THE DEGREE ELEVATION AND L_2 DISTANCE FOR THE RATIONAL BÉZIER CURVES

BYUNG-GOOK LEE, KWAN-PYO KO, AND YUNBEOM PARK

ABSTRACT. An algorithmic approach to degree elevation of rational Bézier curves is presented. The algorithm is based on the matrix representations of the degree elevation processes.

1. Introduction

The Bézier curves are basically and widely used in CAGD – short for Computer Aided Geometric Design. The Bézier curves were independently developed by P. de Casteljau about 1959[2] and by P. Bézier about 1962[1]. The underlying mathematical theory is based on the concept of Bernstein polynomials. De Casteljau directly exploited this relationship; but it was not before 1970 that R. Forrest[7] discovered the connection between Bézier's work and Bernstein polynomials. Bézier and de Casteljau developed their theories as part of CAD systems that were being built up at two French car companies, Rénault and Citroën. The Rénault system UNISURF (by Bézier) was soon described in several publications; this is the reason that the underlying theory now bears Bézier's name.

Bézier curves and surfaces are now established as the mathematical basis of many CAD systems, they have also become a major tool for the development of new methods for curve and surface descriptions. Farin[6] summarize the basic theory of such curves and provide many relevant references.

The nonrational Bézier representation uses Bernstein polynomials as basis functions for the linear space of polynomials. In terms of the Bernstein polynomials of degree n,

$$B_k^n(t) = \binom{n}{k}(1-t)^{n-k}t^k,\ \ 0 \le t \le 1,\ \ k = 0, \cdots, n, \tag{1}$$

a parametric polynomial curve of degree n $(n > 0)$ in the plane, can be expressed as

$$b^n(t) = \sum_{k=0}^{n} b_k B_k^n(t),\ \ b_k \in \mathbb{R}^2. \tag{2}$$

The points b_k, $k = 0, \cdots, n$ are called the *control points* for the polynomial, and the polygon formed by joining successive control points is the *control polygon*.

The fundamental idea of the rational Bézier algorithms is to evaluate and manipulate the curves and surfaces by a small number of control points and weights.

A rational Bézier curve is defined by

$$b^n(t) = \frac{\sum_{k=0}^n w_k b_k B_k^n(t)}{\sum_{k=0}^n w_k B_k^n(t)}, 0 \le t \le 1 \tag{3}$$

where w_k are weights of the Bézier points b_k. Clearly (3) is a genuine generation of (1): we obtain the nonrational case by setting all $w_k = 1$.

This paper describes matrix representation for rational Bézier curves, and the geometric principe behind the algorithm is used to develop other algorithms for these curves.

2. Degree elevation

Suppose we were designing with Bézier curves trying to use a Bézier curve of degree n. After modifying the polygon a few times, it may turn out that a degree n curve does not possess sufficient flexibility to model the desired shape. One way to proceed in such situation is to increase the flexibility of the polygon by adding another vertex to it.

In general degree elevation of nonrational Bézier curves address the following problem.

Problem 1 (Degree Elevation). Let $\{b_k\}_{k=0}^n \subset \mathbb{R}^2$ be a given set of control points which define the rational Bézier curve of degree n

$$b^n(t) = \frac{\sum_{k=0}^n w_k b_k B_k^n(t)}{\sum_{k=0}^n w_k B_k^n(t)}, 0 \le t \le 1,$$

where w_k are weights of the Bézier points b_k. Then find another points set $\{b_k^{(r)}\}_{k=0}^{n+r}$ and weights $\{w_k^{(r)}\}_{k=0}^{n+r}$ defining the rational Bézier curve of higher degree $n+r$ so that the shape of the curve unchanged.

As a first step, to raising the degree of the rational Bézier curve by one. Let us denote the control vertices of the degree elevated curve by $b_k^{(1)}$; they are given by

$$b_k^{(1)} = \frac{\alpha_k w_{k-1} b_{k-1} + (1-\alpha_k) w_k b_k}{\alpha_k w_{k-1} + (1-\alpha_k) w_k}, k = 0, 1, \dots, n+1 \tag{4}$$

and $\alpha_k = k/(n+1)$. The weights $w_k^{(1)}$ of the new control vertices are given by

$$w_k^{(1)} = \alpha_k w_{k-1} + (1-\alpha_k) w_k, k = 0, 1, \dots, n+1. \tag{5}$$

An equivalent form of (4) is

$$w_k^{(1)} b_k^{(1)} = \alpha_k w_{k-1} b_{k-1} + (1-\alpha_k) w_k b_k \tag{6}$$

We can rewrite the formular (5) and (6) as a linear system

$$W^{(1)} = T_n W$$

and

$$W^{(1)} \odot B^{(1)} = T_n (W \odot B)$$

, where the $(n+2)*(n+1)$matrix $T_n =$

$$\frac{1}{n+1}\begin{pmatrix} n+1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & n & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & n-1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n-1 & 2 & 0 \\ 0 & 0 & 0 & \dots & 0 & n & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & n+1 \end{pmatrix}$$

, the $(n+1)-$vector $W = (w_0, w_1, \dots, w_n)^t$, $B = (b_0, b_1, \dots, b_n)^t$, the $(n+2)-$vector $W^{(1)} = (w_0^{(1)}, w_1^{(1)}, \dots, w_{n+1}^{(1)})^t$, $B^{(1)} = (b_0^{(1)}, b_1^{(1)}, \dots, b_{n+1}^{(1)})^t$, and $\odot$ is the Hadamard product : $A \odot B = \{a_{i,j} * b_{i,j}\}$.

We may repeat this process and obtain a sequence of controls points. After r degree elevations, we have a linear system

$$W^{(r)} = T_{n,r} W \tag{7}$$

and

$$W^{(r)} \odot B^{(r)} = T_{n,r}(W \odot B) \tag{8}$$

where the $(n+r+1)*(n+1)$ matrix

$$T_{n,r} = T_{n+r-1} T_{n+r-2} \dots T_{n+1} T_n$$

has elements

$$t_{i+j,i} = \frac{\binom{n}{i}\binom{r}{j}}{\binom{n+r}{i+j}}, \quad \begin{cases} i = 0, 1, \dots, n \\ j = 0, 1, \dots, r. \end{cases}$$

The sum of any row and any column of the matrix $T_{n,r}$ are 1 and $\frac{n+r+1}{n+1}$ respectively, i.e. for any i,

$$\sum_{j=0}^{n} t_{i,j} = 1,$$

and for any j,

$$\sum_{i=0}^{n+r} t_{i,j} = \frac{n+r+1}{n+1}.$$

3. L_2 Distance

For the degree reduction of any given curves, we must compute a distance of two Bézier curves. The most appropriate metric in geometrical terms would be the *Hausdorff distance*[3]. Suppose (M, d) is a metric space with subsets A and B. We define the Hausdorff metric d_H by

$$d_H(A, B) = \max\left\{ \sup_{x\in A} d(x, B), \sup_{y \in B} d(y, A) \right\},$$

where

$$d(x, B) = \inf_{y\in B} d(x, y).$$

If we regard a planar curve as simply a locus of points without any underlying parameterization, the Hausdorff metric for two such curves is essentially the radius of the largest circle with its center on one curve and touching the other curve. For general parametric curves, this measure is truly independent of the relative parameterizations of two curves. Emery[4] presents a method for explicit computation of Hausdorff metric for piecewise linear curves, but the computation of Hausdorff distance d_H of two nonlinear curves is not so easy.

So we define and use the L_2 distance for the rational Bézier curves. If two sets $\{a_k\}_{k=0}^n$ and $\{b_k\}_{k=0}^n$ with weights $\{v_k\}_{k=0}^n$ and $\{w_k\}_{k=0}^n$ represent the control polygons of two different rational Bézier curves of degree n, denoted by a^n and b^n. Then we have,

$$
\begin{aligned}
d_{LS}(a^n, b^n)^2 &= \int_0^1 \|a^n(t) - b^n(t)\|^2 dt \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N} \|a^n(\frac{i}{N}) - b^m(\frac{i}{N})\|^2 \\
&= \lim_{r\to\infty} \frac{1}{n+r} \sum_{i=0}^{n+r} \|a_i^{(r)} - b_i^{(r)}\|^2 \\
&= \lim_{r\to\infty} \frac{1}{n+r} \sum_{i=0}^{n+r} [a_i^{(r)} - b_i^{(r)}][a_i^{(r)} - b_i^{(r)}] \\
&= \lim_{r\to\infty} \frac{1}{n+r} [A^{(r)} - B^{(r)}]^t [A^{(r)} - B^{(r)}] \\
&= \lim_{r\to\infty} \frac{1}{n+r} D^t D \quad \text{where } D = A^{(r)} - B^{(r)}
\end{aligned}
$$

For some large r, it is sufficient from now on to investigate the Euclidian distance with respect to the control points.

Equation (8) can be written in the following form

$$B^{(r)} = [T_{n,r}(W \odot B)] \oslash [T_{n,r}W]$$

where $\oslash$ is the Hadamard division : $A \oslash B = \{a_{i,j}/b_{i,j}\}$.

$$D = A^{(r)} - B^{(r)} = [T_{n,r}(V \odot A)] \oslash [T_{n,r}V] - [T_{n,r}(W \odot B)] \oslash [T_{n,r}W]$$

4. Conclusions

A method for obtaing matrix representation of rational Bézier curves has been presented in this paper. Explicit matrix forms for degree elevation and L_2 distance are given. The matrix evaluations are found to be much faster than recursive evaluations.

References

[1] P. Bézier, *Numerical Control, Mathematics and Applications*, Wiely, New York, 1972.
[2] P. de Casteljau, Outillage méthodes calcul, André Citroën Automobiles SA, Paris, 1959.
[3] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.
[4] J. D. Emery, The definition and computation of a metric on plane curves, *Comput. Aided Des.* 18:25-28 (1986).
[5] G. Farin, Algorithms for rational Bézier curves. *Comput. Aided Des.* 15:73-77 (1983).
[6] G. Farin, *Curves and Surfaces for Computer Aided Geometric Design*, Academic Press, New York, 1988.
[7] A. R. Forrest, Interactive interpolation and approximation by Bézier polynomials, *Comput. Aided Des.* 22:527-537 (1990).
[8] Y. Park and B. G. Lee, The L^2 norm of B'ezier curves, *Korean J. Comp. & Appl. Math*, 3:245-252 (1996).
[9] H. Hagen and G. P. Bonneau, Variational design of smooth rational Bézier curves, *Comput. Aided Geometric Des.* 8:393-399 (1991).
[10] B. K. Choi, W. S. Yoo and C. S. Lee, Matrix representation for NURBS curves and surfaces, *Comput. Aided Des.* 22:235-240 (1990).
[11] L. Piegl, Representation of rational Bézier curves and surfaces by recursive algorithms, *Comput. Aided Des.* 18:361-366 (1990).

Department of Applied Mathematics
Dongseo University
Pusan, 617–716
Korea

Department of Mathematics Education
Seowon University
Chongju, 361–742
Korea