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Constrained polynomial degree reduction in the L_2-norm equals best weighted Euclidean approximation of Bézier coefficients

Young Joon Ahn [a,*], Byung-Gook Lee [b], Yunbeom Park [c], Jaechil Yoo [d]

[a] *Department of Mathematics Education, Chosun University, Gwangju, 501-759, South Korea*
[b] *Division of Internet Engineering, Dongseo University, Busan, 617-716, South Korea*
[c] *Department of Mathematics Education, Seowon University, Cheongju, 361-742, South Korea*
[d] *Department of Mathematics, Dongeui University, Busan, 614-714, South Korea*

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Abstract

In this paper we show that the best constrained degree reduction of a given Bézier curve f of degree from n to m with $C^{\alpha-1}$-continuity at the boundary in L_2-norm is equivalent to the best weighted Euclidean approximation of the vector of Bernstein–Bézier (BB) coefficients of f from the vector of degree raised BB coefficients of polynomials of degree m with $C^{\alpha-1}$-continuity at the boundary.

Keywords: Constrained degree reduction; Legendre; Bernstein; Bézier curves; Weights

1. Introduction

Degree reduction of Bézier curves is an important problem in CAGD (Computer Aided Geometric Design) or CAD/CAM. In general, degree reduction cannot be done exactly so that it invokes approximation problems. Thus many efforts and proposals for dealing with the problems have been made in the recent twenty years or so. They are classified by different norm in which the distance between polynomials is measured, e.g., in L_∞-norm (Eck, 1993; Lachance, 1988; Watkins and Worsey, 1988), in L_2-norm (Lee and Park, 1997; Lee et al., 2002; Lutterkort et al., 1999; Peters and Reif, 2000), in L_1-norm (Kim and Moon, 1997) or in L_p-norm (Brunnett et al., 1996; Kim et al., 1996), etc. Furthermore, if the error is larger than prespecified tolerance, then subdivision schemes are needed and the best degree-reduced Bézier curves are not continuous at the subdivision points. In many cases of actual CAD/CAM

* Corresponding author.
E-mail address: ahn@chosun.ac.kr (Y.J. Ahn).

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systems, it is required that the curves and surfaces are continuous of order $\alpha \geqslant 1$ (Farin, 1996). Thus the constrained degree reduction of Bézier curves with $C^{\alpha-1}$-continuity at both end points has been developed in many previous literature (Bogaki et al., 1995; Eck, 1995; Kim and Ahn, 2000; Lachance, 1991).

Recently, Yong et al. (2001) presented an algorithm for degree reduction of B-spline using constrained optimization method without subdivision of B-spline into Bézier pieces. Chen and Wang (2002) gave the best multiple degree reduction of Bézier curve with constraints of endpoints continuity in L_2-norm at a time avoiding stepwise computing. Ahn (2003) also presented a good degree reduction of Bézier curve with constraints of endpoints continuity in L_∞-norm.

In particular, Lutterkort et. al (1999) proved that the orthogonal complement of a subspace in the polynomial space of degree n with respect to the L_2-inner product and the Euclidean inner product of BB coefficients are equal. Using this fact they also showed that the best degree reduction of polynomial f of degree n in L_2-norm is equivalent to the best approximation of the vector of BB coefficients of f from all vector of BB coefficients of degree elevated polynomials of degree less than n in the Euclidean norm of the vector. We follow their results in the case of constrained degree reduction. We first show that the orthogonal complement of a subspace in the constrained polynomial space of degree n with respect to L_2-inner product and the weighted Euclidean inner product of BB coefficients are equal for some weights. The weights appear in the representation of the constrained Legendre polynomials in BB coefficients. Using the fact we also show that the best constrained degree reduction of f of degree n with $C^{\alpha-1}$-continuity at boundary in L_2-norm is equivalent to the best approximation of the vector of coefficients from all vectors of coefficients of degree elevated polynomials with $C^{\alpha-1}$-continuity at boundary in weighted Euclidean norm of vectors.

The outline of this paper is as follows. In Section 2, we explain the constrained Legendre polynomials, and their representation in BB form using weights. In Section 3, we show that the orthogonal complement of a subspace in the constrained polynomial space of degree n with respect to L_2-inner product and the weighted Euclidean inner product of BB coefficients are equal. In Section 4, we present a property of the best constrained degree reduction of Bézier curves using the weights. In Section 5, our results are also valid in the case of the best asymmetric constrained degree reduction. In Section 6, our assertions are illustrated.

2. Constrained Legendre polynomials

In this section we explain the best constrained degree reduction of Bézier curve in L_2-norm. It is well known (Watson, 1980; Cheney, 1982) that the best approximation of a given Bézier curve $f(t)$ of degree n by the Bézier curve of degree $n-1$ in L_2-norm is

$$\bar{f}(t) = f(t) - a_n l_n(2t-1), \quad t \in [0,1],$$

where a_n is the leading coefficient of $f(t)$ and

$$l_n(t) = \sum_{i=0}^{n} (-1)^{n+i} \binom{n}{i} B_i^n\left(\frac{t+1}{2}\right)$$

is the Legendre polynomial of degree n. However, the best approximation $\bar{f}(t)$ does not agree with $f(t)$ for $t=0$ and 1. Eck (1995) extended the concept in such a way that the first $\alpha-1$ derivatives of the two curves $f(t)$ and $\bar{f}(t)$ at $t=0$ and 1 agree for the integer $\alpha \geqslant 0$. Then the best approximation of given Bézier curve $f(t)$ of degree n by the Bézier curve of degree $n-1$ in L_2-norm satisfying

$$\frac{\mathrm{d}^i f}{\mathrm{d}t^i}(t) = \frac{\mathrm{d}^i \bar{f}}{\mathrm{d}t^i}(t), \quad i = 0, \ldots, \alpha - 1$$

at $t=0$ and 1 is

$$\bar{f}(t) = f(t) - a_n^\alpha l_n^\alpha(2t-1), \quad t \in [0,1],$$

where $l_n^\alpha(t)$ is called by *constrained Legendre polynomial* (Eck, 1995) and has the Bernstein–Bézier representation

$$l_n^\alpha(t) = \sum_{i=\alpha}^{n-\alpha} (-1)^{n+i} \frac{\binom{n}{i}}{w_i} B_i^n\left(\frac{t+1}{2}\right)$$

with the weights

$$w_i = \frac{\binom{n}{i}^2}{\binom{n}{i-\alpha}\binom{n}{i+\alpha}}.$$

Here a_n^α must be chosen so that $\bar{f}(t)$ is a polynomial of degree $n-1$. The weights play an important role in the following sections.

3. Equivalence of orthogonal complements

Let $\mathbb{P}_n$ be the linear space of polynomials of degree less than or equal to n. Also for $m < n$ and $\alpha = 0, \ldots, [m/2]+1$, let

$$\mathbb{P}_m^\alpha = \left\{ f(t) \in \mathbb{P}_m\colon \frac{\mathrm{d}^i f}{\mathrm{d}t^i}(t) = 0 \text{ at } t = 0, 1 \text{ for } i = 0, \ldots, \alpha - 1 \right\},$$
$$\mathbb{Q}_m^\alpha = \left\{ f(t) \in \mathbb{P}_m\colon f(i) = 0 \text{ for } i = 0, \ldots, \alpha - 1 \text{ and } i = n - \alpha + 1, \ldots, n \right\}.$$

Note that $\mathbb{P}_m^0 = \mathbb{Q}_m^0 = \mathbb{P}_m$. Let B^n and Q^n be the row vectors of Bernstein polynomials and Lagrange polynomials

$$B^n := [B_0^n, \ldots, B_n^n], \quad \text{where } B_i^n(t) := \binom{n}{i} t^i (1-t)^{n-i},$$

$$Q^n := [Q_0^n, \ldots, Q_n^n], \quad \text{where } Q_i^n(t) := \prod_{j=0,\ j\neq i}^{n} \frac{t-j}{i-j}.$$

With $b \in \mathbf{R}^{n+1}$, a column vector of coefficients, we write polynomials in BB form and Lagrange form as $B^n b$ and $Q^n b$, respectively. The latter form is used to relate a discrete polynomial dependence of the coefficients of the vector index to a continuous polynomial. For example, if the coefficients $b(i) = i(i-1)(i-n)(i-n+1)(i+\sqrt{3})$ depend quintically on the index i, then $Q^n(t)b = t(t-1)(t-n)(t-n+1)(t+\sqrt{3})$ is the corresponding quintic polynomial. The following lemma is an extension of Lemma 2.1 in (Lutterkort et al., 1999).

Lemma 3.1. *A polynomial $B^n b$ is of degree $\leqslant m$ with $b(i)=0$ for $i=0,\dots,\alpha-1$ and $i=n-\alpha+1,\dots,n$ if and only if the vector of coefficients is a polynomial of degree $\leqslant m$ with zeros at $i=0,\dots,\alpha-1$ and $i=n-\alpha+1,\dots,n$ in its index, i.e.,*

$$B^n b \in \mathbb{P}_m^\alpha \quad \Leftrightarrow \quad Q^n b \in \mathbb{Q}_m^\alpha.$$

Proof. It is well known (Lutterkort et al., 1999) that

$$B^n b \in \mathbb{P}_m^0 \quad \Leftrightarrow \quad Q^n b \in \mathbb{Q}_m^0.$$

Thus it suffices to show that $\frac{\mathrm{d}^i}{\mathrm{d}t^i} B^n(t)b = 0$ at $t=0,1$ for $i=0,\dots,\alpha-1$ if and only if $Q^n(i)b=0$ for $i=0,\dots,\alpha-1$ and $i=n-\alpha+1,\dots,n$. By the way, they are equivalent to $b(i)=0$ for $i=0,\dots,\alpha-1$ and $i=n-\alpha+1,\dots,n$. Hence we have $B^n b\in\mathbb{P}_m^\alpha$ if and only if $Q^n b\in\mathbb{Q}_m^\alpha$. □

Theorem 3.2. *The orthogonal complements of $\mathbb{P}_m^\alpha$ in $\mathbb{P}_n^\alpha$ with respect to the L_2-inner product*

$$\langle f, g\rangle_L := \int_0^1 f(t)g(t)\,\mathrm{d}t \tag{1}$$

and the weighted Euclidean inner product of the BB coefficients

$$\langle B^n b, B^n c\rangle_w := \sum_{i=\alpha}^{n-\alpha} b_i c_i w_i \tag{2}$$

are equal, where

$$w_i = \frac{\binom{n}{i}^2}{\binom{n}{i-\alpha}\binom{n}{i+\alpha}}.$$

Proof. Denote the orthogonal complement of $\mathbb{P}_m^\alpha$ in $\mathbb{P}_n^\alpha$ with respect to the weighted Euclidean inner product by $\mathbb{P}_{m,n}^\alpha$, and let $B^n u_{m-2\alpha+1},\dots,B^n u_{n-2\alpha}$ be some basis of this space. By equality of dimensions it suffices to show that $\mathbb{P}_{m,n}^\alpha$ is contained in the orthogonal complement with respect to the L_2-inner product, i.e., the polynomials $B^n u_k$ have to be L_2-orthogonal to all polynomials in $\mathbb{P}_m^\alpha$,

$$\left\langle B^n u_j, t^{\alpha+i}(1-t)^\alpha\right\rangle_L = 0, \quad 0\leqslant i\leqslant m-2\alpha < j\leqslant n-2\alpha.$$

Defining the column vector p_i by

$$p_i(k) := \frac{1}{w_k}\int_0^1 B_k^n(t)t^{\alpha+i}(1-t)^\alpha\,\mathrm{d}t,$$

we rewrite $\langle B^n u_j, t^{\alpha+i}(1-t)^\alpha\rangle_L = \langle B^n u_j, B^n p_i\rangle_w$. By definition, the latter expression vanishes if and only if $B^n p_i\in\mathbb{P}_m^\alpha$, and by Lemma 3.1, this is equivalent to $Q^n p_i\in\mathbb{Q}_m^\alpha$. In other words, we have to show that $p_i(k)$ is a polynomial in k of degree $\leqslant m$ with zeros at $k=0,\dots,\alpha-1$, and $n-\alpha+1,\dots,n$ for all $i=0,\dots,m-2\alpha$. We have

$$p_i(k) = \frac{1}{w_k}\int_0^1 B_k^n(t)t^{\alpha+i}(1-t)^\alpha\,\mathrm{d}t = \frac{\binom{n}{k-\alpha}\binom{n}{k+\alpha}}{\binom{n+2\alpha+i}{k+\alpha+i}\binom{n}{k}}\int_0^1 B_{k+\alpha+i}^{n+2\alpha+i}(t)\,\mathrm{d}t = \frac{\binom{n}{k-\alpha}\binom{n}{k+\alpha}}{\binom{n+2\alpha+i}{k+\alpha+i}\binom{n}{k}}\frac{1}{n+2\alpha+i+1}$$

$$= \frac{n!}{(n+2\alpha+i+1)!}\prod_{l_1=1}^{\alpha}(k-\alpha+l_1)\prod_{l_2=1}^{\alpha}(n-k-\alpha+l_2)\prod_{l_3=1}^{i}(k+\alpha+l_3)$$

using the formula

$$\int_0^1 B_k^d(t)\,\mathrm{d}t = \frac{1}{d+1}.$$

Thus $p_i(k)$ is a polynomial of degree $\leqslant m$ and $p_i(k)=0$ for $k=0,\dots,\alpha-1$ and $n-\alpha+1,\dots,n$. □

4. Constrained degree reduction

Corollary 4.1. *Given a polynomial $B^n b$ of degree n, the approximation problem*

$$\min_{p\in\mathbb{P}_m}\left\{\|B^n b - p\|:\ \frac{\mathrm{d}^i}{\mathrm{d}t^i}B^n(t)b = \frac{\mathrm{d}^i}{\mathrm{d}t^i}p(t) \text{ at } t=0,1 \text{ for } i=0,\dots,\alpha-1\right\}$$

has the same minimizer for the norm induced either by the L_2-inner product (1) *or the weighted Euclidean inner product* (2).

Proof. Let f^α be a polynomial of degree m satisfying

$$\frac{\mathrm{d}^i}{\mathrm{d}t^i}B^n(t)b = \frac{\mathrm{d}^i}{\mathrm{d}t^i}f^\alpha(t)$$

at $t=0,1$ for $i=0,\dots,\alpha-1$. Then the polynomial $B^n b - f^\alpha \in \mathbb{P}_n^\alpha$ can be decomposed uniquely according to

$$B^n b - f^\alpha = p^\alpha + q^\alpha, \quad p^\alpha \in \mathbb{P}_m^\alpha,\ q^\alpha \in \mathbb{P}_{m,n}^\alpha$$

and, by the orthogonality, p^α is the minimizer of

$$\min_{p^\alpha\in\mathbb{P}_m^\alpha}\|B^n b - f^\alpha - p^\alpha\|$$

for both norm. For all $p\in\mathbb{P}_m$ satisfying

$$\frac{\mathrm{d}^i}{\mathrm{d}t^i}B^n(t)b = \frac{\mathrm{d}^i}{\mathrm{d}t^i}p(t)$$

at $t=0,1$ for $i=0,\dots,\alpha-1$, we have

$$\|B^n b - p\| = \left\|B^n b - f^\alpha - (p - f^\alpha)\right\| \geqslant \|B^n b - f^\alpha - p^\alpha\|$$

since $p - f^\alpha \in \mathbb{P}_m^\alpha$. Thus $p = p^\alpha + f^\alpha \in \mathbb{P}_m$ is the wanted solution for both norms. □

Corollary 4.2. *Denote by $\mathcal{P}_{m,n}^\alpha$ the linear operator mapping polynomials $B^n b\in\mathbb{P}_n$ to their best constrained L_2-norm or weighted Euclidean approximant $p\in\mathbb{P}$. Then*

$$\mathcal{P}_{m,n}^\alpha = \mathcal{P}_{m,\ell}^\alpha\mathcal{P}_{\ell,n}^\alpha, \quad m\leqslant\ell\leqslant n.$$

5. Asymmetric constrained degree reduction

In this section, we extend the results in above sections to the case of asymmetric constraint. For the nonnegative integers α and β satisfying $\alpha+\beta \leqslant m+1$, let

$$\mathbb{P}_m^{\alpha,\beta} = \left\{ f(t) \in \mathbb{P}_m \colon \frac{\mathrm{d}^i f}{\mathrm{d}t^i}(0) = 0 \text{ for } i=0,\dots,\alpha-1, \text{ and } \frac{\mathrm{d}^i f}{\mathrm{d}t^i}(1) = 0 \text{ for } i=0,\dots,\beta-1 \right\}$$
$$\mathbb{Q}_m^{\alpha,\beta} = \left\{ f(t) \in \mathbb{P}_m \colon f(i) = 0 \text{ for } i=0,\dots,\alpha-1 \text{ and } i=n-\beta+1,\dots,n \right\}.$$

The following lemma follows directly from Lemma 3.1.

Lemma 5.1. $B^n b \in \mathbb{P}_m^{\alpha,\beta} \Leftrightarrow Q^n b \in \mathbb{Q}_m^{\alpha,\beta}$.

The proof of the following corollary is also obtained easily from the proof of Theorem 3.2.

Corollary 5.2. *The orthogonal complements of* $\mathbb{P}_m^{\alpha,\beta}$ *in* $\mathbb{P}_n^{\alpha,\beta}$ *with respect to the* L_2*-inner product and the weighted Euclidean inner product of the BB coefficients*

$$\langle B^n b, B^n c\rangle_w := \sum_{i=\alpha}^{n-\beta} b_i c_i w_i \tag{3}$$

are equal, where

$$w_i = \frac{\binom{n}{i}^2}{\binom{n}{i-\alpha}\binom{n}{i+\beta}}. \tag{4}$$

Proof. By simple calculations, we have for $i=0,\dots,m-\alpha-\beta$ and $k=0,\dots,\alpha-1$ and $n-\beta+1,\dots,n$,

$$\begin{aligned} p_i(k) &:= \frac{1}{w_k}\int_0^1 B_k^n(t) t^{\alpha+i}(1-t)^\beta \,\mathrm{d}t \\ &= \frac{n!}{(n+\alpha+\beta+i+1)!}\prod_{l_1=1}^{\alpha}(k-\alpha+l_1)\prod_{l_2=1}^{\beta}(n-k-\beta+l_2)\prod_{l_3=1}^{i}(k+\alpha+l_3). \end{aligned}$$

Thus the assertion follows from the same way of the proof of Theorem 3.2. □

Corollary 5.3. *Given a polynomial* $B^n b$ *of degree* n, *the approximation problem*

$$\min_{p\in\mathbb{P}_m}\left\{ \|B^n b - p\| \colon \frac{\mathrm{d}^i}{\mathrm{d}t^i}B^n(t)b = \frac{\mathrm{d}^i}{\mathrm{d}t^i}p(t) \text{ at } t=0 \text{ for } i=0,\dots,\alpha-1, \right.$$
$$\left. \text{and at } t=1 \text{ for } i=0,\dots,\beta-1 \right\}$$

has the same minimizer for the norm induced either by the L_2*-inner product* (1) *or the weighted Euclidean inner product* (3).

Corollary 5.4. *Denote by $\mathcal{P}_{m,n}^{\alpha,\beta}$ the linear operator mapping polynomials $B^n b \in \mathbb{P}_n$ to their best constrained L_2-norm or the weighted Euclidean approximant $p \in \mathbb{P}$. Then*

$$\mathcal{P}_{m,n}^{\alpha,\beta} = \mathcal{P}_{m,\ell}^{\alpha,\beta}\mathcal{P}_{\ell,n}^{\alpha,\beta}, \quad m \leqslant \ell \leqslant n.$$

6. Examples

In practice, one is often interested in the BB form $p = B^m c$ of the constrained best degree reduction from the polynomial $B^n b$. In order to compare coefficients, p has to be represented in terms of B^n, i.e., $p = B^n c^{(r)}$. The degree raising $(n+1) \times (m+1)$ matrix $T_{m,r}$ for mapping the BB coefficients c to $c^{(r)}$ has elements

$$T_{m,r}(i,j) = \frac{\binom{m}{j}\binom{r}{i-j}}{\binom{m+r}{i}}, \quad i = 0,1,\ldots,m+r \text{ and } j = 0,1,\ldots,m.$$

Then, with $\|.\|_w$ denoting the weighted Euclidean norm (3) in $\mathbf{R}^{n+1}$, degree reduction amounts to solving the least squares problem

$$\min_{c\in\mathbf{R}^{m+1}} \|b - T_{m,r}c\|_w$$

with $C^{\alpha-1}$-continuity at $t = 0$ and $C^{\beta-1}$-continuity at $t = 1$.

To solve the least squares problem explicitly, let

$$d = b - T_{m,r}c$$

where b is the given vector, and c is the unknown vector. Considering $C^{\alpha-1}$-continuity at $t = 0$ and $C^{\beta-1}$-continuity at $t = 1$, we can impose $\frac{n!}{(n-i)!}\Delta^i b_0 = \frac{m!}{(m-i)!}\Delta^i c_0$ for $i = 0,1,\ldots,\alpha-1$ and $\frac{n!}{(n-i)!}\Delta^i b_{n-i} = \frac{m!}{(m-i)!}\Delta^i c_{m-i}$ for $i = 0,1,\ldots,\beta-1$.

With these conditions, we solve $b - T_{m,r}c$ and split it in two parts, namely the known part $\tilde{b}$ and the unknown part $\tilde{c}$ i.e.,

$$\tilde{d} = \tilde{b} - T_{m,r}\tilde{c}.$$

Let $A^{\alpha,\beta}$ be the submatrix of the $(n+1) \times (n+1)$ matrix A obtained by extracting rows α through $(n+1-\beta)$ and columns α through $(n+1-\beta)$ and let $v^{\alpha,\beta}$ be the subvector of the vector v obtained by extracting rows α through $(n+1-\beta)$. Let W_n be the diagonal matrix whose diagonal elements are given by (4). Then the solution $\tilde{c}^{\alpha,\beta}$ is given by the pseudo inverse $P_{m,r}^{\alpha,\beta}$ of the degree raising matrix,

$$\tilde{c}^{\alpha,\beta} = P_{m,r}^{\alpha,\beta}\tilde{b}^{\alpha,\beta} = \left(\left(T_{m,r}^{\alpha,\beta}\right)^T W_n^{\alpha,\beta} T_{m,r}^{\alpha,\beta}\right)^{-1}\left(T_{m,r}^{\alpha,\beta}\right)^T W_n^{\alpha,\beta}\tilde{b}^{\alpha,\beta}.$$

Example 1 ($n = 5$, $r = 1$, C^0*-continuity*, $\alpha = \beta = 1$).

$$d = b - T_{4,1}c,$$

$$d = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix},$$

where $c_0 = b_0$, and $c_4 = b_5$.

$$\tilde{d} = \tilde{b} - T_{4,1}\tilde{c},$$

$$\tilde{d} = \begin{pmatrix} 0 \\ b_1 - \frac{1}{5}b_0 \\ b_2 \\ b_3 \\ b_4 - \frac{1}{5}b_5 \\ 0 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ c_1 \\ c_2 \\ c_3 \\ 0 \end{pmatrix}.$$

$$\tilde{d}^{1,1} = \tilde{b}^{1,1} - T_{4,1}^{1,1}\tilde{c}^{1,1},$$

$$\tilde{d}^{1,1} = \begin{pmatrix} b_1 - \frac{1}{5}b_0 \\ b_2 \\ b_3 \\ b_4 - \frac{1}{5}b_5 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 4 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

$$P_{4,1}^{1,1} = \left(\left(T_{4,1}^{1,1}\right)^T W_5^{1,1} T_{4,1}^{1,1}\right)^{-1}\left(T_{4,1}^{1,1}\right)^T W_5^{1,1} = \begin{pmatrix} \frac{55}{48} & \frac{5}{24} & -\frac{5}{24} & \frac{5}{48} \\ -\frac{5}{12} & \frac{5}{6} & \frac{5}{6} & -\frac{5}{12} \\ \frac{5}{48} & -\frac{5}{24} & \frac{5}{24} & \frac{55}{48} \end{pmatrix},$$

where

$$W_5^{1,1} = \begin{pmatrix} \frac{5}{2} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \frac{5}{2} \end{pmatrix}.$$

$$\tilde{c}^{1,1} = P_{4,1}^{1,1}\tilde{b}^{1,1}.$$

Example 2 ($n = 5$, $r = 1$, $\alpha = 1$, $\beta = 2$).

$$d = b - T_{4,1}c,$$

$$d = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix},$$

where $c_0 = b_0$, $c_3 = b_5 - \frac{5}{4}(b_5 - b_4)$, and $c_4 = b_5$.

$$\tilde{d} = \tilde{b} - T_{4,1}\tilde{c},$$

$$\tilde{d} = \begin{pmatrix} 0 \\ b_1 - \frac{1}{5}b_0 \\ b_2 \\ b_3 - \frac{1}{2}b_4 + \frac{1}{10}b_5 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ c_1 \\ c_2 \\ 0 \\ 0 \end{pmatrix}.$$

$$\tilde{d}^{1,2} = \tilde{b}^{1,2} - T_{4,1}^{1,2}\tilde{c}^{1,2},$$

$$\tilde{d}^{1,2} = \begin{pmatrix} b_1 - \frac{1}{5}b_0 \\ b_2 \\ b_3 - \frac{1}{2}b_4 + \frac{1}{10}b_5 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 4 & 0 \\ 2 & 3 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

$$P_{4,1}^{1,2} = \left(\left(T_{4,1}^{1,2}\right)^T W_5^{1,2} T_{4,1}^{1,2}\right)^{-1}\left(T_{4,1}^{1,2}\right)^T W_5^{1,2} = \begin{pmatrix} \frac{35}{36} & \frac{5}{9} & -\frac{5}{9} \\ -\frac{5}{27} & \frac{10}{27} & \frac{35}{27} \end{pmatrix},$$

where

$$W_5^{1,2} = \begin{pmatrix} \frac{5}{2} & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 10 \end{pmatrix}.$$

$$\tilde{c}^{1,2} = P_{4,1}^{1,2}\tilde{b}^{1,2}.$$

As an example in Fig. 1, consider the polynomial $B^5 b$ with BB coefficients

$$b = [0, 1, 4, 2, 5, 0]^{\mathrm{t}}.$$

The best approximation $B^4 c$ with C^0-continuity at $t = 0$ and C^1-continuity at $t = 1$ has coefficients

$$c = [0, 125/36, 35/54, 25/4, 0].$$

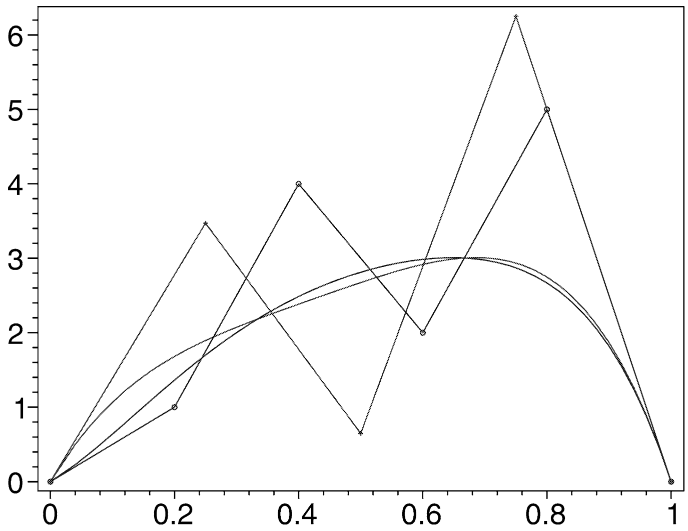

Fig. 1. Degree reduction of Bézier curve from degree five to degree four with constraints $\alpha = 1$ and $\beta = 2$.

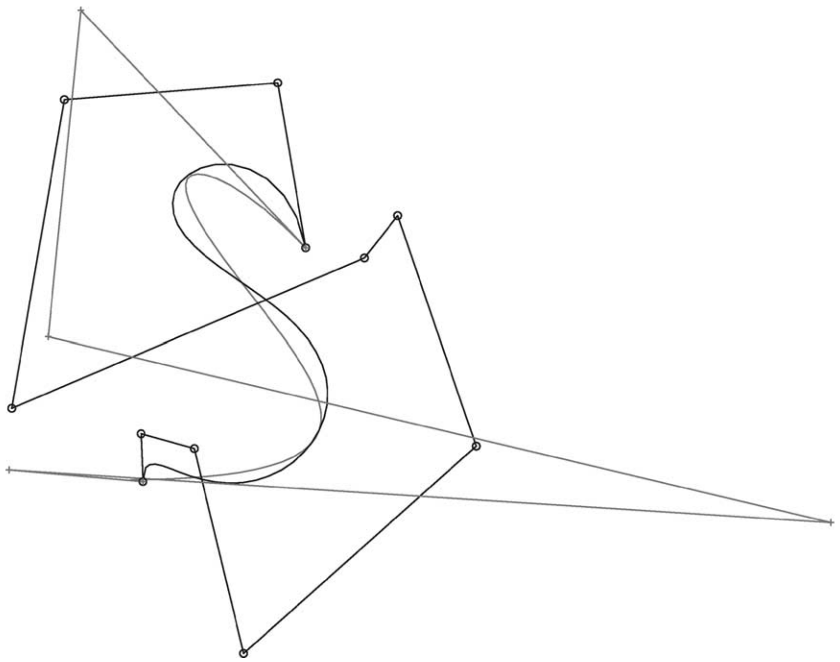

Fig. 2. Degree reduction of planar Bézier curve from degree ten to degree five with constraint $\alpha = \beta = 1$: the small circles are the control points of given Bézier curve of degree ten and the small crosses are the control points of degree-reduced Bézier curve of degree five.

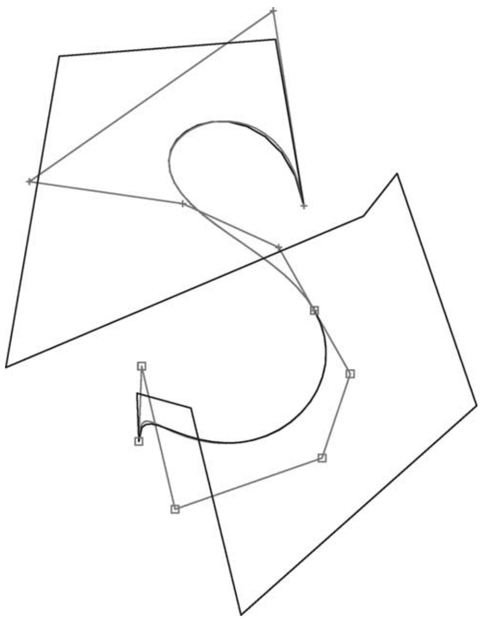

Fig. 3. Degree reductions after subdivision of the planar Bézier curve of degree ten: the small crosses and the small boxes are the control points of the degree-reduced Bézier curves of degree five with constraints $(\alpha, \beta) = (1, 2)$ and $(\alpha, \beta) = (2, 1)$, respectively.

Example 3 ($n = 10$, $r = 5$, $\alpha = 1, \beta = 2$ *and* $\alpha = 2, \beta = 1$). In this example we give the plane Bézier curve of degree ten which is a part of the out-lines of font "S". The best multi-degree reduction ($r = 5$) with C^0-continuity at boundary points ($\alpha = \beta = 1$) is shown in Fig. 2. After the subdivision at $t = 1/2$, two Bézier curve of degree ten are approximated by the Bézier curves of degree five using best multi-degree reduction with C^1-continuity at the subdivision point and C^0-continuity at the boundary points as shown in Fig. 3.

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