CAGDT Manual

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CAGDT(TM) – short for Computer Aided Geometric Design tutorial Tools – is concerned with the approximation and representation of curves and surfaces that arise when these objects have to be processed by a computer. CAGDT is the interactive tools that illustrate the concepts of CAGD in a powerful,engaging way. It provides full support for all of the basic topics of CAGD including Bézier curves, B–spline curves, NURBS curves and a variety of surface forms. The properties, behavior, and use of these curves and surfaces are discussed in depth. CAGDT is graphic tools for Microsoft Windows 95.

MainMenu	–	SubMenu
File	–	Curves, Surfaces, Open..., Save, SaveAs..., Output..., Close output, Exit
Edit	–	New, Copy, Paste, Control points..., Weights..., Knots..., Save
Bezier	–	Bezier, Degree elevation, Degree reduction, Rec. to triangular, Control points..., Save
RBezier	–	Rational Bezier, Degree elevation, Polynomial approx, Control points..., Weights..., Save
Bspline	–	Bspline, Degree elevation, Degree reduction, Chaikin′s method, Chaikin′s knot elevation, Chaikin′s knot reduction, Knot insert, Knot delete, Control points..., Weights..., Save
NURBS	–	NURBS, Degree elevation, Degree reduction, Knot insert, Knot delete, Control points..., Weights..., Knots..., Save

1. Control Points

- Click the left mouse button on the window to define the control points.
- To move a control point to a different position. 1) Point the cursor at

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the control point. 2) Press and hold the left mouse button. 3) Drag the control point by moving the mouse. 4) Release the mouse button.

- To remove a control point. 1) Point the cursor at the control point. 2) Press the right mouse button.

2. File Menu

- The **Open** command is used to select the filename containing a control data. If loading is successful, the same filename will be used in the **File/Save** command.
- The **Save** command saves user-defined control data to the filename obtained in the last **File/Open** execution.
- The **Save as** command lets you rename your current control data filename. It calls Window's Common "Save As" Dialog for such purposes.
- The **Output** command obtains the filename where the current output data is to be saved. This command redirects the evaluation results of display to a file. It will continue to capture output data until the **File/Close** Output is selected.

3. Edit Menu

- The **New** command removes the current control data and replaces it with the background color.
- The **Copy** command sends the current control data to the Windows clipboard. The **Paste** Command retrieves the contents of the clipboard.
- The **Control points** obtains the control data. This technique is useful when you want to enter data directly.
- **Weights** This dialog window obtains the weights data of rational curves.
- **Knots** This dialog window obtains the knots data of curves.
- **Save** To apply new control data after operation(reduction, elevation). Click the save menu.

4. Control Points Dialog Box

- 3차원 control point 를 등록할 수 있는 $7 * 6$ 입력창을 제공한다. $7 * 6$ 을 초과하는 데이터는 일반 편집기를 이용하여 파일(*.cnt)로 작성하여야 한다.
- order 는 B-spline, NURBS에서 사용하고, degree 는 degree elevation, reduction 에서 사용한다.

- grid By default, grid is set to 50 points. A higher grid will produce more accurate curve and surface plots, but will take longer.
- Scale, TX, TY, Phi, Tht These values control the way the 3-d coordinates of the plot are mapped into the 2-d screen space where Phi and Tht control the rotation angles (in degrees) along a virtual 3-d coordinate system. Phi is bounded to the [0:180] range with a default of 60 degrees, while Tht is bounded to the [0:360] range with a default of 30 degrees. Scale controls the scaling of the entire surfaces. Translations cause an object to be displaced in a specific direction by a specific amount.
- Click the x-axis and y-axis check box. The 는 rectangle 곡면에서 degree reduction, elevation 할 축을 지정한다. 즉, x-axis 를 선택하면 x 축으로 reduction, elevation operation 을 행한다.
- Click the triangle check box. Then the Bezier command draw Bezier triangular surfaces.

5. Bézier Curve and Surfaces

- Click control points anywhere on the window. The Bezier command draw a bezier curve. Now you can click on any of the control points that you have already created and drag it around. Notice how the Bezier curve is redrawn according to the new control points.

DEFINITION 1 (BÉZIER CURVE) *A Bézier curve of degree n can be represented by*

$$b(u) = \sum_{i=0}^{n} b_i B_{i,n}(u),\ 0 \le u \le 1$$

where the $\{b_i\}$ are the **control points***(forming a* **control polygon***), and $\{B_{i,n}(u)\}$ are the nth-degree Bernstein polynomials given by*

$$B_{i,n}(u) = \binom{n}{i}(1-u)^{n-i}u^i.$$

- The degree elevation command increase the flexibility of the control polygon by adding another vertex to it. We may repeat this process

PROBLEM 1 (DEGREE ELEVATION[1]) *Find another points set $\{b_i^{(r)}\}_{i=0}^{n+r}$ defining the Bézier curve of higher degree $n+r$ so that the shape of the curve unchanged.*

- The degree reduction is the process of elevation. In general, exact degree reduction is not possible. Therefore, degree reduction can only be viewed as a method to approximate a given curve by one of lower degree. We have several schemes producing solutions for this approximation problem. Simple([1] p.51), Eck([2]), LSE([4]). Constrained LSE([5]).

PROBLEM 2 (DEGREE REDUCTION[1]) *Find another points set* $\{a_i\}_{i=0}^{m}$ *defining the approximative Bézier curve* $a(u) = \sum_{i=0}^{m} a_i B_{i,m}(u),\ 0 \le u \le 1$ *of lower degree* $m < n$ *so that a suitable distance function* $d(b^n, a^m)$ *between* b^n *and* a^m *is minimized*

PROBLEM 3 (L_2 DEGREE REDUCTION[2,3,4]) *Find another points set* $\{a_i\}_{i=0}^{m}$ *so that the least squares distance*

$$d_2(b^n, a^m) = \sqrt{\int_0^1 ||b^n(t) - a^m(t)||^2 dt}$$

between b^n *and* a^m *is minimized, where* $||\cdot||$ *denotes the Euclidean distance.*

THEOREM 1 (L_2 DISTANCE[4]) *The* L_2 *distance between the two Bézier curves* b^n *and* a^m *is*

$$d_2(b^n, a^m) = d_2(b^n, a^{(r)}) = \sqrt{D^t Q_n D},$$

where $n - m = r$, $D = B - T_{m,r}A$, $B = (b_0, \ldots, b_n)^t$ *and* $A = (a_0, \ldots, a_m)^t$.

PROBLEM 4 (CONSTRAINED L_2 DEGREE REDUCTION[5]) *Suppose we wish to choose the vector A using the least-squares method where C is subjected to cosistent linear-equality restrictions. Then we have the problem*

$$\begin{array}{ll} \textit{Minimize} & D^t Q_n D \\ \textit{subject to} & CA = R \end{array}$$

EXAMPLE 1 (C_0 CONDITION[5]) $n = 4$, $m = 3$, *degree* $= 1$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}, \text{and } R = \begin{pmatrix} b_0 \\ 0 \\ 0 \\ b_4 \end{pmatrix}$$

EXAMPLE 2 (C_1 CONDITION[5]) $n = 4$, $m = 3$, *degree* $= 1$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}, \text{and } R = \begin{pmatrix} b_0 \\ lt \\ rs \\ b_4 \end{pmatrix}$$

where l,r : left, right direction vectors and t,s : scalars

EXAMPLE 3 (C_2 CONDITION[5]) *$n = 7$, $m = 6$, degree $= 1$ C_2 condition*

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, A = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix}, \text{and } R = \begin{pmatrix} b_0 \\ l_1 t_1 \\ l_2 t_2 \\ 0 \\ r_2 s_2 \\ r_1 s_1 \\ b_7 \end{pmatrix}$$

where l_1, r_1 : left, right 1th derivative vectors, l_2, r_2 : left, right 2th derivative vectors, and t_1, t_2, s_1, s_2 : scalars.

- **File** Select the Surfaces menu from the File menu. Define the 3-dim control points and data in Control Points dialog window. Then the Bezier command draw Bezier surfaces.

DEFINITION 2 (BÉZIER RECTANGLE) *A Bézier rectangle of degree (m, n) can be represented by*

$$\mathbf{b}^{m,n}(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{b}_{i,j} B_i^m(u) B_j^n(v), \ 0 \le u, v \le 1,$$

where $\mathbf{b}_{i,j}$ $(i = 0, \cdots, m, j = 0, \cdots, n)$ are control points.

DEFINITION 3 (BÉZIER TRIANGULAR) *A Bézier triangular can be written in terms of Bernstein polynomials:*

$$\mathbf{b}^n(\mathbf{u}) = \sum_{|\mathbf{i}|=n} \mathbf{b}_{\mathbf{i}} B_{\mathbf{i}}^n(\mathbf{u}),$$

where $B_{\mathbf{i}}^n(\mathbf{u}) = \binom{n}{\mathbf{i}} u^i v^j w^k = \frac{n!}{i!j!k!} u^i v^j w^k$ are the bivariate Bernstein polynomials, and $\mathbf{u} = (u, v, w)$ are barycentric coordinates with respect to the vertices of the triangle. The $\mathbf{b}_{\mathbf{i}} \in R^3$ are the triangular array of control points which define the surface.

- We can also use degree elevation and reduction operation in Bézier surface

PROBLEM 5 (L_2 DEGREE REDUCTION OF BÉZIER RECTANGULAR[6]) *Find another points set $\{\mathbf{a}_{i,j}\}_{i,j=0}^{p,q}$ so that the least squares distance*

$$d_2(\mathbf{b}^{m,n}, \mathbf{a}^{p,q}) = \sqrt{\int_0^1 \int_0^1 ||\mathbf{b}^{m,n}(u, v) - \mathbf{a}^{p,q}(u, v)||^2 du dv}$$

between $\{\mathbf{b}_{i,j}\}_{i,j=0}^{m,n}$ and $\{\mathbf{a}_{i,j}\}_{i,j=0}^{p,q}$is minimized.

PROBLEM 6 (L_2 DEGREE REDUCTION OF BÉZIER TRIANGULAR[6]) *Find another points set* $\{\mathbf{a_i}\}_{|\mathbf{i}|=m}$ *so that the least squares distance*

$$d_2(\mathbf{b}^n, \mathbf{a}^m) = \sqrt{\oint ||\mathbf{b}^n(\mathbf{u}) - \mathbf{a}^m(\mathbf{u})||^2 d\mathbf{u}}$$

between $\{\mathbf{b_i}\}_{|\mathbf{i}|=n}$ *and* $\{\mathbf{a_i}\}_{|\mathbf{i}|=m}$*is minimized.*

[1] G. FARIN , *Curves and Surfaces in Computer Aided Geometric Design* ,Academic Press , New York , 1988.
[2] M.ECK, *Least squares degree reduction of Bézier curves*, Comput. Aided Des., 27 (1995), pp. 8455–851.
[3] BYUNG-GOOK LEE AND YUNBEOM PARK, *The L_2 norm of Bézier curves*, Korean J. Com. & Appl. Math. Vol. 3 (1996), pp. 245–252.
[4] BYUNG-GOOK LEE AND YUNBEOM PARK, *The Distance for Bézier Curves and Degree reduction*, Bull. of the Australian Math. Soc. 56 (1997), pp. 507–515.
[5] BYUNG-GOOK LEE AND YUNBEOM PARK, *Constrained Degree reduction of Bézier Curves*, TBA.
[6] BYUNG-GOOK LEE AND YUNBEOM PARK, *Least Squares Degree Reduction of Bézier Rectangles and Triangles*, TBA.

6. Rational Bézier Curves and Surfaces

- control point 들을 먼저 선택한 후 **Rational Bezier** 선택하면 rational Bézier curve 를 그려준다. rational Bézier curve 의 정의는 아래와 같다.

DEFINITION 4 (RATIONAL BÉZIER CURVE) *A rational Bézier Curve can be written in terms of Bernstein polynomials:*

$$b(u) = \frac{\sum_{i=0}^{n} w_i b_i B_{i,n}(u)}{\sum_{i=0}^{n} w_i B_{i,n}(u)},\ 0 \le u \le 1$$

where the w_i *are the* **weights**.

- 이때 weight 들은 **Weights** 에서 지정할 수 있으며, 기본 값은 1 이다. rational Bézier curve 에서 degree elevation 을 수행하려면 **Degree elevation** 을 선택하면 된다.
- **Bezier approximation**에서는 rational Bézier curve를 non-rational Bézier curve 로 근사한 곡선을 찾아 준다. 다음 3 가지 방법으로 구할 수 있다. **Least-square method, Consistent equation** and **Hybrid Bézier curve**.
- **Least-square method** 는 만약 degree 4 인 rational Bézier curve 에서 degree 를 2 로 지정하여 approximation 하면 degree 가 6인 non-rational Bézier curve 중 approximation dimension 만큼 degree elevation 한 control point 들의 L_2 distance 가 가장 적은 곡선을 찾아 준다.

PROBLEM 7 (APPROXIMATION BY NONRATIONAL BÉZIER CURVE[7,8]) *Find another points set* $\{a_i\}_{i=0}^m$ *defining the approximative polynomial Bézier curve of higher degree* $m > n$ *so that the least squares distance function*

$$d_2(b^n, a^m) = \sqrt{\int_0^1 \|b^n(t) - a^m(t)\|^2 dt}$$

between b^n *and* a^m *is minimized on the interval* $[0, 1]$.

PROBLEM 8 (CONSISTENT EQUATION[8]) *In general, we will not be able to converse rational Bézier curve to polynomial one. But, we can be approximately represented as a polynomial Bézier curve of sufficiently large degree* m*,*

$$\frac{\sum_{i=0}^n w_i b_i B_i^n(t)}{\sum_{i=0}^n w_i B_i^n(t)} \approx \sum_{j=0}^m a_j B_j^m(t) \text{ for some large } m.$$

PROBLEM 9 (HYBRID BÉZIER CURVE[9]) *Any rational Bézier curve can be split into a polynomial and a rational part as follows. The polynomial part can be of arbitrary degree* m*. But one of the polynomial control points,* p_s*, is given special treatment: the point* p_s *will not be constant; instead it will vary along a rational Bézier curve of the original degree* n *and with the original weights* w_i*. The hybrid form of a rational Bézier curve takes the form*

$$b^n(t) = \sum_{i=0, i!=s}^m p_i B_i^m(t) + \frac{\sum_{i=0}^n w_i c_i B_i^n(t)}{\sum_{i=0}^n w_i B_i^n(t)} B_s^m(t)$$

- **File** 메인 메뉴에서 **Surfaces** 를 선택한후 **Control points** 를 이용하여 3차원 데이터값을 입력한 후 **Rational Bezier** 선택하면 rational Bézier surface 를 그려준다. rational Bézier surface 의 정의는 아래와 같다.

DEFINITION 5 (RATIONAL BÉZIER RECTANGLE) *A rational Bézier rectangle of degree* (m, n) *can be represented by*

$$\mathbf{b}^{m,n}(u, v) = \frac{\sum_{i=0}^m \sum_{j=0}^n w_{i,j} \mathbf{b}_{i,j} B_i^m(u) B_j^n(v)}{\sum_{i=0}^m \sum_{j=0}^n w_{i,j} B_i^m(u) B_j^n(v)},$$

where the $w_{i,j}$ *are the weights.*

DEFINITION 6 (RATIONAL BÉZIER TRIANGULAR) *A rational Bézier triangular can be written in terms of Bernstein polynomials:*

$$\mathbf{b}^n(\mathbf{u}) = \frac{\sum_{|\mathbf{i}|=n} w_{\mathbf{i}} \mathbf{b}_{\mathbf{i}} B_{\mathbf{i}}^n(\mathbf{u})}{\sum_{|\mathbf{i}|=n} w_{\mathbf{i}} B_{\mathbf{i}}^n(\mathbf{u})},$$

where the $w_{\mathbf{i}}$ *are the weights associated with the control vertices.*

[7] Byung-Gook Lee, Kwan-Pyo Ko and YunBeom Park, *The Degree Elevation and L_2 distance for the Rational Bézier curves*, *Dongseo University Theses Collection*, 3 (1997), pp. 75–79.
[8] Byung-Gook Lee and YunBeom Park, *Approximate Conversion of Rational Bézier Curves*, *KSIAM* 2 (1998), pp. 89–94.
[9] T. Sederberg and M. Kakimoto, *Approximating rational curves using polynomial curves.* In G. Farin, editor. *NURBS for Curve and Surface Design*, SIAM, 1991, pp. 149–158.

7. B-spline Curves and Surfaces

- control point 들을 먼저 선택한 후 **B-spine** 선택하면 B-spline Curve 를 그려준다. B-spline curve 의 정의는 아래와 같다.

Definition 7 (B-spline Curve) *A pth degree B-spline curve is defined by*

$$C^n(u) = \sum_{i=0}^{n} P_i N_{i,p}(u),$$

where the $\{P_i\}$ are the control points, and the B-spline basis functions of degree p defined recursively as

$$N_{i,0}(t) = \begin{cases} 1 & u_i \le u < u_{i+1} \\ 0 & o.w. \end{cases}$$

$$N_{i,p}(t) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u)$$

where u_i are the so-called knots forming a knot vector

$$U = \{u_0, u_1, \ldots, u_m\} \quad u_i \le u_{i+1} \quad i = 0, 1, \ldots, m-1.$$

- 이때 **Knots** 을 이용하여 knot vector 를 먼저 지정하여야한다. default 는 uniform knot vector 이다. 언제든지 **Reset** 메뉴를 이용하면 control point 의 개수와 선택된 order 값을 이용하여 uniform knot vector 값을 생성해 준다. 만약 control points 의 개수가 5이고 order 가 4(degree p=3) 이면 5+4=9개의 knots 가 필요하며 다음과 같은 knot vector 를 자동적으로 생성한다 $U = \{0, 0, 0, 0, 1, 2, 2, 2, 2\}$.
- Bspine degree elevation 과 reduction 을 수행하려면 **Degree Elevation, Degree Reduction** 을 선택하면 된다. 이때 elevation 과 reduction degree 는 knots 의 degree 에서 지정할 수 있다.

Problem 10 (B-spline Degree Elevation[11]) *Since a B-spline curve is a piecewise polynomial curve, it must be possible to evaluate its degree from p to $p + r$. That is, there must exist control points $\tilde{P}$ and a knot vector $\tilde{U}$ such that*

$$C^n(u) = \tilde{C}^{\tilde{n}}(u) = \sum_{i=0}^{\tilde{n}} \tilde{P}_i N_{i,p+r}(u)$$

The computing of $\tilde{n}, \tilde{P}$, and $\tilde{U}$ is referred to as degree elevation

PROBLEM 11 (B-SPLINE DEGREE REDUCTION[11]) *Find another points set* $\{Q_i\}_{i=0}^{l}$ *defining the approximative B-spline curve* $C_Q^l(u)$ *of lower degree* $q < p$ *so that the least square distance function*

$$d_{LS}(C^n, C_Q^l) = \sqrt{\int ||C^n(u) - C_Q^l(u)||^2 du}$$

between C^n *and* C_Q^l *is minimized.*

- Chaikin's method 에서는 Yamaguch "Curves and Surfaces in Computer Aided Geometric Design" P.322의 Fig 6.54 방법을 Bspline-Knots 에서 지정한 Chaikin's 의 값만큼 반복하여 수행한다.
- **Chaikin's knot elevation** 를 선택하면 uniform knot vector 인 경우 uniform 을 유지하면서 knot elevation 을 할 수 있다. 즉, order 가 4인 경우, $\{0,0,0,0,1,2,\ 2,2,2\}$ 에서 $\{0,0,0,0,1/2,1,3/2,2,2,2,2\}$ 로 2개의 knot elevation 이 이루어진다. 0 과 1 사이에 1/2, 1 과 2 사이에 3/2 추가. control points 의 증가개수는 $4 \rightarrow 5$, $5 \rightarrow 7$, $6 \rightarrow 9$, $7 \rightarrow 11$, $n-3$ 개씩 증가한다[12].
- **Chaikin's knot reduction** 를 선택하면 **knot elevation** 역으로 수행한다.
- **Knot insert, Knot delete** 에서는 B-spline curve 에서 knot 을 insert 와 delete 를 수행한다. insert 나 delete 할 knot 의 값은 **Knots** 에서 지정한다.
- **File** 메인 메뉴에서 **Surfaces** 를 선택한후 **Control Points** 를 이용하여 3차원 데이터값을 입력하고 **Knots** 에서 knot vector 값을 지정한 후 B-spline 선택하면 B-spline surface 를 그려준다. B-spline surface 의 정의는 아래와 같다.

DEFINITION 8 (B-SPLINE SURFACE) *A B-spline surface is obtained by taking a bidirectional net of control points, two knot vectors, and the products of the univariate B-spline functions*

$$S(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} P_{i,j} N_{i,p}(u) N_{j,q}(v)$$

with

$$U = \{\underbrace{0,0,\ldots,0}_{p+1}, u_{p+1}, \ldots, u_{r-p-1}, \ldots, \underbrace{1,1,\ldots,1}_{p+1}\}$$

$$V = \{\underbrace{0,0,\ldots,0}_{q+1}, v_{q+1}, \ldots, v_{s-q-1}, \ldots, \underbrace{1,1,\ldots,1}_{q+1}\}$$

U has $r+1$ *knots, and V has* $s+1$.

[10] *L.Piegl and W.Tiller, The NURBS Book, P.141-161, Springer, 1995*
[11] *B.G.Lee and Y.Park, Degree elevation and reduction of B-spline curves, TBA*
[12] *B.G.Lee and Y.Park, Uniform B-spline curve generation by geometrical processing, TBA*

8. NURBS Curves and Surfaces

- control point 들을 먼저 선택한 후 **NURBS** 선택하면 NURBS Curve 를 그려 줍니다. 이때 **Knots** 와 **Weights** 를 이용하여 knot vector 와 weights 들을 먼저 지정하여야한다. default 는 uniform knot vector 와 nonrational (weights=1) 이다. NURBS(Non-Uniform Rational B-Spline) Curve 의 정의는 아래와 같다.

Definition 9 (NURBS curve) *A pth-degree NURBS curve is defined by*

$$C^n(u) = \frac{\sum_{i=0}^{n} w_i P_i N_{i,p}(u)}{\sum_{i=0}^{n} w_i N_{i,p}(u)} \quad 0 \le u \le 1$$

where the $\{w_i\}$ *are the weights, the* $\{P_i\}$ *are the control points (forming a control polygon), and* $\{N_{i,p}\}$ *are the pth-degree B-spline basis functions defined on the nonuniform knot vector*

$$U = \{0, \dots, 0, u_{p+1}, \dots, u_{m-p-1}, 1, \dots, 1\}.$$

- Degree elevation 과 reduction 을 수행하려면 **Degree elevation, Degree reduction** 을 선택하면 된다. 이때 elevation 과 reduction degree 는 **Knots** 에서 지정할 수 있다.
- **Knot insert, Knot delete** 에서는 NURBS curve 에서 knot insert 와 delete 를 수행한다. Insert 나 delete 할 knot 의 값은 **Knots** 에서 지정한다.
- **File** 메인 메뉴에서 **Surfaces** 를 선택한후 **Control Points** 를 이용하여 3차원 데이터값을 입력한 후 **BURBS** 선택하면 NURBS surface 를 그려준다. NURBS surface 의 정의는 아래와 같다.

Definition 10 (NURBS Surface) *A NURBS surface of degree p in the u direction and degree q in the v direction is a bivariate vector-valued piecewise rational function of the form*

$$S(u,v) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u) N_{j,q}(v) w_{i,j} P_{i,j}}{\sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u) N_{j,q}(v) w_{i,j}}$$

The $\{P_{i,j}\}$ *form a bidirectional control net, the* $\{w_{i,j}\}$ *are the weights, and the* $\{N_{i,p}(u)\}$ *and* $\{N_{j,q}(v)\}$ *are the nonrational B-spline basis functions defined on the knot vectors*

$$U = \{\underbrace{0, 0, \dots, 0}_{p+1}, u_{p+1}, \dots, u_{r-p-1}, \dots, \underbrace{1, 1, \dots, 1}_{p+1}\}$$

$$V = \{\underbrace{0, 0, \ldots, 0}_{q+1}, v_{q+1}, \ldots, v_{s-q-1}, \ldots, \underbrace{1, 1, \ldots, 1}_{q+1}\}$$

where $r = n + p + 1$ *and* $s = m + q + 1$.