A Cubic Quasi-interpolant

by

Byung-Gook Lee

Dongseo University, Pusan, Korea
http://kowon.dongseo.ac.kr/∼lbg/

1. Quasi-interpolants

We shall approximate f by an approximation $P_d f$ in a spline space $\mathbf{S}_{\mathbf{d},\tau}$ on the form

$$P_d f(x) = \sum_{j=1}^{n} (\lambda_j f) B_{j,d},$$

where $\{B_{j,d}\}$ is a sequence of B-splines of order d for the knot vector $\tau = (\tau_j)_{j=1}^{n+d+1}$.

2. How to construct $\lambda_j f$

$\lambda_j f$ are appropriate linear functionals chosen so that:

- $P_d f$ can be applied to a large class of functions including, for example, continuous functions

- $P_d f$ is local in the sense that $(P_d f)(x)$ depends only on values of f in a small neighborhood of x

We shall construct $\lambda_j f$ on the form

$$\lambda_j f = \sum_{k=1}^{m_j} w_{j,k} f(x_{j,k}),$$

where $\{x_{j,k}\}_{k=1}^{m_j}$ are given data points in the vicinity of the support $[\tau_j, \tau_{j+d+1}]$ of the B-spline $B_{j,d}$.

3. How to choose $w_{j,k}$

In order to make $P_d f$ approximate smooth functions f well, we shall choose $w_{j,k}$ such that

$$P_d f = f, \text{ all } f \in \mathbf{S}_{d,\tau}.$$

This was studied by Lyche, T. and L. L. Schumaker (1975), they restrict the data points $\{x_{j,k}\}$ to one subinterval $[\tau_l, \tau_{l+1}]$ of $[\tau_j, \tau_{j+d+1}]$. Here we want to enlarge the subinterval.

4. P_d : reproduce spline space

To find the $w_{j,k}$ so that $P_d f$ reproduces all splines in $\mathbf{S_{d,\tau}}$. By the local support properties of the B-splines, it is enough to require

$$\lambda_j B_{i,d} = \sum_{k=1}^{m_j} w_{j,k} B_{i,d}(x_{j,k}) = \delta_{j,i},$$

$i = \mu - d, \ldots, \nu - 1$, where $[\tau_\mu, \tau_\nu]$ is the smallest subinterval of the knot vector that contains given data point $\{x_{j,k}\}_{k=1}^{m_j}$.

This is a linear system of :

- $\nu - \mu + d$ linear equations
- m_j unknowns $(w_{j,1}, w_{j,2}, \ldots, w_{j,m_j})$

To solve the linear system, we need :

- at least $\nu - \mu + d$ data points ($m_j \geq \nu - \mu + d$)
- the Schoenberg-Whitney nesting conditions

If we have more than $\nu - \mu + d$ data points, we can use the least squares minimum norm solution.

5. A Cubic Quasi-Interpolant

Suppose we have $m \geq 7$ data points $(x_j, y_j)_{j=1}^{m}$ sampled from a function f, where m is assumed to be odd and $\mathbf{x} = \{x_j\}_{j=1}^{m}$ are increasing sequence. From $\mathbf{x}$ we form the knot vector

$$\tau = (\tau_j)_{j=1}^{n+4} = (x_1, x_1, x_1, x_1, x_4, x_6, \ldots, x_{m-3}, x_m, x_m, x_m, x_m),$$

where $n = (m + 3)/2$.

We shall design a cubic quasi-interpolant that to each $\lambda_j f$ depend on (at most) 5 data points in $[\tau_{j+1}, \tau_{j+3}]$.

Explicit expressions for $w_{j,k},\ j = 3, \ldots, n-2$

Coefficient matrix :

$$A_j = \begin{pmatrix} B_{j-2,3}(x_{j,1}) & B_{j-2,3}(x_{j,2}) & B_{j-2,3}(x_{j,3}) & B_{j-2,3}(x_{j,4}) & B_{j-2,3}(x_{j,5}) \\ B_{j-1,3}(x_{j,1}) & B_{j-1,3}(x_{j,2}) & B_{j-1,3}(x_{j,3}) & B_{j-1,3}(x_{j,4}) & B_{j-1,3}(x_{j,5}) \\ B_{j,3}(x_{j,1}) & B_{j,3}(x_{j,2}) & B_{j,3}(x_{j,3}) & B_{j,3}(x_{j,4}) & B_{j,3}(x_{j,5}) \\ B_{j+1,3}(x_{j,1}) & B_{j+1,3}(x_{j,2}) & B_{j+1,3}(x_{j,3}) & B_{j+1,3}(x_{j,4}) & B_{j+1,3}(x_{j,5}) \\ B_{j+2,3}(x_{j,1}) & B_{j+2,3}(x_{j,2}) & B_{j+2,3}(x_{j,3}) & B_{j+2,3}(x_{j,4}) & B_{j+2,3}(x_{j,5}) \end{pmatrix}$$

Solution of linear system :

$$w_j = \frac{1}{3\eta_j \phi_j \theta_{j,1} \theta_{j,2}} \begin{pmatrix} (\phi_j + \theta_{j,2} + \phi_j \theta_{j,2})\theta_{j,2} \\ -(\phi_j + \theta_{j,2} + \phi_j \theta_{j,2})(1+\theta_{j,1})^3 \theta_{j,2} \\ \gamma_j \\ -(1+\theta_{j,1}+\phi_j)\phi_j^2 \theta_{j,1}(1+\theta_{j,2})^3 \\ (1+\theta_{j,1}+\phi_j)\phi_j^2 \theta_{j,1}\theta_{j,2}^3 \end{pmatrix}$$

where $\eta_j = \phi_j + \theta_{j,2} + \phi_j\theta_{j,2} + (1+\theta_{j,1}+\phi_j)\theta_{j,2}$, $\gamma_j = (\phi_j + \theta_{j,2} + \phi_j\theta_{j,2})(3(1+\theta_{j,1}+\phi_j) + \theta_{j,1}^2)\theta_{j,1}\theta_{j,2} + \phi_j(1+\theta_{j,1}+\phi_j)(3(\phi_j+\theta_{j,2}+\phi_j\theta_{j,2})\theta_{j,2}+\phi_j)\theta_{j,1}$.

$$\theta_{j,1} = \frac{x_{j,2}-x_{j,1}}{x_{j,3}-x_{j,2}}, \theta_{j,2} = \frac{x_{j,4}-x_{j,3}}{x_{j,5}-x_{j,4}}, \phi_j = \frac{\tau_{j+3}-\tau_{j+2}}{\tau_{j+2}-\tau_{j+1}}.$$

Explicit expressions for $w_{j,k},\ j = 2, n-1$ (Boundary case)

Coefficient matrix :

$$A_2 = \begin{pmatrix} B_{1,3}(x_1) & B_{1,3}(x_2) & B_{1,3}(x_3) & B_{1,3}(x_4) \\ B_{2,3}(x_1) & B_{2,3}(x_2) & B_{2,3}(x_3) & B_{2,3}(x_4) \\ B_{3,3}(x_1) & B_{3,3}(x_2) & B_{3,3}(x_3) & B_{3,3}(x_4) \\ B_{4,3}(x_1) & B_{4,3}(x_2) & B_{4,3}(x_3) & B_{4,3}(x_4) \end{pmatrix}$$

Solution of linear system :

$$w_2 = \frac{1}{3(1+\theta_{2,1})(1+\theta_{2,2})\phi_2} \begin{pmatrix} -(1+2\theta_{2,1}+\theta_{2,2}+\phi_2)(1+\theta_{2,2}) \\ (1+\theta_{2,1})^2(1+\theta_{2,2}+\phi_2)^2 \\ -\theta_{2,1}^2(1+\theta_{2,2})(1+\theta_{2,2}+\phi_2)^2 \\ (1+\theta_{2,1})^2\theta_{2,2}\phi_2^2 \end{pmatrix}$$

where

$$\theta_{2,1} = \frac{x_2 - x_1}{x_3 - x_2}, \theta_{2,2} = \frac{x_3 - x_2}{x_4 - x_3}, \phi_2 = \frac{x_2 - x_1}{x_4 - x_3}.$$

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Theorem 1 *(A Cubic Quasi-interpolant)*

Suppose that f is a function in $C^4_\Delta[x_1, x_m]$. There exists a constant $K > 0$, independent of f, such that the error $P_3 f - f$ is bounded by

$$||P_3 f - f||_{\infty,[x_1,x_m]} \leq K\theta^2 \phi h^4 ||D^4 f||_{\infty,[x_1,x_m]},$$

where

$$\theta = \max_{1\leq j\leq n}\{\theta_{j,1}, \theta_{j,1}^{-1}\}, \theta_{j,1} = \frac{x_{j,2} - x_{j,1}}{x_{j,3} - x_{j,2}},$$

$$\phi = \max_{1\leq j\leq n}\{\phi_j, \phi_j^{-1}\}, \phi_j = \frac{\tau_{j+3} - \tau_{j+2}}{\tau_{j+2} - \tau_{j+1}},$$

$$h = \max_{1\leq j\leq m-1} x_{j+1} - x_j,$$

and

$$K = 356.$$

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A Cubic Quasi-Interpolant

- local property
- the same order as the best spline approximation
- can be computed directly without solving systems of equations

This work is in collaboration with Tom Lyche and Knut Mørken (Oslo University, Norway).