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Application of Legendre–Bernstein basis transformations to degree elevation and degree reduction

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Abstract

We study the relationship of transformations between Legendre and Bernstein basis. Using the relationship, we present a simple and efficient method for optimal multiple degree reductions of Bézier curves with respect to the L_2-norm.
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1. Introduction

We can express a polynomial curve with an appropriate basis for its use. The use of orthogonal basis such as Chebyshev and Legendre polynomial permits optimal degree reduction to exchange, convert or reduce data, or compare geometric entities which is an important task in CAGD (Li and Zhang, 1998; Mazure, 1999). For example, we have seen the use of Chebyshev and Legendre polynomial in degree reduction schemes (Watkins and Worsey, 1988; Eck, 1993, 1995). On the other hand, the Bernstein form of a polynomial having the recursive formula and the property of partition of unity offers valuable insight into its geometrical behavior, and has won widespread acceptance as the basis for Bézier curves and surfaces in CAGD (Farin, 1993). But Bernstein polynomials are not orthogonal. So the basis transformation is important and has been studied in many ways. Farouki (2000) found the explicit form of the basis transformation between Legendre and Bernstein basis.

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In this paper, we find the relationships between the Gram matrix Q_n and the Legendre–Bernstein basis transformation matrix M_n, the M_n and the Bernstein–Legendre basis transformation matrix M_n^{-1}, and the orthogonal matrix U_n and M_n. We also obtain the relationships between the basis transformation matrices M_n, M_n^{-1} and the degree elevation matrix T_n, the basis transformation matrices and the degree reduction matrix.

This paper is organized as follows. We explain the degree n Legendre and Bernstein basis, and their transformations in Section 2. We discuss the relationship among transformations, M_n, M_n^{-1}, eigenvalues of Gram matrix and an orthogonal matrix U_n in Section 3. We present the explicit method to degree elevation and degree reduction of Bézier curves in Sections 4 and 5.

2. Legendre and Bernstein basis

The Bézier representation uses Bernstein polynomials as basis functions for the linear space of polynomials. In terms of the Bernstein polynomials of degree n,

$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i, \quad i = 0, 1, \ldots, n,$$

a parametric polynomial curve $P_n(t)$ of degree n $(n > 0)$ in the plane can be expressed as

$$P_n(t) = \sum_{i=0}^{n} c_i B_i^n(t), \quad 0 \leqslant t \leqslant 1,$$

where the $\{c_i\}_{i=0}^n$ are the set of $(n+1)$ control points.

The product of Bernstein polynomials is

$$B_i^n(t)B_j^m(t) = \frac{\binom{n}{i}\binom{m}{j}}{\binom{n+m}{i+j}} B_{i+j}^{n+m}(t) \tag{1}$$

and the integration is

$$\int_0^1 B_k^n(t)\,\mathrm{d}t = \frac{1}{n+1}. \tag{2}$$

The Legendre polynomials constitute an orthonormal basis that is well suited to least-squares approximation. To emphasize symmetry properties they are traditionally defined on the interval $[-1, +1]$, but for our purpose it is preferable to map this to $[0, 1]$. The Legendre polynomials $L_n(t)$ on $t \in [0, 1]$ can be generated by the explicit form

$$L_n(t) = \sqrt{2n+1} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{i, i} \left(t^2 - t\right)^i (2t-1)^{n-2i},$$

where $\binom{n}{i,j} = \frac{n!}{i!j!(n-i-j)!}$. This gives, in the first few instances,

$$L_0(t) = 1,$$
$$L_1(t) = \sqrt{3}(2t-1),$$

$$L_2(t) = \sqrt{5}\bigl(6t^2 - 6t + 1\bigr),$$
$$L_3(t) = \sqrt{7}\bigl(20t^3 - 30t^2 + 12t - 1\bigr).$$

The orthonormality of these polynomials is expressed by the relation

$$\int_0^1 L_j(t)L_k(t)\,\mathrm{d}t = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases} \tag{3}$$

Consider a polynomial $P_n(t)$ of degree n, expressed in the degree n Bernstein and Legendre basis on $t \in [0, 1]$:

$$P_n(t) = \sum_{j=0}^{n} c_j B_j^n(t) = \sum_{k=0}^{n} l_k L_k(t).$$

We are interested in the linear transformation

$$c_j = \sum_{k=0}^{n} M_n(j,k) l_k, \quad j = 0, 1, \ldots, n,$$

that maps the Legendre coefficients $l_0, l_1, \ldots, l_n$ into the Bernstein coefficients $c_0, c_1, \ldots, c_n$, and its inverse. Writing $c = [c_0, c_1, \ldots, c_n]^{\mathrm{t}}$ and $l = [l_0, l_1, \ldots, l_n]^{\mathrm{t}}$, we may express this in vector-matrix form as

$$c = M_n l.$$

Then we have the following theorem, see (Farouki, 2000).

Theorem 1. *The Legendre polynomial $L_k(t)$ can be expressed in the Bernstein basis $B_0^n(t), B_1^n(t), \ldots, B_n^n(t)$ of degree n as*

$$L_k(t) = \sqrt{2k+1} \sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} B_i^k(t)$$
$$= \sum_{j=0}^{n} \frac{\sqrt{2k+1}}{\binom{n}{j}} \sum_{i=\max(0,j+k-n)}^{\min(j,k)} (-1)^{k+i} \binom{k}{i}\binom{k}{i}\binom{n-k}{j-i} B_j^n(t).$$

The elements of the matrix M_n that transforms the Legendre coefficients of degree n polynomials into the Bernstein coefficients according to equation, are given for $0 \leqslant j, k \leqslant n$ by

$$M_n(j,k) = \frac{\sqrt{2k+1}}{\binom{n}{j}} \sum_{i=\max(0,j+k-n)}^{\min(j,k)} (-1)^{k+i} \binom{k}{i}\binom{k}{i}\binom{n-k}{j-i}. \tag{4}$$

For Bernstein to Legendre transformation matrix M_n^{-1}, see (Farouki, 2000).

Theorem 2. *The elements of the inverse M_n^{-1} are given for $0 \leqslant j, k \leqslant n$ by*

$$M_n^{-1}(j,k) = \frac{\sqrt{2j+1}}{n+j+1} \frac{1}{\binom{n+j}{n}} \sum_{i=0}^{j} (-1)^{j+i} \binom{j}{i}\binom{k+i}{k}\binom{n-k+j-i}{n-k}. \tag{5}$$

Example 1.

$$M_1=\begin{bmatrix}1 & -\sqrt{3}\\ 1 & \sqrt{3}\end{bmatrix},\qquad M_1^{-1}=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6}\end{bmatrix},$$

$$M_2=\begin{bmatrix}1 & -\sqrt{3} & \sqrt{5}\\ 1 & 0 & -2\sqrt{5}\\ 1 & \sqrt{3} & \sqrt{5}\end{bmatrix},\qquad M_2^{-1}=\begin{bmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ -\frac{\sqrt{3}}{6} & 0 & \frac{\sqrt{3}}{6}\\ \frac{\sqrt{5}}{30} & -\frac{2\sqrt{5}}{30} & \frac{\sqrt{5}}{30}\end{bmatrix},$$

$$M_3=\begin{bmatrix}1 & -\sqrt{3} & \sqrt{5} & -\sqrt{7}\\ 1 & -\frac{\sqrt{3}}{3} & -\sqrt{5} & 3\sqrt{7}\\ 1 & \frac{\sqrt{3}}{3} & -\sqrt{5} & -3\sqrt{7}\\ 1 & \sqrt{3} & \sqrt{5} & \sqrt{7}\end{bmatrix},\qquad M_3^{-1}=\begin{bmatrix}\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ -\frac{3\sqrt{3}}{20} & -\frac{\sqrt{3}}{20} & \frac{\sqrt{3}}{20} & \frac{3\sqrt{3}}{20}\\ \frac{\sqrt{5}}{20} & -\frac{\sqrt{5}}{20} & -\frac{\sqrt{5}}{20} & \frac{\sqrt{5}}{20}\\ -\frac{\sqrt{7}}{140} & \frac{3\sqrt{7}}{140} & -\frac{3\sqrt{7}}{140} & \frac{\sqrt{7}}{140}\end{bmatrix}.$$

3. L_2-norm of the polynomial P_n

We compute the L_2-norm of a Bézier curve of degree n. From these equations (1) and (2), we obtain the following computation for the L_2-norm of the polynomial P_n with Bernstein basis:

$$\begin{aligned}\|P_n\|_2^2 &= \int_0^1\left|\sum_{i=0}^{n} c_i B_i^n(t)\right|^2 \mathrm{d}t = \int_0^1 \sum_{i,j} c_i c_j B_i^n(t) B_j^n(t)\,\mathrm{d}t\\ &= \sum_{i,j} c_i c_j \int_0^1 \frac{\binom{n}{i}\binom{n}{j}}{\binom{2n}{i+j}} B_{i+j}^{2n}(t)\,\mathrm{d}t = \frac{1}{2n+1}\sum_{i,j} c_i c_j \frac{\binom{n}{i}\binom{n}{j}}{\binom{2n}{i+j}}.\end{aligned}$$

Let the elements of the Gram matrix Q_n of the Bernstein basis be the

$$Q_n(i,j)=\frac{1}{2n+1}\frac{\binom{n}{i}\binom{n}{j}}{\binom{2n}{i+j}},\quad i,j=0,1,\ldots,n. \tag{6}$$

Then the L_2-norm of the polynomial P_n is

$$\|P_n\|_2^2 = c^{\mathrm{t}} Q_n c. \tag{7}$$

Here are some examples of Q_n.

Example 2.

$$Q_1=\begin{bmatrix}\frac{1}{3} & \frac{1}{6}\\ \frac{1}{6} & \frac{1}{3}\end{bmatrix},\qquad Q_2=\begin{bmatrix}\frac{1}{5} & \frac{1}{10} & \frac{1}{30}\\ \frac{1}{10} & \frac{2}{15} & \frac{1}{10}\\ \frac{1}{30} & \frac{1}{10} & \frac{1}{5}\end{bmatrix},\qquad Q_3=\begin{bmatrix}\frac{1}{7} & \frac{1}{14} & \frac{1}{35} & \frac{1}{140}\\ \frac{1}{14} & \frac{3}{35} & \frac{9}{140} & \frac{1}{35}\\ \frac{1}{35} & \frac{9}{140} & \frac{3}{35} & \frac{1}{14}\\ \frac{1}{140} & \frac{1}{35} & \frac{1}{14} & \frac{1}{7}\end{bmatrix}.$$

From the definition of the Gram matrix Q_n and the mathematical induction, all the upper left submatrices of the Gram matrix Q_n have positive determinants. So Q_n is a real symmetric positive definite matrix, see (Lee and Park, 1997). Thus it can be diagonalized by an orthogonal matrix U_n (i.e., $U_n^{-1} = U_n^{\mathrm{t}}$) whose column vectors are orthonormal eigenvectors of Q_n, that is,

$$Q_n = U_n D_n U_n^{\mathrm{t}},$$

where D_n is the diagonal matrix with positive eigenvalues of the matrix Q_n.

The following theorem is the direct result from Proposition 10 in (Lyche and Scherer, 2000).

Theorem 3. *From the Gram matrix Q_n given by* (6) *we have*

$$Q_n M_n = M_n D_n,$$

where $\lambda_k = \frac{1}{2n+1} \frac{\binom{2n+1}{n-k}}{\binom{2n}{n}}$ $(k = 0, 1, \ldots, n)$ *are eigenvalues of the Gram matrix Q_n.*

From the orthonormality (3) of Legendre basis, we obtain the following computation for the L_2-norm of the polynomial P_n with Legendre basis:

$$\|P_n\|_2{}^2 = \int_0^1 \left| \sum_{i=0}^{n} l_i L_i(t) \right|^2 \mathrm{d}t = \sum_{i,j} l_i l_j \int_0^1 L_i(t) L_j(t)\, \mathrm{d}t = l^{\mathrm{t}} l. \tag{8}$$

From Theorem 3, we get the following theorem that describes the relationship among M_n^{-1}, M_n and D_n.

Theorem 4. *For the Bernstein to Legendre transformation matrix M_n^{-1} we have*

$$M_n^{-1} = D_n M_n^{\mathrm{t}}.$$

Proof. From (7) and (8), we have

$$c^{\mathrm{t}} Q_n c = l^{\mathrm{t}} l.$$

By the definition of M_n^{-1}, we can also express the L_2-norm of the polynomial P_n as

$$c^{\mathrm{t}} Q_n c = c^{\mathrm{t}} \big(M_n^{-1}\big)^{\mathrm{t}} M_n^{-1} c. \tag{9}$$

By Theorem 3 and (9) we obtain

$$Q_n = M_n D_n M_n^{-1} = \big(M_n^{-1}\big)^{\mathrm{t}} M_n^{-1}.$$

Multiplying both sides by M_n and considering the transpose of both sides, we complete the proof. □

The following theorem enables us to compute U_n with the explicit forms of M_n and D_n.

Theorem 5. *For the orthogonal matrix U_n of the Gram matrix Q_n we have*

$$U_n = M_n \sqrt{D}_n.$$

Proof. From Theorem 4, we have

$$Q_n = M_n D_n M_n^{-1} = M_n D_n D_n M_n^{\mathrm{t}} = M_n \sqrt{D}_n D_n \left(M_n \sqrt{D}_n\right)^{\mathrm{t}}.$$

And we can easily check the orthogonality of $M_n \sqrt{D}_n$

$$\left(M_n \sqrt{D}_n\right)^{-1} = \sqrt{D}_n^{\;-1} M_n^{-1} = \sqrt{D}_n M_n^{\mathrm{t}} = \left(M_n \sqrt{D}_n\right)^{\mathrm{t}}.$$

This completes the proof. □

Here are some examples of U_n and D_n.

Example 3.

$$U_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad D_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{6} \end{bmatrix},$$

$$U_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}, \qquad D_2 = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{30} \end{bmatrix},$$

$$U_3 = \begin{bmatrix} \frac{1}{2} & -\frac{3}{\sqrt{20}} & \frac{1}{2} & -\frac{1}{\sqrt{20}} \\ \frac{1}{2} & -\frac{1}{\sqrt{20}} & -\frac{1}{2} & \frac{3}{\sqrt{20}} \\ \frac{1}{2} & \frac{1}{\sqrt{20}} & -\frac{1}{2} & -\frac{3}{\sqrt{20}} \\ \frac{1}{2} & \frac{3}{\sqrt{20}} & \frac{1}{2} & \frac{1}{\sqrt{20}} \end{bmatrix}, \qquad D_3 = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{3}{20} & 0 & 0 \\ 0 & 0 & \frac{1}{20} & 0 \\ 0 & 0 & 0 & \frac{1}{140} \end{bmatrix}.$$

4. Degree elevation

For raising the degree of Bézier curve by one without changing the shape of the curve. We can show that new vertices $c_i^{(1)}$ are obtained from the old polygon by piecewise linear interpolation at the parameter values $i/(n+1)$, see (Farin, 1993).

$$c_i^{(1)} = \frac{i}{n+1} c_{i-1} + \left(1 - \frac{i}{n+1}\right) c_i, \quad i = 0, 1, \ldots, n+1. \tag{10}$$

We can rewrite the formula (10) as a linear system $T_n c = c^{(1)}$, where the $(n+2) \times (n+1)$ matrix T_n is

$$T_n = \frac{1}{n+1} \begin{pmatrix} n+1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & n & 0 & \ldots & 0 & 0 & 0 \\ 0 & 2 & n-1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & n-1 & 2 & 0 \\ 0 & 0 & 0 & \ldots & 0 & n & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 & n+1 \end{pmatrix}$$

and the $(n+1)$ vector c and the $(n+2)$ vector $c^{(1)}$ are

$$c = (c_0, c_1, \ldots, c_n)^{\mathrm{t}},$$
$$c^{(1)} = \left(c_0^{(1)}, c_1^{(1)}, \ldots, c_n^{(1)}\right)^{\mathrm{t}}.$$

We may repeat this process and then obtain a sequence of control points. After r degree elevations, we have a linear system $T_{n,r}c = c^{(r)}$, where the $(n+r+1)\times(n+1)$ matrix

$$T_{n,r} = T_{n+r-1}T_{n+r-2}\ldots T_{n+1}T_n$$

has elements

$$T_{n,r}(i,j) = \frac{\binom{n}{j}\binom{r}{i-j}}{\binom{n+r}{i}}, \quad i = 0,1,\ldots,n+r \text{ and } j = 0,1,\ldots,n.$$

By the orthogonality of Legendre basis, the degree elevation of a polynomial with Legendre basis is given by

$$l = (l_0, l_1, \ldots, l_n)^{\mathrm{t}},$$
$$l^{(1)} = (l_0, l_1, \ldots, l_n, 0)^{\mathrm{t}}.$$

After r degree elevations, we have a linear system $\tilde{I}_{n,r}l = l^{(r)}$, where the $(n+r+1)\times(n+1)$ matrix $\tilde{I}_{n,r}$ has elements

$$\tilde{I}_{n,r}(i,j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

After transforming the Bernstein coefficients to the Legendre coefficients by M_n^{-1}, the degree elevation by $\tilde{I}_{n,r}$, and finding the Bernstein coefficients by M_{n+r}, we obtain the following theorem.

Theorem 6. *The degree elevation matrix $T_{n,r}$ can be expressed in M_n^{-1}, $\tilde{I}_{n,r}$ and M_{n+r} as*

$$T_{n,r} = M_{n+r}\tilde{I}_{n,r}M_n^{-1}.$$

5. Degree reduction

When we find the best approximation in the sense of L_2-norm, in general, the degree reduction of Bézier curves address the following problem.

Problem 1 (L_2 *degree reduction*). Let $\{c_i\}_{i=0}^n$ be a given set of control points which define the Bézier curve

$$c^n(t) = \sum_{i=0}^{n} c_i B_i^n(t)$$

of degree n. Then find another point set $\{b_i\}_{i=0}^m$ defining the approximative Bézier curve

$$b^m(t) = \sum_{i=0}^{m} b_i B_i^m(t)$$

of lower degree $m < n$ so that an L_2-distance function $d_2(b^m, c^n)$ between b^m and c^n is minimized.

The L_2-distance of the two Bézier curves b^m and c^n is defined as following:

$$d_2^2\left(b^m, c^n\right) = \int_0^1 \left|b^m(t) - c^n(t)\right|^2 \mathrm{d}t = \int_0^1 \left| \sum_{i=0}^{m} b_i B_i^m(t) - \sum_{i=0}^{n} c_i B_i^n(t) \right|^2 \mathrm{d}t.$$

Using the matrix $T_{m,r}$, we can elevate the degree of b^m from m to n, where $r = n - m$,

$$b^{(r)} = T_{m,r} b.$$

Then the curve b^m of degree m is rewritten as a curve of degree n

$$b^m(t) = b^{(r)}(t) = \sum_{i=0}^{n} b_i^{(r)} B_i^n(t),$$

and the distance is

$$d_2^2\left(b^m, c^n\right) = d_2^2\left(b^{(r)}, c^n\right) = \int_0^1 \left| \sum_{i=0}^{n} b_i^{(r)} B_i^n(t) - \sum_{i=0}^{n} c_i B_i^n(t) \right|^2 \mathrm{d}t = \int_0^1 \left| \sum_{i=0}^{n} \left(b_i^{(r)} - c_i\right) B_i^n(t) \right|^2 \mathrm{d}t.$$

Thus we obtain the following theorem for the L_2-distance between the Bézier curve b^m of degree m and the Bézier curve c^n of degree n.

Theorem 7. *The L_2-distance between the two Bézier curves b^m and c^n is*

$$d_2^2\left(b^m, c^n\right) = d_2^2\left(b^{(r)}, c^n\right) = A^{\mathrm{t}} Q_n A,$$

where $A = c - T_{m,r} b$, $b = (b_0, b_1, \ldots, b_m)^{\mathrm{t}}$ and $c = (c_0, c_1, \ldots, c_n)^{\mathrm{t}}$.

For developing the method, rewrite $d_2^2(b^m, c^n)$.

$$\begin{aligned} d_2^2\left(b^m, c^n\right) &= A^{\mathrm{t}} Q_n A \\ &= [c - T_{m,r} b]^{\mathrm{t}} Q_n [c - T_{m,r} b] \\ &= c^{\mathrm{t}} Q_n c - 2 b^{\mathrm{t}} T_{m,r}^{\mathrm{t}} Q_n c + b^{\mathrm{t}} T_{m,r}^{\mathrm{t}} Q_n T_{m,r} b. \end{aligned}$$

One method of obtaining the vector b is so-called the method of least squares (Lee and Park, 1997; Lutterkort et al., 1999). This method consists of minimizing $A^{\mathrm{t}} Q_n A$ with respect to b. We choose the vector $\hat{b}$ as that the value of b minimizes $A^{\mathrm{t}} Q_n A$. Equating $\partial(A^{\mathrm{t}} Q_n A)/\partial b$ to zero and writing the resulting equations in terms of $\hat{b}$, we find that these equations are

$$T_{m,r}^{\mathrm{t}} Q_n T_{m,r} \hat{b} = T_{m,r}^{\mathrm{t}} Q_n c.$$

They are known as the normal equations.

Theorem 8. *The $(n+1) \times (n+1)$ matrix $T_{n-1}^{\mathrm{t}} Q_n T_{n-1}$ has the following property*:

$$T_{n-1}^{\mathrm{t}} Q_n T_{n-1} = Q_{n-1}.$$

Proof.

$$\begin{aligned} T_{n-1}^{\mathrm{t}} Q_n T_{n-1}(i,j) &= \frac{1}{2n+1} \sum_{l=0}^{n+1} \frac{\binom{n-1}{i}\binom{1}{l-i}}{\binom{n}{l}} \sum_{k=0}^{n+1} \frac{\binom{n}{l}\binom{n}{k}}{\binom{2n}{l+k}} \frac{\binom{n-1}{j}\binom{1}{k-j}}{\binom{n}{k}} \\ &= \frac{1}{2n+1} \binom{n-1}{i}\binom{n-1}{j} \sum_{l=0}^{n+1}\sum_{k=0}^{n+1} \frac{\binom{1}{l-i}\binom{1}{k-j}}{\binom{2n}{l+k}} \\ &= \frac{1}{2n-1} \frac{\binom{n-1}{i}\binom{n-1}{j}}{\binom{2n-2}{i+j}}, \quad i,j = 0,1,\ldots,n-1. \qquad \square \end{aligned}$$

From Theorem 8, we have $T_{m,r}^{\mathrm{t}} Q_n T_{m,r} = Q_m$. Hence, the real symmetric positive definite matrix $T_{m,r}^{\mathrm{t}} Q_n T_{m,r}$ is invertible. Provided $(T_{m,r}^{\mathrm{t}} Q_n T_{m,r})^{-1}$ exists, we have the unique solution for $\hat{b}$,

$$\hat{b} = \left(T_{m,r}^{\mathrm{t}} Q_n T_{m,r}\right)^{-1} T_{m,r}^{\mathrm{t}} Q_n c. \tag{11}$$

The approximate curve given by (11) is the best approximation with respect to the L_2-norm.

By the orthogonality of Legendre basis, the degree reduction of a polynomial with Legendre basis is given by

$$l = (l_0, l_1, \ldots, l_n)^{\mathrm{t}},$$
$$l^{(-1)} = (l_0, l_1, \ldots, l_{n-1})^{\mathrm{t}}.$$

After r degree reductions, we have a linear system

$$\tilde{I}_{n,-r} l = l^{(-r)},$$

where the $(n-r+1) \times (n+1)$ matrix $\tilde{I}_{n,-r}$ is

$$\tilde{I}_{n,-r} = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{pmatrix}. \tag{12}$$

After transforming the Bernstein coefficients to the Legendre coefficients by M_n^{-1}, the degree reduction by $\tilde{I}_{n,-r}$, and finding the Bernstein coefficients by M_m, we obtain the following theorem.

Theorem 9. *The degree reduction matrix can be expressed in* M_n^{-1} , $\tilde{I}_{n,-r}$ *and* M_m *as*

$$\left(T_{m,r}^{\mathrm{t}} Q_n T_{m,r}\right)^{-1} T_{m,r}^{\mathrm{t}} Q_n = M_m \tilde{I}_{n,-r} M_n^{-1}.$$

For the degree reduction with Bernstein basis, we can use the explicit matrix forms of M_m, M_n^{-1} and $\tilde{I}_{n,-r}$ to compute $M_m \tilde{I}_{n,-r} M_n^{-1}$ given by the formula (4), (5) and (12), respectively. Therefore, our method using the relationship of transformations between Legendre and Bernstein basis is a simple and efficient method for optimal multiple degree reductions with respect to the L_2-norm. However, this best approximation does not in general interpolate the given curve at its endpoints. Thus we have to consider the smoothness of our method for the practical use.

Here is the example given by the explicit matrix form and we have the same results with (Lutterkort et al., 1999).

Example 4 (*Parametric case, $n = 4$*).

$$c = [0, 1, 2, 1, 0]^{\mathrm{t}},$$
$$l = M_4^{-1} c = \left[\frac{4}{5}, 0, -\frac{6\sqrt{5}}{35}, 0, \frac{2}{105}\right]^{\mathrm{t}},$$
$$\|c^4\|_2^2 = \left(\frac{4}{5}\right)^2 + \left(\frac{6\sqrt{5}}{35}\right)^2 + \left(\frac{2}{105}\right)^2,$$
$$M_2 \tilde{I}_{4,-2} M_4^{-1} = \frac{1}{35}\begin{bmatrix} 31 & 9 & -3 & -5 & 3 \\ -13 & 17 & 27 & 17 & -13 \\ 3 & -5 & -3 & 9 & 31 \end{bmatrix},$$
$$l^{(-2)} = \tilde{I}_{4,-2} M_4^{-1} c = \left[\frac{4}{5}, 0, -\frac{6\sqrt{5}}{35}\right]^{\mathrm{t}},$$
$$c^{(-2)} = M_2 \tilde{I}_{4,-2} M_4^{-1} c = \left[\frac{-2}{35}, \frac{88}{35}, \frac{-2}{35}\right]^{\mathrm{t}},$$
$$\|c^{(-2)}\|_2^2 = \left(\frac{4}{5}\right)^2 + \left(\frac{6\sqrt{5}}{35}\right)^2.$$

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References

Eck, M., 1993. Degree reduction of Bézier curves. Computer Aided Geometric Design 10, 237–251.
Eck, M., 1995. Least squares degree reduction of Bézier curves. Computer-Aided Design 27, 845–851.
Farin, G., 1993. Curves and Surfaces for Computer Aided Geometric Design, 3rd Edition. Academic Press, Boston.
Farouki, R.T., 2000. Legendre–Bernstein basis transformations. J. Comput. Appl. Math. 119, 145–160.
Lee, B.G., Park, Y., 1997. Distance for Bézier curves and degree reduction. Bull. Australian Math. Soc. 56, 507–515.
Li, Y.M., Zhang, X.Y., 1998. Basis conversion among Bézier, Tchebyshev and Legendre. Computer Aided Geometric Design 15, 637–642.
Lutterkort, D., Peters, J., Reif, U., 1999. Polynomial degree reduction in the L_2-norm equals best Euclidean approximation of Bézier coefficients. Computer Aided Geometric Design 16, 607–612.
Lyche, T., Scherer, K., 2000. On the p-norm condition number of the multivariate triangular Bernstein basis. J. Comput. Appl. Math. 119, 259–273.
Mazure, M.-L., 1999. Chebyshev–Bernstein bases. Computer Aided Geometric Design 16, 649–669.
Watkins, M., Worsey, A., 1988. Degree reduction for Bézier curves. Computer-Aided Design 20, 398–405.