

A simple matrix form for degree reduction of Bézier curves using Chebyshev–Bernstein basis transformations

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Abstract

We use the matrices of transformations between Chebyshev and Bernstein basis and the matrices of degree elevation and reduction of Chebyshev polynomials to present a simple and efficient method for r times degree elevation and optimal r times degree reduction of Bézier curves with respect to the weighted L_2 -norm for the interval $[0, 1]$, using the weight function $w(x) = 1/\sqrt{4x - 4x^2}$. The error of the degree reduction scheme is given, and the degree reduction with continuity conditions is also considered.

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Keywords: Bézier curves; Chebyshev polynomials; Basis transformations; Degree elevation; Degree reduction; Continuity conditions

1. Introduction

In CAGD there are a variety of basis form involving the most used basis of monomials, Bernstein polynomials, and B-spline basis functions. Usually, we need to work with data and basis created by other CAD systems. And thus a corresponding operation must be done on basis. This operation may include degree raising, degree reduction, or basis transformation between different kinds of basis. Conversion of basis, which is not always exact, is sometimes needed to convert data to be handled on our own system. Transformation of basis is exact for polynomials of the same degree between the monomials, Bernstein and B-spline basis functions. It is important to do these operations in an efficient and easy way [4,7,9].

The Bernstein basis have very important properties like the recursive relation, the symmetric properties, and making partition of unity. These properties offer valuable insight into its geometrical behavior, and has won widespread acceptance as the basis for Bézier curves and surfaces in CAGD [3]. But Bernstein polynomials are not orthogonal. On the other hand, the use of orthogonal basis such as Chebyshev polynomial permits finding the polynomial of best approximation.

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The weight functions are used quite naturally in many practical problems of approximation theory and are connected with the theory of special functions. We may introduce a weight function to get better approximation to one part of the curve than another. The Chebyshev and Legendre polynomials are used in many degree reduction schemes [11,1,2]. So, we need to transform between these basis forms. Also the basis transformation is important and has been studied in many ways. Rababah [9] found the explicit form for the matrix of Chebyshev–Bernstein basis transformation. Other related results are given in [5,8].

In this paper, we use the matrices of transformations between Chebyshev and Bernstein basis to present a simple and efficient method for optimal r times degree reduction of Bézier curves with respect to the Chebyshev weighted L_2 -norm. Furthermore, we consider r times degree reduction of Bézier curves with continuity conditions at the end points, and find the error.

2. Chebyshev and Bernstein polynomials

The Bézier representation of a parametric polynomial curve $P_n(x)$ of degree n is expressed using the Bernstein polynomials as basis in the form:

$$P_n(x) = \sum_{i=0}^n c_i B_i^n(x), \quad 0 \leq x \leq 1,$$

where the $\{c_i\}_{i=0}^n$ is the set of $(n + 1)$ Bézier points, and

$$B_i^n(x) = \binom{n}{i} (1-x)^{n-i} x^i, \quad i = 0, 1, \dots, n$$

are the Bernstein polynomials of degree n .

The area under a Bernstein polynomial $B_k^n(x)$, $k = 0, 1, \dots, n$ of degree n is given by

$$\int_0^1 B_k^n(x) dx = \frac{1}{n+1}. \tag{1}$$

The product of Bernstein polynomials of degree n and m is also a Bernstein polynomial of degree $n + m$ and given by

$$B_i^n(x) B_j^m(x) = \frac{\binom{n}{i} \binom{m}{j}}{\binom{n+m}{i+j}} B_{i+j}^{n+m}(x). \tag{2}$$

For more on the Bernstein polynomials and Bézier curves, see [3].

The Chebyshev polynomials are orthonormal on the interval $[0, 1]$ with respect to the weight function $w(x) = 1/\sqrt{4x - 4x^2}$, that is

$$\int_0^1 \frac{T_j(x) T_k(x)}{\sqrt{4x - 4x^2}} dx = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases} \tag{3}$$

Because of symmetry properties, the Chebyshev polynomials are usually defined on the interval $[-1, +1]$. The Bernstein polynomials are traditionally defined on the interval $[0, 1]$. However, for our purposes in this article, we use the interval $[0, 1]$ for both polynomials. The Chebyshev polynomials $T_n(x)$ on $x \in [0, 1]$ have the following Rodrigues type formula

$$T_n(x) = \frac{2}{\sqrt{\pi}} \frac{(-1)^n 2^n}{(2n-1)!!} \sqrt{x-x^2} \frac{d^n}{dx^n} (x-x^2)^{n-\frac{1}{2}}, \quad n = 1, 2, \dots,$$

where $(2n - 1)!! = (2n - 1)(2n - 3)(2n - 5) \cdots 1$. The first few terms are given by

$$\begin{aligned} T_0(x) &= \frac{\sqrt{2}}{\sqrt{\pi}}, \\ T_1(x) &= \frac{2}{\sqrt{\pi}}(2x - 1), \\ T_2(x) &= \frac{2}{\sqrt{\pi}}(8x^2 - 8x + 1), \\ T_3(x) &= \frac{2}{\sqrt{\pi}}(32x^3 - 48x^2 + 18x - 1). \end{aligned}$$

For more on the Chebyshev polynomials, see [10].

3. Basis transformation

We consider a polynomial $P_n(x)$ of degree n , expressed in terms of the Bernstein and the Chebyshev polynomials:

$$P_n(x) = \sum_{j=0}^n c_j B_j^n(x) = \sum_{k=0}^n t_k T_k(x).$$

We consider the linear transformation of the Chebyshev coefficients t_0, t_1, \dots, t_n into the Bernstein coefficients c_0, c_1, \dots, c_n as follows:

$$c_j = \sum_{k=0}^n M_n(j, k) t_k, \quad j = 0, 1, \dots, n.$$

The transformation above can be expressed in the following matrix form:

$$c = M_n t,$$

where $c = [c_0, c_1, \dots, c_n]^t$ and $t = [t_0, t_1, \dots, t_n]^t$.

Then the elements of the matrix $M_n(j, k)$, $0 \leq j, k \leq n$ are given in the following formula, see [9]:

$$M_n(j, k) = \frac{(2 - \sqrt{2})\delta_k + \sqrt{2}}{\sqrt{\pi}} \frac{1}{\binom{n}{j}} \sum_{i=\max(0, j+k-n)}^{\min(j, k)} (-1)^{k+i} \binom{2k}{2i} \binom{n-k}{j-i}, \quad (4)$$

where $\delta_k = 0$, if $k = 0$, and $\delta_k = 1$, otherwise.

The elements of the Bernstein to Chebyshev transformation matrix $M_n^{-1}(j, k)$, $0 \leq j, k \leq n$ are given by, see [9]

$$M_n^{-1}(j, k) = \frac{\delta_j + 1}{4^{n+j}} \binom{n}{k} \sum_{i=0}^j (-1)^{j+i} \frac{\binom{2j}{2i} \binom{2k+2i}{k+i} \binom{2n-2k+2j-2i}{n-k+j-i}}{\binom{n+j}{k+i}}. \quad (5)$$

Example 1. The Chebyshev–Bernstein transformation matrices M_n and M_n^{-1} for $n = 1, 2, 3$ are as follows:

$$M_1 = \frac{\sqrt{2}}{\sqrt{\pi}} \begin{bmatrix} 1 & -\sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix}, \quad M_1^{-1} = \frac{\sqrt{\pi}}{\sqrt{2}} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix},$$

$$M_2 = \frac{\sqrt{2}}{\sqrt{\pi}} \begin{bmatrix} 1 & -\sqrt{2} & \sqrt{2} \\ 1 & 0 & -3\sqrt{2} \\ 1 & \sqrt{2} & \sqrt{2} \end{bmatrix}, \quad M_2^{-1} = \frac{\sqrt{\pi}}{\sqrt{2}} \begin{bmatrix} \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ -\frac{\sqrt{2}}{4} & 0 & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{16} & -\frac{\sqrt{2}}{8} & \frac{\sqrt{2}}{16} \end{bmatrix},$$

$$M_3 = \frac{\sqrt{2}}{\sqrt{\pi}} \begin{bmatrix} 1 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 1 & -\frac{\sqrt{2}}{3} & -\frac{5\sqrt{2}}{3} & 5\sqrt{2} \\ 1 & \frac{\sqrt{2}}{3} & -\frac{5\sqrt{2}}{3} & -5\sqrt{2} \\ 1 & \sqrt{2} & \sqrt{2} & \sqrt{2} \end{bmatrix}, \quad M_3^{-1} = \frac{\sqrt{\pi}}{\sqrt{2}} \begin{bmatrix} \frac{5}{16} & \frac{3}{16} & \frac{3}{16} & \frac{5}{16} \\ -\frac{15\sqrt{2}}{64} & -\frac{3\sqrt{2}}{64} & \frac{3\sqrt{2}}{64} & \frac{15\sqrt{2}}{64} \\ \frac{3\sqrt{2}}{32} & -\frac{3\sqrt{2}}{32} & -\frac{3\sqrt{2}}{32} & \frac{3\sqrt{2}}{32} \\ -\frac{\sqrt{2}}{64} & \frac{3\sqrt{2}}{64} & -\frac{3\sqrt{2}}{64} & \frac{\sqrt{2}}{64} \end{bmatrix}.$$

4. Weighted L_2 -norm

The weighted L_2 -norm of the Bézier curve P_n in the Bernstein basis form is given by

$$\|P_n\|_w^2 = \int_0^1 \frac{|\sum_{i=0}^n c_i B_i^n(x)|^2}{\sqrt{4x - 4x^2}} dx. \tag{6}$$

Simplifying this using (1) and (2) gives

$$\|P_n\|_w^2 = c^t Q_n c, \tag{7}$$

where

$$Q_n(i, j) = \frac{\Gamma(2n - i - j + \frac{1}{2})\Gamma(i + j + \frac{1}{2})}{2\Gamma(2n + 1)} \binom{n}{i} \binom{n}{j}, \quad i, j = 0, 1, \dots, n \tag{8}$$

are the elements of the Gram matrix Q_n of the Bernstein basis. The matrix Q_n is a real symmetric matrix, as a consequence of the symmetry of the combinatorial function. The matrix Q_n is also a positive definite matrix, as a consequence of the positivity of the left-hand side in the definition. Thus Q_n is a real symmetric positive definite matrix.

We give some examples of the Gram matrix Q_n .

Example 2. The Gram matrices Q_n for $n = 1, 2, 3$ are

$$Q_1 = \frac{\pi}{16} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad Q_2 = \frac{\pi}{256} \begin{bmatrix} 35 & 10 & 3 \\ 10 & 12 & 10 \\ 3 & 10 & 35 \end{bmatrix}, \quad Q_3 = \frac{\pi}{2048} \begin{bmatrix} 231 & 63 & 21 & 5 \\ 63 & 63 & 45 & 21 \\ 21 & 45 & 63 & 63 \\ 5 & 21 & 63 & 231 \end{bmatrix}.$$

Let D_n be the similar matrix of the Gram matrix Q_n with M_n . Then

$$D_n = M_n^{-1} Q_n M_n. \tag{9}$$

We give some examples of D_n .

Example 3. The elements of D_n , $n = 1, 2, 3$ are given by

$$D_1 = \frac{\pi}{8} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_2 = \frac{\pi}{128} \begin{bmatrix} 22 & 0 & \sqrt{2} \\ 0 & 16 & 0 \\ \sqrt{2} & 0 & 3 \end{bmatrix}, \quad D_3 = \frac{\pi}{1024} \begin{bmatrix} 136 & 0 & 12\sqrt{2} & 0 \\ 0 & 117 & 0 & 3 \\ 12\sqrt{2} & 0 & 36 & 0 \\ 0 & 3 & 0 & 5 \end{bmatrix}.$$

We use the orthonormality property (3) of the Chebyshev polynomials to get the weighted L_2 -norm of the polynomial P_n in the Chebyshev basis form:

$$\|P_n\|_w^2 = \int_0^1 \frac{|\sum_{i=0}^n t_i T_i(x)|^2}{\sqrt{4x - 4x^2}} dx = \sum_{i,j} t_i t_j \int_0^1 \frac{T_i(x) T_j(x)}{\sqrt{4x - 4x^2}} dx = t^t t. \tag{10}$$

From the similar matrix D_n in (9), the relationship among the transformation matrix M_n^{-1} from Bernstein to Chebyshev basis, M_n and D_n is given by

$$M_n^{-1} = D_n M_n^t. \tag{11}$$

5. Degree elevation

The degree elevation problem is concerned with writing a given Bézier curve in basis of degree n into basis of degree $n + 1$ without changing the curve. The new vertices $c_i^{(1)}$ of the new polygon are calculated from the following formula, see [3]:

$$c_i^{(1)} = \frac{i}{n+1} c_{i-1} + \left(1 - \frac{i}{n+1}\right) c_i, \quad i = 0, 1, \dots, n+1. \tag{12}$$

Rewriting the formula (12) in the matrix form $c^{(1)} = T_n c$, where $c = (c_0, c_1, \dots, c_n)^t$, $c^{(1)} = (c_0^{(1)}, c_1^{(1)}, \dots, c_{n+1}^{(1)})^t$, and the $(n + 2) \times (n + 1)$ matrix T_n is given by

$$T_n = \frac{1}{n+1} \begin{pmatrix} n+1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & n & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & n-1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n-1 & 2 & 0 \\ 0 & 0 & 0 & \dots & 0 & n & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & n+1 \end{pmatrix}.$$

This process can be repeated r times to obtain a new sequence of control points $c^{(r)}$ for the degree elevated curve, where $c^{(r)} = T_{n,r} c$, and the $(n + r + 1) \times (n + 1)$ matrix $T_{n,r}$ is given by

$$T_{n,r} = T_{n+r-1} T_{n+r-2} \dots T_{n+1} T_n$$

and has the elements

$$T_{n,r}(i, j) = \frac{\binom{n}{j} \binom{r}{i-j}}{\binom{n+r}{i}}, \quad i = 0, 1, \dots, n+r \text{ and } j = 0, 1, \dots, n.$$

Elevating the degree of the polynomial P_n with coefficients $t = (t_0, t_1, \dots, t_n)^t$ gives the polynomial of degree $n + 1$ with coefficients $t^{(1)} = (t_0, t_1, \dots, t_n, 0)^t$.

Continuing the process of degree elevation r times to the polynomial P_n , we get the polynomial of degree $n + r$ with coefficients $t^{(r)} = (t_0, t_1, \dots, t_n, 0, \dots, 0)^t$.

The r times degree elevation can be written in the matrix form: $t^{(r)} = I_{n,r} t$, where the matrix $I_{n,r}$ of dimension $(n + r + 1) \times (n + 1)$ has the elements

$$I_{n,r}(i, j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

The r times degree elevation matrix $T_{n,r}$ can be obtained by first transforming the Bernstein coefficients to the Chebyshev coefficients using M_n^{-1} , and then the r times degree elevation using $I_{n,r}$, and finally finding the Bernstein coefficients using M_{n+r} . This gives the following formula:

$$T_{n,r} = M_{n+r} I_{n,r} M_n^{-1}. \tag{13}$$

6. Degree reduction

While the degree elevated curve can be obtained exactly, the degree reduced curve is obtained by a sense of approximation. In this context, we use the best approximation in the sense of the weighted L_2 -norm.

Given the Bézier curve $c^n(x)$ of degree n ,

$$c^n(x) = \sum_{i=0}^n c_i B_i^n(x),$$

which has the set of Bézier points $\{c_i\}_{i=0}^n$. Find a Bézier curve $b^m(x)$ of degree m , where $m < n$,

$$b^m(x) = \sum_{i=0}^m b_i B_i^m(x),$$

which has the set of Bézier points $\{b_i\}_{i=0}^m$, so that the weighted L_2 -norm between b^m and c^n is a minimum.

The weighted L_2 -norm of c^n and b^m is given by

$$\|b^m - c^n\|_w^2 = \int_0^1 \frac{|b^m(x) - c^n(x)|^2}{\sqrt{4x - 4x^2}} dx.$$

Elevating the degree of b^m from m to n using the matrix $T_{m,r}$, where $r = n - m$, gives

$$b^{(r)} = T_{m,r} b.$$

This last step rewrites the curve b^m of degree m as a curve of degree n

$$b^m(x) = b^{(r)}(x) = \sum_{i=0}^n b_i^{(r)} B_i^n(x),$$

and hence, the weighted L_2 -norm is given by

$$\|b^m - c^n\|_w^2 = \|b^{(r)} - c^n\|_w^2 = \int_0^1 \frac{|\sum_{i=0}^n (b_i^{(r)} - c_i) B_i^n(x)|^2}{\sqrt{4x - 4x^2}} dx.$$

Invoking (8) into the last equation gives the weighted L_2 -norm between the Bézier curves b^m and c^n in the following formula:

$$\|b^m - c^n\|_w^2 = \|b^{(r)} - c^n\|_w^2 = A^t Q_n A, \tag{14}$$

where $A = c - T_{m,r} b$, $b = (b_0, b_1, \dots, b_m)^t$ and $c = (c_0, c_1, \dots, c_n)^t$.

Substituting $A = c - T_{m,r} b$ in $\|b^m - c^n\|_w^2$ and doing some algebraic manipulations gives:

$$\|b^m - c^n\|_w^2 = c^t Q_n c - 2b^t T_{m,r}^t Q_n c + b^t T_{m,r}^t Q_n T_{m,r} b. \tag{15}$$

The error, defined above, is a function of the elements of the vector b . To find the minimum, we use the method of least squares approximation to find the vector \hat{b} that minimizes the last formula. We insist that the first partial derivatives $\partial(A^t Q_n A) / \partial b$ are equal zero. This process leads to the normal equations:

$$T_{m,r}^t Q_n T_{m,r} \hat{b} = T_{m,r}^t Q_n c.$$

Since $T_{m,r}^t Q_n T_{m,r} = Q_m$, and the latter matrix is positive definite. Thus $T_{m,r}^t Q_n T_{m,r}$ is invertible. Hence, the normal equations are uniquely solvable and have the solution:

$$\hat{b} = Q_m^{-1} T_{m,r}^t Q_n c. \tag{16}$$

The Bézier curve with Bézier points given in (16) is the best approximation curve in the least-squares sense with respect to the weighted L_2 -norm. A similar solution using the Legendre–Bernstein basis transformation is also given in [6].

We consider the polynomial P_n in the Chebyshev basis form with coefficients $t = (t_0, t_1, \dots, t_n)^t$, then the degree reduction with respect to the weighted L_2 -norm is reduced to the polynomial of degree $n - 1$ with coefficients $t^{(-1)} = (t_0, t_1, \dots, t_{n-1})^t$.

Applying the process of degree reduction r times to the polynomial P_n with respect to the weighted L_2 -norm we get the polynomial of degree $n - r$ with coefficients $t^{(-r)} = (t_0, t_1, \dots, t_{n-r})^t$.

This process can be written in the following matrix form:

$$t^{(-r)} = I_{n,-r} t,$$

where the $(n - r + 1) \times (n + 1)$ matrix $I_{n,-r}$ is given by

$$I_{n,-r} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix}. \tag{17}$$

The r times degree reduction matrix $R_{m,r}$ is obtained by first transforming the Bernstein coefficients to the Chebyshev coefficients by M_n^{-1} , and then the r times degree reduction by $I_{n,-r}$, and finally converting to the Bernstein coefficients by M_m . This is summarized in the following theorem.

Theorem 1. *The r times degree reduction matrix $P_{m,r}$ can be calculated using the matrices M_n^{-1} , $I_{n,-r}$ and M_m as follows*

$$R_{m,r} = Q_m^{-1} T_{m,r}^t Q_n = M_m I_{n,-r} M_n^{-1}.$$

However, for the error ϵ_w of this r times degree reduction with respect to the weighted L_2 -norm is given in the following theorem.

Theorem 2. *The error of the solution \hat{b} of the r -times degree reduction with respect to the weighted L_2 -norm is given by*

$$\epsilon_w^2 = c^t E_{m,r} c,$$

where

$$E_{m,r} = Q_n \left[I - T_{m,r} \left(T_{m,r}^t Q_n T_{m,r} \right)^{-1} T_{m,r}^t Q_n \right].$$

Example 4. The r times degree reduction matrix $P_{m,r}$ and the error matrix $E_{m,r}$ for $m = 1, 2, 3$ and $r = 1$ are

$$R_{1,1} = \begin{bmatrix} \frac{7}{8} & \frac{1}{4} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{1}{4} & \frac{7}{8} \end{bmatrix}, \quad E_{1,1} = \frac{\pi}{64} \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}.$$

$$R_{2,1} = \begin{bmatrix} \frac{31}{32} & \frac{3}{32} & -\frac{3}{32} & \frac{1}{32} \\ -\frac{1}{4} & \frac{3}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{32} & -\frac{3}{32} & \frac{3}{32} & \frac{31}{32} \end{bmatrix}, \quad E_{2,1} = \frac{5\pi}{4096} \begin{bmatrix} \frac{1}{20} & -\frac{3}{20} & \frac{3}{20} & -\frac{1}{20} \\ -\frac{3}{20} & \frac{9}{20} & -\frac{9}{20} & \frac{3}{20} \\ \frac{3}{20} & -\frac{9}{20} & \frac{9}{20} & -\frac{3}{20} \\ -\frac{1}{20} & \frac{3}{20} & -\frac{3}{20} & \frac{1}{20} \end{bmatrix}.$$

$$R_{3,1} = \begin{bmatrix} \frac{127}{128} & \frac{1}{32} & -\frac{3}{64} & \frac{1}{32} & -\frac{1}{128} \\ -\frac{33}{128} & \frac{33}{32} & \frac{29}{64} & -\frac{29}{96} & \frac{29}{384} \\ \frac{29}{384} & -\frac{29}{96} & \frac{29}{64} & \frac{33}{32} & -\frac{33}{128} \\ -\frac{1}{128} & \frac{1}{32} & -\frac{3}{64} & \frac{1}{32} & \frac{127}{128} \end{bmatrix}, \quad E_{3,1} = \frac{35\pi}{131,072} \begin{bmatrix} \frac{1}{70} & -\frac{2}{35} & \frac{3}{35} & -\frac{2}{35} & \frac{1}{70} \\ -\frac{2}{35} & \frac{8}{35} & -\frac{12}{35} & \frac{8}{35} & -\frac{2}{35} \\ \frac{3}{35} & -\frac{12}{35} & \frac{18}{35} & -\frac{12}{35} & \frac{3}{35} \\ -\frac{2}{35} & \frac{8}{35} & -\frac{12}{35} & \frac{8}{35} & -\frac{2}{35} \\ \frac{1}{70} & -\frac{2}{35} & \frac{3}{35} & -\frac{2}{35} & \frac{1}{70} \end{bmatrix}.$$

7. Degree reduction with end points interpolation

The curves $C^n(x)$ and $b^m(x)$ are continuous at the end points 0 and 1 if $b_0 = c_0$, and $b_m = c_n$. Thus, the degree reduction with continuity at the end points is given in the following theorem.

Theorem 3. *The weighted L_2 -norm between the continuous (at both end points) Bézier curves b^m and c^n is given by*

$$\|b^m - c^n\|_w^2 = \|b^{(r)} - c^n\|_w^2 = A^{*t} Q_n^* A^*,$$

where $A^* = c^* - T_{m,r}^* b$, $b = (b_0, b_1, \dots, b_m)^t$ and $c^* = (c_1, \dots, c_{n-1})^t$. Q_n^* is obtained by eliminating the first and last rows and columns from Q_n . And $T_{m,r}^*$ is obtained by eliminating the first and last rows from $T_{m,r}$.

Substituting for A^* and using the least-squares approximation gives the normal equations

$$T_{m,r}^{(*)t} Q_n^* c^* = T_{m,r}^{(*)t} Q_n^* T_{m,r}^{(*)} \hat{b}^* + T_{m,r}^{(*)t} Q_n^* T_{m,r}^+ b^+,$$

where $T_{m,r}^+$ contains the first and the last columns of $T_{m,r}^*$. b^+ contains the first and the last elements of b . $T_{m,r}^{(*)}$ is obtained by eliminating the first and the last rows and columns from $T_{m,r}$. This leads to the following theorem.

Theorem 4. *The unique solution of r times degree reduction with continuity at the end points is given by*

$$\hat{b}^* = (T_{m,r}^{(*)t} Q_n^* T_{m,r}^{(*)})^{-1} [T_{m,r}^{(*)t} Q_n^* c^* - T_{m,r}^{(*)t} Q_n^* T_{m,r}^+ b^+]. \tag{18}$$

On the other hand, since $T_k(0) = (-1)^k$, $T_k(1) = 1$, the curves

$$\sum_{k=0}^m t_k^* T_k(x), \quad \sum_{k=0}^n t_k T_k(x)$$

are continuous at $x = 0, 1$ if

$$\sum_{k=0}^m t_k^* T_k(0) = \sum_{k=0}^n t_k T_k(0), \quad \text{and} \quad \sum_{k=0}^m t_k^* T_k(1) = \sum_{k=0}^n t_k T_k(1),$$

i.e. if

$$\sum_{k=0}^m (-1)^k t_k^* = \sum_{k=0}^n (-1)^k t_k, \quad \text{and} \quad \sum_{k=0}^m t_k^* = \sum_{k=0}^n t_k.$$

Solving the first equation for t_0^* , and solving the second equation for t_1^* gives

$$t_0^* = t_0 + \sum_{k=m+1}^n (-1)^k t_k + \sum_{k=1}^m (-1)^k (t_k - t_k^*),$$

$$t_1^* = t_1 + (t_0 - t_0^*) + \sum_{k=m+1}^n t_k + \sum_{k=2}^m (t_k - t_k^*).$$

Applying the process of degree reduction r times, we get the following linear system for the Bézier points

$$t^{(-r)} = I_{n,-r} t,$$

where the matrix $I_{n,-r}$ is given in (17). Consequently, we get

$$t_0^* = t_0 + (t_1^* - t_1) + \sum_{k=m+1}^n (-1)^k t_k,$$

$$t_1^* = t_1 + (t_0 - t_0^*) + \sum_{k=m+1}^n t_k.$$

Solving these equations, we get

$$t_0^* = t_0 + \frac{1}{2} \sum_{k=m+1}^n ((-1)^k + 1)t_k,$$

$$t_1^* = t_1 - \frac{1}{2} \sum_{k=m+1}^n ((-1)^k - 1)t_k.$$

This leads to the solution in the following theorem.

Theorem 5. *The solution of the r times degree reduction with continuity conditions at the end points is*

$$M_m I_{n,-r} \left[M_{n,-}^{-1} c + t_+^* \right], \quad (19)$$

where $M_{n,-}^{-1}$ is obtained by equating the elements of the first and second rows of M_n^{-1} equal zero, and $t_+^* = (t_0^*, t_1^*, 0, \dots, 0)$.

The solutions in (18) and (19) represent two different approaches for the r times degree reduction with continuity conditions at the end points.

Acknowledgement

Lee was supported by Korea Science and Engineering Foundation Grant (R05-2004-000-10968-0) and Yoo was supported by Dongeui University.

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