$$

with $\hat{a}(w) = \sum_{\alpha\in\mathbb{Z}} a_\alpha e^{-iw\alpha}$. Now by (4.37) $\hat{\varphi}(w)\frac{1-e^{-iw}}{w} = \hat{\phi}_{\mathbf{a}}(w)$. Multiplying (4.38) by $\frac{w}{1-e^{-iw}}$ we obtain

$$\hat{\varphi}(w) = \frac{1}{2}\ \frac{2\hat{a}(\frac{w}{2})}{1+e^{-i\frac{w}{2}}}\hat{\varphi}\Big(\frac{w}{2}\Big) = \frac{1}{2}\hat{q}\Big(\frac{w}{2}\Big)\hat{\varphi}\Big(\frac{w}{2}\Big)$$

proving (4.36).

From Theorems 4.25, 4.26, 4.23 and 4.11 we conclude

Corollary 4.27 *Let $S_{\mathbf{a}}$ be C^1 and L_∞-stable. Then $q(z) = \frac{2a(z)}{(1+z)^2}$ is a Laurent polynomial and $S_{\mathbf{q}}$ is contractive.*

This Corollary together with Corollary 4.14 implies

Corollary 4.28 *Let $S_{\mathbf{a}}$ be convergent and L_∞-stable. Then the contractivity of $S_{2^m a(z)(1+z)^{-(m+1)}}$ is necessary and sufficient for $S_{\mathbf{a}}$ to be C^m.*

4.3 Analysis of bivariate stationary schemes via difference schemes

The analysis of convergence and smoothness of subdivision schemes defined on regular grids, which is of interest to geometric modeling in $\mathbb{R}^3$, is in the case $s = 2$. Thus for the sake of simplicity of presentation, we limit the discussion to this case. The results are easily extended to $s > 2$. Here we present similar analysis tools to those in the univariate, stationary case for bivariate, stationary subdivision schemes defined on regular quad-meshes and on regular triangulations. When the symbol is factorizable to enough linear factors, (each a univariate smoothing factor in some direction in $\mathbb{Z}^2$), the analysis is almost as simple as in the univariate case [4, 32]. This factorization is not the result of (4.10) or of the smoothness of the limit functions, as in the univariate case, but is an additional assumption, which holds for many of the schemes in use. In fact the same factorization of non-stationary symbols, leads to similar results, also for non-stationary schemes. When the symbol is not factorizable to univariate smoothing factors, (4.10) leads to non-unique matrix difference schemes, and the theory of the univariate case can be extended to this case [4, 32, 55] (see §4.3.2).

4.3.1 Analysis of schemes with factorizable symbols

The necessary condition for the convergence of a bivariate scheme $S_{\mathbf{a}}$ defined on $\mathbb{Z}^2$, which is obtained from (4.10), is

$$\sum_{\beta\in\mathbb{Z}^2} a_{\alpha-2\beta} = 1, \quad \alpha \in \big\{(0,0),(0,1),(1,0),(1,1)\big\} . \tag{4.39}$$

These conditions imply

$$a(1,1) = 4, \quad a(-1,1) = 0, \quad a(1,-1) = 0, \quad a(-1,-1) = 0 . \tag{4.40}$$

In contrast to the univariate case ($s = 1$), in the bivariate case ($s = 2$), the necessary condition (4.39) and the derived conditions on $a(z)$, (4.40), do not imply a factorization of the mask to linear factors.

If the factorization

$$a(z) = (1+z_1)^m(1+z_2)^m b(z), \qquad z = (z_1, z_2) , \tag{4.41}$$

is imposed, then with $m = 1$ the convergence can be analyzed almost as in the univariate case, and similarly the smoothness if $m > 1$.

Theorem 4.29 *Let $S_{\mathbf{a}}$ have a symbol of the form (4.41) with $m = 1$. If the schemes with the symbols $a_1(z) = \frac{a(z)}{1+z_1} = (1+z_2)b(z)$, $a_2(z) = \frac{a(z)}{1+z_2} = (1+z_1)b(z)$ are both contractive, then $S_{\mathbf{a}}$ is convergent. Conversely, if S_a is convergent then S_{a_1} and S_{a_2} are contractive.*

The proof of this theorem is similar to the proof of Theorem 4.11, due to the observation that for $\Delta_1\mathbf{f} = \{f_{i,j} - f_{i-1,j} : i,j \in \mathbb{Z}\}$, and $\Delta_2\mathbf{f} = \{f_{i,j} - f_{i,j-1} : i,j \in \mathbb{Z}\}$,

$$S_{\mathbf{a}_\ell}\Delta_\ell\mathbf{f} = \Delta_\ell S_{\mathbf{a}}\mathbf{f} , \qquad \ell = 1,2 .$$

Thus convergence is checked in this case as contractivity of two subdivision schemes $S_{\mathbf{a_1}}, S_{\mathbf{a_2}}$. For schemes having the symmetry of the square grid (topologically equivalent rules for the computation of vertices corresponding to edges), then $a_1(z_1, z_2) = a_2(z_2, z_1)$, and the contractivity of only one scheme has to be checked. Note that the factorization in (4.41) has then the symmetry of $\mathbb{Z}^2$.

For the smoothness result, we introduce the inductive definition of differences: $\Delta^{[i,j]} = \Delta_1\Delta^{[i-1,j]}$, $\Delta^{[i,j]} = \Delta_2\Delta^{[i,j-1]}$, $\Delta^{[1,0]} = \Delta_1$, $\Delta^{[0,1]} = \Delta_2$.

Theorem 4.30 *Let $a(z)$ be factorizable as in (4.41). If the schemes with the masks*

$$a_{i,j}(z) = \frac{2^{i+j}a(z)}{(1+z_1)^i(1+z_2)^j}, \quad i,j = 0,\dots,m \tag{4.42}$$

are convergent, then

$$\frac{\partial^{i+j}}{\partial t_1^i \partial t_2^j}(S_{\mathbf{a}}^\infty \mathbf{f}^0)(t) = (S_{\mathbf{a_{i,j}}}^\infty \Delta_1^i\Delta_2^j \mathbf{f}^0)(t)\ , \quad i,j = 0,\dots,m. \tag{4.43}$$

In particular $S_{\mathbf{a}}$ is C^m.

In geometric modeling the required smoothness of surfaces is at least C^1 and at most C^2. To verify that a scheme $S_{\mathbf{a}}$ generates C^1 limit functions, with the aid of the last two theorems, we have to assume a symbol of the form

$$a(z) = (1+z_1)^2(1+z_2)^2 b(z)\ ,$$

and to check the contractivity of the three schemes with symbols

$$2(1+z_1)(1+z_2)b(z), \quad 2(1+z_2)^2 b(z), \quad 2(1+z_1)^2 b(z)\ .$$

This analysis applies also to tensor-product schemes, but is not needed, since if $a(z) = a_1(z_1)a_2(z_2)$ is the symbol of a tensor-product scheme, then $\phi_{\mathbf{a}}(t_1,t_2) = \phi_{\mathbf{a_1}}(t_1)\cdot\phi_{\mathbf{a_2}}(t_2)$, and its smoothness properties are derived from those of $\phi_{\mathbf{a_1}}, \phi_{\mathbf{a_2}}$.

Similar results hold for schemes defined on regular triangulations. For the topology of a regular triangulation, we regard the subdivision scheme as operating on the 3-directional grid. (The vertices of $\mathbb{Z}^2$ with edges in the directions $(1,0),(0,1),(1,1)$.)

Since the 3-directional grid can be regarded also as $\mathbb{Z}^2$, (4.39) and (4.40) hold for convergent schemes on this grid.

A scheme for regular triangulations treats each edge in the 3-directional grid in the same way with respect to the topology of the grid. The symbol of such a scheme, when being factorizable, has the form

$$a(z) = (1+z_1)^m(1+z_2)^m(1+z_1z_2)^m b(z)\ . \tag{4.44}$$

Example 4.31 *The symbol of the butterfly scheme on the three directional grid has the form [80]*

$$a(z) = \frac{1}{2}(1+z_1)(1+z_2)(1+z_1z_2)(1-wc(z_1,z_2))(z_1z_2)^{-1} \tag{4.45}$$

with

$$c(z_1,z_2) = 2z_1^{-2}z_2^{-1} + 2z_1^{-1}z_2^{-2} - 4z_1^{-1}z_2^{-1} - 4z_1^{-1} - 4z_2^{-1} + 2z_1^{-1}z_2 + 2z_1z_2^{-1} + 12 - 4z_1 - 4z_2 - 4z_1z_2 + 2z_1^2z_2 + 2z_1z_2^2 \ . \quad (4.46)$$

Convergence analysis for schemes with factorizable symbols of the form (4.44) is similar to that for schemes with symbols of the form (4.41).

Theorem 4.32 *Let $S_{\mathbf{a}}$ have the symbol*

$$a(z) = (1+z_1)(1+z_2)(1+z_1z_2)b(z) \ . \quad (4.47)$$

$S_{\mathbf{a}}$ is convergent if and only if the schemes with symbols

$$a_1(z) = \frac{a(z)}{1+z_1}, \quad a_2(z) = \frac{a(z)}{1+z_2}, \quad a_3(z) = \frac{a(z)}{1+z_1z_2} \quad (4.48)$$

are contractive. If any two of these schemes are contractive then the third is also contractive.

Note that

$$S_{\mathbf{a_3}}\Delta_3\mathbf{f} = \Delta_3 S_{\mathbf{a}}\mathbf{f} \ ,$$

with $(\Delta_3\mathbf{f})_{i,j} = f_{i,j} - f_{i-1,j-1}$. Thus if two of the schemes $S_{\mathbf{a_i}}$, $i=1,2,3$ are contractive then the differences in two linearly independent directions tend to zero as $k\to\infty$, which implies as in the proof of Theorem 4.11, the uniform convergence of the bi-linear interpolants to the data $\{\mathbf{f}^k\}_{k\in\mathbb{Z}_+}$.

The smoothness analysis for a scheme with a symbol (4.47) is different from that for schemes with symbols as in (4.41).

Theorem 4.33 *Let $S_{\mathbf{a}}$ have the symbol* (4.47)*, and let $a_i(z)$, $i=1,2,3$ be as in* (4.48)*. Then $S_{\mathbf{a}}$ generates C^1 limit functions, if the schemes with the symbols $2a_i(z)$, $i=1,2,3$, are convergent. If any two of these schemes are convergent then the third is also convergent. Moreover,*

$$\frac{\partial}{\partial t_i}(S_{\mathbf{a}}^\infty\mathbf{f}^0)(t) = (S_{\mathbf{2a_i}}\Delta_i\mathbf{f}^0)(t), \quad i=1,2$$
$$\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right)(S_{\mathbf{a}}^\infty\mathbf{f}^0)(t) = (S_{\mathbf{2a_3}}\Delta_3\mathbf{f}^0)(t) \ .$$

The verification, based on Theorems 4.32 and 4.33, that the scheme $S_{\mathbf{a}}$ with symbol (4.47) is C^1, requires to check the contractivity of the three schemes with symbols,

$$2(1+z_1)b(z), \quad 2(1+z_2)b(z), \quad 2(1+z_1z_2)b(z) \ .$$

If these three schemes are contractive, then $S_{\mathbf{a}}$ generates C^1 limit functions. For $a(z)$ with the symmetries of the 3-directional grid, it is sufficient to check the contractivity of only one of the three schemes, as is easily observed in the next example.

Example 4.34 To verify that the butterfly scheme generates C^1 limit functions, we use the fact that the symbol $a(z)$ of the butterfly scheme, given in (4.45), is of the form (4.47). In view of the observation following Theorem 4.33, we have to check the contractivity of the three schemes with symbols

$$q_i(z) = (1+z_i)\big(1 - wc(z_1,z_2)\big)(z_1z_2)^{-1} \ , \quad i = 1,2$$
$$q_3(z) = (1+z_1z_2)\big(1 - wc(z_1,z_2)\big)(z_1z_2)^{-1} \ .$$

Noting that

$$c(z_1,z_2) = c(z_2,z_1) = c(z_1z_2, z_1^{-1}) \ ,$$

and that the factor $(z_1z_2)^{-1}$ in a symbol does not affect the norm of the corresponding subdivision operator, it is sufficient to verify the contractivity of $S_{\mathbf{r}}$, where

$$r(z) = (1+z_1)\big(1 - wc(z_1,z_2)\big) = \sum_{\alpha\in\mathbb{Z}^2} r_\alpha z^\alpha \ .$$

Now

$$\|S_{\mathbf{r}}\|_\infty = \max_{\ell,k\in\{0,1\}} \Big(\sum_{i,j\in\mathbb{Z}} |r_{k+2i,\ell+2j}| \Big) \ ,$$

and since

$$\sum_{i,j\in\mathbb{Z}} |r_{2i,2j}| = |1-8w| + |8w|$$

$\|S_r\|_\infty \geq 1$ for all values of w.

Next, we show that for $w > 0$ small enough $\|S_{\mathbf{r}}^2\|_\infty < 1$ [80]. Ignoring coeifficients of $r^{[2]}(z)$ which are not $O(1)$, and computing the others up to order $O(w)$, we get

$$\begin{aligned} r^{[2]}(z) = r(z)r(z^2) &= (1+z_1+z_1^2+z_1^3)\big(1 - wc(z_1,z_2) - wc(z_1^2,z_2^2) + O(w^2)\big) \\ &= \sum_{i,j\in\mathbb{Z}} r_{ij}^{[2]} z_1^i z_2^j \ . \end{aligned}$$

Thus for $j \neq 0$, $r_{i,j}^{[2]} = O(w)$ while $r_{i,0}^{[2]} = 1 + O(w)$, $i = 0,1,2,3$. From this we conclude that it is sufficient to show that for small enough w

$$\sum_{i,j\in\mathbb{Z}} \big|r_{\ell+4i,4j}^{[2]}\big| < 1, \quad \ell = 0,1,2,3 \ .$$

In case $\ell = 0$, all the non-zero coefficients $\{r_{4i,4j}^{[2]}\}$ are

$$r_{0.0}^{[2]} = 1 - 16w + O(w^2), \quad r_{4,0}^{[2]} = 8w + O(w^2), \quad r_{4,4}^{[2]} = r_{0,-4}^{[2]} = -2w + O(w^2) \ .$$

Hence for $w > 0$ small enough

$$\sum_{i,j\in\mathbb{Z}} \big|r_{4i,4j}^{[2]}\big| = |1-16w| + 12|w| + O(w^2) < 1 \ .$$

In case $\ell = 1$, the relevant coefficients are

$$r^{[2]}_{1,0} = 1 - 12w + O(w^2), \quad r^{[2]}_{5,0} = 4w + O(w^2), \quad r^{[2]}_{5,4} = r^{[2]}_{1,-4} = -2w + O(w^2) \ ,$$

and for $w > 0$ small enough,

$$\sum_{i,j\in\mathbb{Z}} |r_{1+4i,4j}| = |1 - 12w| + 8|w| + O(w^2) < 1 \ .$$

The cases $\ell = 2$ and $\ell = 3$ are similar to the cases $\ell = 1$ and $\ell = 0$, respectively. Thus for $w > 0$ small enough the limit surfaces/functions generated by the butterfly scheme on regular triangulations are C^1.

An explicit value of w_0, such that for $w \in (0, w_0)$ the butterfly scheme generates C^1 limit functions on regular triangulations is computed in [48]. The computation shows that $w_0 > \frac{1}{16}$. The value $w = \frac{1}{16}$ is of special importance, since for this value the butterfly scheme on $\mathbb{Z}^2$ reproduces cubic polynomials, while for $w \neq \frac{1}{16}$ the scheme reproduces only linear polynomials. These properties are related to the approximation properties of the scheme (see §7).

4.3.2 Analysis of general bivariate schemes defined on $\mathbb{Z}^2$

The necessary conditions in the bivariate case (4.39) imply four conditions on the symbol (4.40).

These four conditions, lead to a subdivision scheme with a matrix mask, for the vector of first differences

$$\mathbf{\Delta f} = \left[(\mathbf{\Delta f})_{ij} = \left(\begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} \mathbf{f} \right)_{ij} = \begin{pmatrix} f_{ij} - f_{i-1,j} \\ f_{ij} - f_{i,j-1} \end{pmatrix} : (i,j) \in \mathbb{Z}^2 \right] \ . \tag{4.49}$$

Contrary to the univariate case, this matrix mask is *not uniquely* determined. The matrix mask can be derived with the help of the following lemma,

Lemma 4.35 *Let $p(z) = p(z_1, z_2)$ be a Laurent polynomial, satisfying*

$$p(1,1) = p(-1,1) = p(1,-1) = p(-1,-1) = 0 \ . \tag{4.50}$$

Then there exist Laurent polynomials, p_1, p_2, such that

$$p(z) = (1 - z_1^2)p_1(z) + (1 - z_2^2)p_2(z) \ . \tag{4.51}$$

The "factorization" in (4.51) is not unique, since $(1-z_1^2)(1-z_2^2)q(z)$, with q a Laurent polynomial, can be added to the first term on the right-hand side of (4.51) and subtracted from the second.

The proof of the lemma follows from two observations: (a) The Laurent polynomial

$$P(z) = \frac{1}{2}\big[(1 + z_2)p(z_1, 1) + (1 - z_2)p(z_1, -1)\big]$$

coincides with $p(z)$ for $z_2 = 1$ and for $z_2 = -1$, and therefore there exists a Laurent polynomial $r(z)$ such that

$$p(z) - P(z) = (1 - z_2^2)r(z) \ .$$

(b) $P(z)$ is a Laurent polynomial, which is divisible by $(1 - z_1^2)$, since $P(\pm 1, z_2) \equiv 0$, in view of (4.50).

The last lemma leads to the "factorization" in (4.52).

Theorem 4.36 *Let $a(z) = a(z_1, z_2)$ satisfy* (4.40), *and let*

$$\begin{aligned}(1 - z_1)a(z) &= b_{11}(z)(1 - z_1^2) + b_{12}(z)(1 - z_2^2) \ , \\ (1 - z_2)a(z) &= b_{21}(z)(1 - z_1^2) + b_{22}(z)(1 - z_2^2) \ ,\end{aligned} \tag{4.52}$$

with $\{b_{ij}, i, j = 1, 2\}$ Laurent polynomials. Then

$$\mathbf{\Delta} R_{\mathbf{a}}\mathbf{f} = R_{\mathbf{B}}\mathbf{\Delta f} \ , \tag{4.53}$$

where $R_{\mathbf{B}}$ is the refinement rule

$$(R_{\mathbf{B}}\mathbf{v})_\alpha = \sum_{\beta \in \mathbb{Z}^2} B_{\alpha - 2\beta} v_\beta, \quad \alpha \in \mathbb{Z}^2 \tag{4.54}$$

with the matrix symbol

$$B(z) = \sum_{\alpha \in \mathbb{Z}^2} B_\alpha z^\alpha = \begin{pmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{pmatrix} \tag{4.55}$$

and with $\mathbf{v}$ a sequence of vectors in $\mathbb{R}^2$,

$$\mathbf{v} = \{v_\alpha : v_\alpha \in \mathbb{R}^2, \alpha \in \mathbb{Z}^2\} \ .$$

Sketch of the proof: The formalism of z-transforms is the tool for proving the theorem. Observing that

$$L(\mathbf{\Delta f}; z) = \begin{pmatrix} 1 - z_1 \\ 1 - z_2 \end{pmatrix} L(\mathbf{f}; z)$$

and recalling the basic relation in (2.25)

$$L(R_{\mathbf{a}}\mathbf{f}; z) = a(z) L(\mathbf{f}; z^2) \ ,$$

we obtain from (4.52),

$$\begin{pmatrix} 1 - z_1 \\ 1 - z_2 \end{pmatrix} L(R_{\mathbf{a}}\mathbf{f}; z) = \begin{pmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{pmatrix} \begin{pmatrix} 1 - z_1^2 \\ 1 - z_2^2 \end{pmatrix} L(\mathbf{f}; z^2) \ ,$$

which is equivalent to (4.53), (4.54). In the following we denote by $S_{\mathbf{B}}$ the stationary scheme with the refinement rule $R_{\mathbf{B}}$ in (4.54). Theorem 4.36 leads, as in the univariate case, to

Corollary 4.37 *Let $S_{\mathbf{a}}$ be a bivariate subdivision scheme satisfying* (4.39). *Then $S_{\mathbf{a}}$ is convergent if and only if $S_{\mathbf{B}}$ is contractive for all initial data of the form $\mathbf{\Delta f}$.*

A sufficient condition for convergence, is thus, the contractivity of the scheme $S_{\mathbf{B}}$. This can be verified by considering the numbers $\|S_{\mathbf{B}}^M\|_\infty$ for $M = 1, 2, \ldots$. Here again the formalism of z-transforms leads to the symbol of S_B^M as

$$B^{[M]}(z) = B(z)B^{[M-1]}(z^2) = B(z)B(z^2)\cdots B(z^{2^{M-1}}) ,$$

where the order of the factors in the matrix product is significant. The norm of $S_{\mathbf{B}}^M$ is given by [55, 32],

$$\|S_{\mathbf{B}}^M\|_\infty = \max_{\alpha\in E_2^M} \left\| \sum_{\beta\in\mathbb{Z}^2} |B^{[M]}_{\alpha-2^M\beta}| \right\|_\infty$$

where $|A|$ denotes a matrix with elements which are the absolute values of the corresponding elements in the matrix A, where $\|A\|_\infty$ denotes the L_∞-norm of the matrix A, and where $E_2^M = \left\{\alpha = (\alpha_1, \alpha_2) : 0 \le \alpha_1 < 2^M, 0 \le \alpha_2 < 2^M\right\}$. Thus a similar algorithm to the one given in the univariate case (see §4.2.1), applies also in the bivariate case, although it is based only on a sufficient condition and on a non unique "factorization". It is possible to use optimization techniques to find among all possible "factorization" the one that minimizes $\min\{\|S_{\mathbf{B}}^M\|_\infty : 1 \le M \le M_0\}$ with $2 \le M_0 \le 10$ [61].

The C^1 analysis is based on the result,

Theorem 4.38 *Let $S_{\mathbf{a}}$ be a convergent subdivision scheme. If $\frac{1}{2}S_{\mathbf{B}}$ with $\mathbf{B}$ given by* (4.55), (4.52) *is convergent for initial data of the form $\mathbf{\Delta f}$, then $S_{\mathbf{a}}$ is C^1.*

This result is analogous to Theorem 4.13 in the univariate case. Similarly to the univariate case

$$\left(\frac{1}{2}S_{\mathbf{B}}\right)^\infty \mathbf{\Delta f}^0 = \begin{pmatrix}\partial_1\\ \partial_2\end{pmatrix} S_{\mathbf{a}}^\infty \mathbf{f}^0 \ . \tag{4.56}$$

Equation (4.56) holds, only if

$$\sum_{\alpha\in\mathbb{Z}^2} B_{\gamma-2\alpha} = I_{2\times 2}, \quad \gamma \in \left\{(0,0),(0,1),(1,0),(1,1)\right\} \tag{4.57}$$

which follows from the linear independence of the two components of $(\frac{1}{2}S_{\mathbf{B}})^\infty\mathbf{\Delta f} = (\partial_1 S_{\mathbf{a}}^\infty \mathbf{f}, \partial_2 S_{\mathbf{a}}^\infty \mathbf{f})^T$ for generic $\mathbf{f}$. From (4.57) and Lemma 4.35 follows the existence of a matrix subdivision scheme $S_{\mathbf{C}}$, for the vectors $2^{-k}\Delta^2\mathbf{f}^k$,

$$\mathbf{\Delta}^2\mathbf{f} = \begin{pmatrix}\Delta_1\\ \Delta_2\end{pmatrix} \mathbf{\Delta f} \in \mathbb{R}^4 \ ,$$

with $\mathbf{C}$ a mask of matrices of order 4×4 with symbol $\begin{pmatrix} C^{(1,1)}(z) & C^{(1,2)}(z) \\ C^{(2,1)}(z) & C^{(2,2)}(z)\end{pmatrix}$. Here $C^{(i,j)}(z)$ is a matrix of order 2×2 defined by the "factorization"

$$\begin{pmatrix}1-z_1\\ 1-z_2\end{pmatrix}\frac{1}{2}b_{ij}(z) = C^{(i,j)}(z)\begin{pmatrix}1-z_1^2\\ 1-z_2^2\end{pmatrix} .$$

If $S_{\mathbf{C}}$ is contractive then $S_{\frac{1}{2}\mathbf{B}}$ is convergent and $S_{\mathbf{a}}$ is C^1. The same ideas can be further extended to deal with higher orders of smoothness [55, 32].

5 Analysis by local matrix operators

Given masks $\{\mathbf{a}^k\}$ of the same finite support, the corresponding refinement rules (2.2) and their representations in matrix form (2.18) are local. For the subdivision scheme $S_{\{\mathbf{a}^k\}}$, this locality is also expressed by the compact supports of the corresponding basic limit functions $\{\phi_k : k \in \mathbb{Z}_+\}$, and the representations (2.10) of the limit functions $S_k^\infty \mathbf{f}^0$.

5.1 The local matrix operators in the univariate setting

To simplify the presentation we deal here with the case $s = 1$. The results extend to $s > 1$.

The locality of $R_{\mathbf{a}^k}$ can be more pronouncely expressed in terms of two finite dimensional matrices, which are both sections of the bi-infinite matrix A^k in (2.18). First we obtain the two finite dimensional matrices. Consider

$$S_0^\infty \mathbf{f}^0 = S_{\{\mathbf{a}^k\}}^\infty \mathbf{f}^0 = \sum_{\alpha\in\mathbb{Z}} f_\alpha^0 \phi_0(\cdot - \alpha) = \sum_{\alpha\in\mathbb{Z}} f_\alpha^k \phi_k(2^k \cdot - \alpha) \ , \tag{5.1}$$

and its restriction to a unit interval. Due to the finite support of ϕ_0, there exists a finite set $I \subset \mathbb{Z}$, such that

$$S_{\{\mathbf{a}^k\}}^\infty \mathbf{f}^0 \Big|_{[j,j+1]} = \sum_{\alpha - j \in I} f_\alpha^0 \phi_0(\cdot - \alpha). \tag{5.2}$$

Thus the vector $\{f_\alpha^0 : \alpha - j \in I\}$ determines completely the limit function in $[j, j+1]$. By the same reasoning, and since by the assumption on the supports of the masks $\sigma(\phi_k) = \sigma(\phi_0)$, $k \in \mathbb{Z}_+$, we get in view of (5.1), that the vector $\{f_\alpha^k : \alpha - j \in I\}$ with $\mathbf{f}^k = R_{\mathbf{a}^{k-1}} \cdots R_{\mathbf{a}^0} f^0$ determines the limit function in $[j, j+1]2^{-k}$. Again, by the linearity of $\{R_{\mathbf{a}^k} : k \in \mathbb{Z}_+\}$, there exists a linear map from $\{f_\alpha^{k-1} : \alpha \in I\}$ to $\{f_\alpha^k : \alpha \in I\}$, which is a square matrix of dimension $|I|$. We denote it by A_0^k. Similarly there is a linear transformation from $\{f_a^{k-1} : \alpha \in I\}$ to $\{f_\alpha^k : \alpha - 1 \in I\}$ which is denoted by A_1^k. Note that by the uniformity of $R_{\mathbf{a}^k}$, A_ε^k maps the vector $\{f_\alpha^{k-1} : \alpha - j \in I\}$ to $\{f_\alpha^k : \alpha - j - \varepsilon \in I\}$, $\varepsilon = 0, 1$. It is easy to conclude from the definition of A_0^k, A_1^k as linear operators, that the matrices A_0^k, A_1^k are finite sections of the bi-infinite matrix A^k in (2.19),

$$\begin{aligned} (A_0^k)_{\alpha\beta} &= a_{\alpha-2\beta}^k, \quad \alpha, \beta \in I \ , \\ (A_1^k)_{\alpha\beta} &= a_{\alpha+1-2\beta}^k, \quad \alpha, \beta \in I \ . \end{aligned} \tag{5.3}$$

In the following we show how to get the value $(S_{\{\mathbf{a}^k\}} f^0)(x)$ for $x \in \mathbb{R}$ in terms of the matrices $\{A_0^k, A_1^k : k \in \mathbb{Z}_+\}$. It is enough to consider the interval $[0, 1)$.

For $x \in [0, 1)$, we use the diadic representation $x = \sum_{i=1}^\infty d_i 2^{-i}$, $d_i \in \{0, 1\}$, and obtain [4],

$$\left(S_{\{\mathbf{a}^k\}}^\infty f^0\right)(x) = \lim_{k\to\infty} A_{d_{k+1}}^k A_{d_k}^{k-1} \cdots A_{d_1}^0 \mathbf{f}_{[0,1]}^0 \tag{5.4}$$

with the vector $\mathbf{f}^0_{[0,1)} = \{f^0_\alpha : \alpha \in I\}$. Note that the finite product $A^k_{d_{k+1}} \cdots A^0_{d_1} \mathbf{f}^0_{[0,1)}$ is a vector which determines the limit function in an interval of the form $[j, j+1]2^{-k-1}$ containing x. Thus the convergence and smoothness of the limit function generated by $S_{\{\mathbf{a}^k\}}$ can be deduced from the set of finite dimensional matrices

$$\{A^k_0, A^k_1 : k \in \mathbb{Z}_+\} \tag{5.5}$$

and their infinite products of the form appearing in (5.4). In the stationary case there are only two matrices A_0, A_1, and all possible infinite products of them have to be considered.

5.2 Convergence and smoothness of univariate stationary schemes in terms of finite matrices

In the stationary case the value $(S^\infty_{\mathbf{a}}\mathbf{f}^0)(x)$ for $x = \sum_{j=1}^\infty d_j 2^{-j} \in [0,1)$, $d_j \in \{0,1\}$ is given by

$$(S^\infty_{\mathbf{a}}\mathbf{f}^0)(x) = \lim_{k\to\infty} A_{d_k} \cdots A_{d_1} \mathbf{f}^0_{[0,1)} \tag{5.6}$$

with $\mathbf{f}^0_{[0,1)} = \{f^0_\alpha : \alpha \in I\}$.

Note that $S_{\mathbf{a}}$ is contractive if and only if the joint spectral radius of A_0, A_1, $\rho_\infty(A_0, A_1)$, is less than 1, where

$$\rho_\infty(A_0, A_1) = \sup_{k\in\mathbb{Z}_+\setminus 0} \left(\sup\left\{ \|A_{\varepsilon_k} A_{\varepsilon_{k-1}} \cdots A_{\varepsilon_1}\|_\infty : \varepsilon_i \in \{0,1\}, i = 1, \dots, k \right\} \right)^{\frac{1}{k}} . \tag{5.7}$$

Thus the conditions for convergence and smoothness of a stationary scheme given in section 4.2.1, which can be expressed as the contractivity of a related scheme, can be formulated in terms of the joint spectral radius of two finite matrices. (see e.g. [17]). It is easy to conclude that $\rho_\infty(A_0, A_1) \geq \max\{\rho(A_0), \rho(A_1)\}$, where $\rho(A)$ is the spectral radius of the matrix A. From this inequality and from the necessity of the contractivity condition, we obtain necessary conditions for convergence and smoothness (for the later only in case of L_∞-stability), which are easy to check.

Such necessary conditions are important in the design of new schemes, in the sense that "bad" schemes can be easily excluded. For example, if $a(z) = \frac{(1+z)^2}{2} b(z)$, and $S_{\mathbf{a}}$ is an interpolatory scheme, then $\rho(B_0) < 1$, and $\rho(B_1) < 1$ (with B_0 and B_1 the local matrix operators corresponding to $S_{\mathbf{b}}$) are necessary for $S_{\mathbf{a}}$ to be C^1.

Here we formulate an open problem; What are the conditions for the contractivity of $S_{\{\mathbf{a}^k\}}$ in terms of the matrices $\{A^k_0, A^k_1 : k \in \mathbb{Z}_+\}$?

5.3 L_p-convergence and p-smoothness of univariate stationary schemes, in terms of finite matrices

There is a vast literature (See e.g. [97, 57, 59, 93, 52, 54, 53], and references therein) on the convergence in the L_p-norm of subdivision schemes, and on the p-smoothness of refinable functions. One central method of analysis, is in terms of the p-norm joint spectral radius of two operators restricted to a finite dimensional space.

Let A_0, A_1, be matrices of order $n \times n$. Their p-norm joint spectral radius is

$$\rho_p(A_0, A_1) = \sup_{k \in \mathbb{Z}_+ \backslash 0} \left(\left(\sum_{\varepsilon_1, \ldots, \varepsilon_k \in \{0,1\}} \|A_{\varepsilon_k} \cdots A_{\varepsilon_1}\|_p^p \right)^{\frac{1}{p}} \right)^{\frac{1}{k}} , \quad 1 \leq p < \infty .$$

For $\phi \in L_p(\mathbb{R})$ of compact support, the p-smoothness exponent is defined as

$$\nu_p(\phi) = \sup\{v \geq 0 : \|\Delta_h^n \phi\|_p \leq Ch^v\}$$

for some constant $C > 0$ and for large enough n, where $\Delta_n \phi = \phi - \phi(\cdot - h)$ and $\Delta_h^n = \Delta_h \Delta_h^{n-1}$.

Here we bring one result from [57], which is in some sense an extension of Theorem 4.12 in section 4.2.1.

Theorem 5.1 *Let* $\mathbf{a}$ *be a finitely supported mask such that* $\sum_{i \in \mathbb{Z}} a_i = 2$. *Let* $\phi_{\mathbf{a}}$ *be a non-trivial solution of the refinement equation* $\phi_{\mathbf{a}} = \sum_{i \in \mathbb{Z}} a_i \phi_{\mathbf{a}}(2 \cdot -i)$. *If there exists* $C > 0$ *such that* $\|\Delta R_{\mathbf{a}}^n \boldsymbol{\delta}\|_p^{\frac{1}{n}} \leq C 2^{\frac{1}{p} - \mu}$ *for* $0 < \mu \leq 1$ *and* $1 \leq p < \infty$, *then* $v_p(\phi_{\mathbf{a}}) = \mu$.

Since $\|\Delta R_{\mathbf{a}}^n \boldsymbol{\delta}\|_p^{\frac{1}{n}} = \rho_p(A_0|_V, A_1|_V)$ with $V = \{\mathbf{u} \in \mathbb{R}^{|I|} : \sum_{i \in I} u_i = 0\}$ [57], the condition of the above theorem can be formulated in terms of two finite dimensional matrices, which are the restrictions of two operators to a finite dimentional subspace. In [53] an algorithm is presented for computing $v_2(\phi_{\mathbf{a}})$ efficiently, for $\phi_{\mathbf{a}}$ a multivariate refinable function corresponding to a dilation matrix M and a mask $\mathbf{a}$ both with the same symmetries. For symmetric interpolatory masks there is also an algorithm for the computation of $\nu_\infty(\phi_{\mathbf{a}})$. The situation in the multivariate case is much more complex; there are $|detM|$ operators, and the finite dimensional subspace to which these operators are restricted, is quite complicated.

6 Extraordinary point analysis

For all the types of subdivision schemes that are defined over nets of arbitrary topology, as describes in section 3.5, the refined nets are regular nets, excluding a fixed number of extraordinary (irregular) points of valency $\neq 6$, in the case of triangular nets, and of valency $\neq 4$, in the case of quadrilateral nets. The smoothness analysis of subdivision schemes over nets of arbitrary topology is thus decomposed into two stages. First, the analysis over the regular part is completed, using the tools described in §4 and in §5. After verifying the smoothness over the regular part, we are left with a finite number of isolated points of unknown regularity. The regularity analysis at the extraordinary points has been studied by several authors, starting with the pioneering eigenvalue analysis work by Doo and Sabin [27], through the works by Ball and Storey [2, 1], and completed by Prautzsch [86], Reif [88] and Zorin [99]. It is based on the observation that the regularity of the surface is known over a ring of

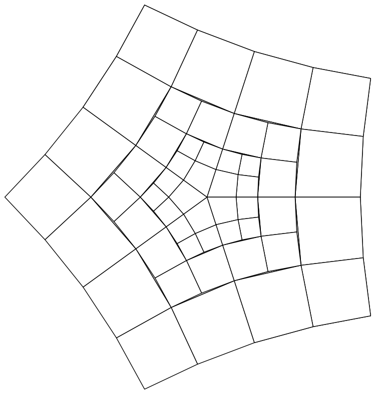

Figure 13: Three rings of patches.

patches Q^k encircling the extraordinary point, and there is a linear transformation T mapping the patches Q^k onto a refined ring of patches, Q^{k+1}. Figure 13 displays a graphical description of three rings of patches around a vertex of valency five. The rings, each composed of 15 quadrilaterals, are self similar, of reducing sizes.

The closure of the union of these rings defines an extraordinary patch covering a 'hole' in the regular part of the surface, and the smoothness of such a patch is completely characterized by the transformation T. In the following we present the key ingredients of the smoothness analysis of such patches and the main results.

Let us denote the basic limit function of the subdivision on a regular net by ϕ. The ring of patches Q^k may be expressed in terms of the control points P^k influencing this ring. Let $P^k = \{P_1^k, P_2^k, ..., P_N^k\} \subset \mathbb{R}^3$ be the control points generating Q^k, and let the transformation T be the square matrix such that $P^{k+1} = TP^k$.

Each patch in the ring $Q_\ell^k \in Q^k$ is a parametric patch, triangular or quadrilateral, which is a linear combination of translations of $\phi(2^k \cdot)$ multiplying control points $\{P_r^k\}_{r \in I_\ell} \subset P^k$. I.e.,

$$Q^k = \bigcup Q_\ell^k \ , \tag{6.1}$$

where

$$Q_\ell^k = \{q_\ell^k(u,v) \equiv \sum_{r \in I_\ell} P_r^k \phi(2^k u - i_r, 2^k v - j_r) \mid (u,v) \in \Omega\} \ . \tag{6.2}$$

$\Omega = \{(u,v) \mid 0 \le u, v \le 1\}$ for quad-meshes and $\Omega = \{(u,v) \mid 0 \le u, v \ \wedge \ u+v \le 1\}$ for triangular meshes.

Since the regularity of ϕ is assume to be already known, it is clear that the behavior at the extraordinary vertex is completely characterized by the matrix T. It is important to note that the conditions for regularity at the extraordinary vertex do not require the knowledge of the explicit formula of ϕ. Using a proper ordering of the points P^k [27], the matrix T is a block circulant matrix. The eigenvalues analysis of this matrix plays a crucial role in the smoothness analysis, as described in [27], [88], [99], [86]. The results include necessary and sufficient conditions for geometric continuity, i.e., existence of continuous limit normal at the extraordinary vertex and necessary and sufficient conditions for C^m-continuity at an extraordinary vertex - under some assumptions.

Let the eigenvalues $\lambda_0, ..., \lambda_{N-1}$ of T be ordered by modulus,

$$|\lambda_0| \geq |\lambda_1| \geq \cdots \geq |\lambda_{N-1}| \ , \tag{6.3}$$

and denote by $V_0, V_1, ..., V_{N-1} \in \mathbb{R}^N$ the corresponding generalized real eigenvectors, assuming they exist.

As first shown in [27], a necessary condition for the continuity of the normal at an extraordinary point is:

$$\lambda_0 = 1 > |\lambda_1| = |\lambda_2| > |\lambda_3| \ , \quad V_0 = \{1, 1, ..., 1\} \ . \tag{6.4}$$

Assuming (6.4) holds, let us consider the particular initial data $P^0 = \{P_1^0, P_2^0, ..., P_N^0\}$ with

$$P_j^0 = (V_{1,j}, V_{2,j}, 0)^t \ , \tag{6.5}$$

and let us examine the corresponding rings of patches defined by (6.2).

Injectivity and regularity assumption: We assume that each mapping q_ℓ^0 in (6.2) is regular and injective, and that

$$\bigcap_\ell int\{Q_\ell^0\} = \emptyset \ . \tag{6.6}$$

In [88], the collection of mappings $\{q_\ell^0\}$ is termed as 'the characteristic map' and the above assumption is thus referred to as the regularity and injectivity of the characteristic map. The importance of this map is that it defines the natural parametric domain for analyzing the smoothness of the surface at the extraordinary vertex. For a discussion and analysis of the characteristic map see [84]. Under the above assumption the following results hold:

Theorem 6.1 *[88] Let* (6.3) *hold with* $\lambda_1 = \lambda_2$ *being a real eigenvalue of* T *with geometric multiplicity 2, and let the characteristic map be regular and injective. Then the limit surface of the subdivision is a regular* C^1 *manifold in a neighborhood of the extraordinary vertex for almost any initial data.*

The necessary and sufficient conditions for C^m-continuity at an extraordinary vertex were derived independently by Prautzch [86] and Zorin [99]. These result is equivalent to the polynomial reproduction result for uniform stationary C^m schemes on regular meshes.

Theorem 6.2 (C^m conditions) *Let the conditions of Theorem 6.1 hold, Then the limit surface of the subdivision is a regular* C^m *manifold in a neighborhood of the extraordinary vertex for almost any initial data, if and only if the following condition holds:*

For any eigenvalue λ *of* T *satisfying* $|\lambda| > \lambda_1^m$,

a. $|\lambda| = \lambda_1^i$ *for some integer* $0 \le i \le m$,

b. *For the initial data* $P^0 = \{P_1^0, P_2^0, ..., P_N^0\}$ *with*

$$P_j^0 = (V_{1,j}, V_{2,j}, V_j)^t \in \mathbb{R}^3 \ , \tag{6.7}$$

and V *an eigenvector coresponding to* λ, *all the patches* Q_ℓ^0 *defined by* (6.2) *lie on a polynomial surface* $z = p(x,y)$ *in* $\mathbb{R}^3$, *where* p *is a homogeneous polynomial of total degree* i.

Theorem 6.2 does not give explicit constructive conditions that can help us to build C^m scheme. The translation of the conditions in Theorem 6.2 into algebraic conditions on the mask coefficients is rather complicated, and even in the C^2 case is not fully resolved. The partial results in this direction include the construction of schemes with bounded curvatures in [75], and the special patch construction by Prautzch and Umlauf [87]. For some applications it is enough to have curvature integrability of the subdivision surface. Reif and Schröder show in [89] that Catmull-Clark and Loop schemes (among many others) have square integrable principal curvatures.

7 Limit values and approximation order

In this section we discuss two practical issues in the implementation of subdivision algorithms in geometric modeling. One issue is the computation of limit values and limit derivatives of the subdivision process at the diadic points at any refinement level. The other important issue, though not widely attended yet, is how to actually attain the optimal approximation order which is theoretically possible for a given scheme. Namely, how to choose the initial control points so that the limit curve/surface will approximate a desired curve/surface with the highest possible approximation power.

7.1 Limit values and derivative values

We consider here only the stationary case, namely when $\mathbf{a}^k = \mathbf{a}$, $k \in \mathbb{Z}_+$, and assume that the basic limit function $\phi \equiv \phi_{\mathbf{a}}$ is C^m. The support of ϕ is contained in the convex hull of the support of the mask, $\sigma(\mathbf{a})$, by (2.11). Furthermore, by (2.10) we can express the limit function of $S_{\mathbf{a}}$ as

$$f \equiv S_{\mathbf{a}}^\infty \mathbf{f}^0 = \sum_{\alpha \in \mathbb{Z}^s} f_\alpha^0 \phi(\cdot - \alpha) \ . \tag{7.1}$$

Thus, the limit values at the integer points $\beta \in \mathbb{Z}^s$ are given by

$$f(\beta) = \sum_{\alpha \in \mathbb{Z}^s} f_\alpha^0 \phi(\beta - \alpha) \ . \tag{7.2}$$

By (7.2), the knowledge of the values of ϕ at the integer points gives one the possibility of computing the limit values of the subdivision process on the integer grid $\mathbb{Z}^s$, using only the initial control points $\mathbf{f}^0$. Similarly, the limit values on the diadic grid $2^{-k}\mathbb{Z}^s$

are defined by the control points $\mathbf{f}^k$ at level k. In the same way we note that the values of a derivative of f at the integers are linear combinations of the values of the same derivative of ϕ at the integers. The vector of values of ϕ, or of one of its derivatives, at the integer points may be computed each as the eigenvector of a finite matrix.

To see this we recall that ϕ satisfies the refinement equation (2.15), and thus,

$$\partial_\lambda \phi = 2^{|\lambda|} \sum_\alpha a_\alpha \partial_\lambda \phi(2 \cdot - \alpha) \ , \tag{7.3}$$

where $\lambda \in \mathbb{Z}_+^s$, $|\lambda| = \sum_{i=1}^s \lambda_i \leq m$. At integer points $\beta \in \mathbb{Z}^s$ we have the linear relations

$$\partial_\lambda \phi(\beta) = 2^{|\lambda|} \sum_\alpha a_\alpha \partial_\lambda \phi(2\beta - \alpha) = 2^{|\lambda|} \sum_\gamma a_{2\beta-\gamma} \partial_\lambda \phi(\gamma) \ . \tag{7.4}$$

Now, since ϕ is of compact support, there is only a finite number N_ϕ of grid points where ϕ is non-zero. Let $\Omega \equiv \mathbb{Z}^s \bigcap \sigma(\phi)$, then $N_\phi = \#\Omega$. The system of equations (7.4), with $\beta \in \Omega$ is a square $N_\phi \times N_\phi$ eigensystem for the values $\{\partial_\lambda \phi(\beta)\}_{\beta\in\Omega}$, and it has a unique solution if we add the side conditions

$$\sum_{-\beta\in\Omega} \beta^\lambda \partial_\lambda \phi(-\beta) = \lambda\,! \ , \quad \sum_{-\beta\in\Omega} \beta^\mu \partial_\lambda \phi(-\beta) = 0 \ , \quad \mu \neq \lambda \ , \ |\mu| = |\lambda| \ . \tag{7.5}$$

These side conditions, in view of (7.2), guaranty that the $|\lambda|$ order derivatives of $S_{\mathbf{a}}^\infty x^\mu|_{\mathbb{Z}^s}$ are correctly obtained, for $|\mu| = |\lambda|$. For example, in the univariate case, the vector of values $\{\phi(\beta)\}$ is an eigenvector of the matrix U with elements $U_{i,j} = a_{2i-j}$, corresponding to the eigenvalue 1, and with the normalization $\sum \phi(\beta) = 1$. The vector of values $\{\phi'(\beta)\}$ is an eigenvector of U with eigenvalue 2. Implementing this, the rule for computing the limit derivatives of a curve defined by the 4-point scheme (3.18) turns to be [33]:

$$f'(2^{-k}i) = \frac{2^k}{1-4w}[\frac{1}{2}(f_{i+1}^k - f_{i-1}^k) - w(f_{i+2}^k - f_{i-2}^k)] \ . \tag{7.6}$$

The method for computing limit values is actually applied to non-interpolatory subdivision surfaces, so that at all refinement levels the rendered points are on the limit surface. The shading of the surface at each level is done with normals which are the actual normals of the limit surface. A detailed example of computing limit normals at regular points and at extraordinary points is given in [96, **?**] for the case of the butterfly scheme.

7.2 Attaining the optimal approximation order

The term approximation order of a subdivision scheme $S_{\mathbf{a}}$ refers to the rate by which the limit functions generated by $S_{\mathbf{a}}$, from initial data sampled from a smooth enough function f, get closer to f. Namely, the largest exponent r such that

$$\|f - S_{\mathbf{a}}^\infty f|_{h\mathbb{Z}^s}\|_\infty \leq ch^r \ .$$

Yet, this order may be improved (for non-interpolatory schemes) by replacing the initial data $f|_{h\mathbb{Z}^s}$ by $Qf|_{h\mathbb{Z}^s}$ with Q a Teoplitz operator of finite support. Our aim is to find the operator Q which yields the largest approximation rate.

Let us start with an example.

Example 7.1 *Let us consider the case of univariate cubic B-splines with integer knots. It is known that the integer shifts of this cubic B-spline, B_3, span π_3, and this implies that the space generated by the integer shifts of the cubic B-spline has potential approximation order 4. If $f \in C^4(\mathbb{R})$, then the use of function values as control points gives a second order approximation, by the corresponding subdivision scheme*

$$\Big| f(x) - \sum_{j\in\mathbb{Z}} f(jh) B_3(\frac{x}{h} - j) \Big| \leq c_2 h^2 \ . \tag{7.7}$$

However, using as control points the values

$$\tilde{f}_j = (Q_h f)(jh) \equiv -\frac{1}{6} f((j-1)h) + \frac{4}{3} f(jh) - \frac{1}{6} f((j+1)h) \ , \tag{7.8}$$

yields the optimal fourth order approximation,

$$\Big| f(x) - \sum_{j\in\mathbb{Z}} \tilde{f}_j B_3(\frac{x}{h} - j) \Big| \leq c_4 h^4 \ . \tag{7.9}$$

This special choice of Q_h is made so that the approximation scheme in (7.9) reproduces all polynomials in $\pi_3(\mathbb{R})$, namely, $\sum (Q_h p)(jh) B_3(\frac{x}{h} - j) = p(x)$ for any $p \in \pi_3(\mathbb{R})$. Therefore, to approximate a curve $c(t)$ by a cubic spline subdivision, given a sequence of points $\{P_j\}$ ordered on it, then it is better to start the subdivision process with the control points

$$\widetilde{P}_j = -\frac{1}{6} P_{j-1} + \frac{4}{3} P_j - \frac{1}{6} P_{j+1} \ . \tag{7.10}$$

The above idea is extended for general subdivision schemes in [71].

For a given uniform stationary scheme $S_{\mathbf{a}}$ we identify the maximal m such that $\pi_m(\mathbb{R}^s)$ is invariant under $S_{\mathbf{a}}$ in the sense that $S_{\mathbf{a}}^\infty p|_{\mathbb{Z}^s} \in \pi_m(\mathbb{R}^s)$ for any $p \in \pi_m(\mathbb{R}^s)$. Then, the potential approximation order is $m+1$. To achieve this approximation power we look for a Teoplitz operator Q, of minimal support Σ, of the form

$$(Q\mathbf{f}^0)_\alpha = \sum_{\sigma\in\Sigma} q_\sigma f^0_{\alpha-\sigma} \ , \tag{7.11}$$

such that

$$S_{\mathbf{a}}^\infty Q(p|_{\mathbb{Z}^s}) = p \ , \quad \forall p \in \pi_m(\mathbb{R}^s) \ . \tag{7.12}$$

Namely, Q is the inverse of $S_{\mathbf{a}}^{\infty}$ on $\pi_m(\mathbb{R}^s)$. If Q exists then it commutes with $S_{\mathbf{a}}^{\infty}$ on π_m. Therefore, we look for Q such that $QS_{\mathbf{a}}^{\infty}p|_{\mathbb{Z}^s} = p$, $\forall p \in \pi_m(\mathbb{R}^s)$. Using the results of §7.1 we can define the polynomials

$$r_\gamma \equiv S_{\mathbf{a}}^{\infty}\{x^\gamma|_{\mathbb{Z}^s}\} = \sum_{\alpha\in\mathbb{Z}^s} \phi(\alpha)(\cdot - \alpha)^\gamma \quad , \quad \gamma \in \mathbb{Z}^s \ , \ |\gamma| \le m \ ,$$

which constitute a basis of π_m. Now we look for an operator Q such that on π_m it is the inverse of $S_{\mathbf{a}}^{\infty}$, namely,

$$Qr_\gamma = x^\gamma \ , \quad \gamma \in \mathbb{Z}^s \quad , \quad |\gamma| \le m \ . \tag{7.13}$$

This can be formulated as a system of linear equations in the finite dimensional space π_m,

$$\sum_{\sigma\in\Sigma} q_\sigma r_\gamma(x - \sigma) = x^\gamma \ , \quad |\gamma| \le m \ . \tag{7.14}$$

In the above example of the cubic B-splines, the operator Q may be chosen also as $Qf = f - \frac{1}{6}f''$ or as the difference operator given in (7.8). The two options act in the same way on π_3, yet, for the purpose of applying Q on the given data points we need the discrete form (7.8). For further examples and applications see [71].

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