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A ternary 4-point approximating subdivision scheme

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Abstract

In the implementation of subdivision scheme, three of the most important issues are smoothness, size of support, and approximation order. Our objective is to introduce an improved ternary 4-point approximating subdivision scheme derived from cubic polynomial interpolation, which has smaller support and higher smoothness, comparing to binary 4-point and 6-point schemes, ternary 3-point and 4-point schemes (see Table 2). The method is easily generalized to ternary $(2n+2)$-point approximating subdivision schemes. We choose a ternary scheme because a way to get smaller support is to raise arity. And we use polynomial reproduction to get higher approximation order easily.

Keywords: Ternary subdivision scheme; Support; Mask; Approximation order; Convergence and smoothness

1. Introduction

A subdivision scheme is useful for generating smooth curves or surfaces as the limits of sequences of successive refinements.

An n-ary linear subdivision scheme consists of linear combinations of control points in $\mathbb{R}^2$ from level k to generate level $k+1$, i.e., for all k and i, there exist sets of real number (called subdivision masks) $\mathbf{a}^k = \{a_i^k, i \in \mathbb{Z}, k = 0, 1, \ldots\}$ with $\mathbb{Z}$ the set of all integers such that

$$F_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-nj}^k F_j^k.$$

In the relation, the quantity $n(\geqslant 2)$ is termed the arity of the scheme. The constant indicates the number of new points at level $k+1$ inserted between two consecutive points from level k. In the case when $n = 2$ and 3, the subdivision schemes are called binary and ternary, respectively. For more details on general n-ary schemes, we may refer to the thesis of Aspert [1]. Let $\sigma(\mathbf{a}^k) = \{i \in \mathbb{Z} | a_i^k \neq 0\}$ be the support of the mask $\mathbf{a}^k$. When $\sigma(\mathbf{a}^k) \subset K$ for some compact set K, the subdivision scheme is said to have a finite support. If the mask $\mathbf{a}^k$ does not depend on k, i.e., $\mathbf{a}^k = \mathbf{a}$, the scheme is called stationary. Otherwise, it is called non-stationary. Similarly, if the mask is independent of i, the scheme is termed uniform.

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Starting with given control points $F^0 = \{F_i^0 = (i, f_i^0) \in \mathbb{R}^2 : i \in \mathbb{Z}\}$, the stationary ternary subdivision scheme is a process which recursively defines a sequence of control points $F^{k+1} = \{F_i^{k+1} = (x_i^{k+1}, f_i^{k+1}) \in \mathbb{R}^2 : i \in \mathbb{Z}\}$ by a finite linear combination of control points F^k with mask $\mathbf{a} = \{a_i \in \mathbb{R} : i \in \mathbb{Z}\}$;

$$F_i^{k+1} = \sum_{j \in \mathbb{Z}} a_{i-3j} F_j^k, \quad k = 0, 1, 2, \ldots.$$

which is formally denoted by $F^{k+1} = SF^k = S^k F^0$. A point of F^{k+1} is defined by a finite linear combination of points in F^k with three different rules. If a subdivision scheme retains the point of level k as a subset of point of level $k+1$, it is called an interpolating scheme. Otherwise, it is termed approximating.

A ternary subdivision scheme is said to be *uniformly convergent* if for every initial data $F^0 = \{F_i^0 = (i, f_i^0) \in \mathbb{R}^2 : i \in \mathbb{Z}\}$, there is a continuous function $f \in C(\mathbb{R})$ such that

$$\lim_{k \to \infty} \sup_{i \in \mathbb{Z}} |f_i^k - f(x_i^k)| = 0,$$

and such that $f \not\equiv 0$ for some initial data. We denote the function f by $S^\infty F^0$ or $S^\infty f^0$ for the sequence $f^0 = \{f_i^0\}_{i \in \mathbb{Z}}$, and call it a limit function of the subdivision scheme S. For the initial data $f^0 = \{f_i^0 : f_i^0 = \delta_{i,0}, i \in \mathbb{Z}\}$, the limit function, denoted by ϕ, is called the basic limit function. The scheme is said to be C^p if the basic limit function ϕ is C^p.

Three of the most important issues in the implementation of subdivision scheme are smoothness, size of support, and approximation order. The support size of a scheme influences the locality, and the approximation order is related to how closely the original function is reconstructed if an initial data is sampled from an underlying function. The approximation order is measured by the polynomial reproducing property and the smoothness of a subdivision scheme. It is well-known that a higher approximation order does not guarantee a higher regularity. For mathematical theory, approximation order is a more important property than the support size. But for Computer Aided Geometric Design (CAGD), the latter is the more important concept. However, the creation of highly smooth curves or surfaces via a subdivision scheme and the shortness of the support size of its mask are two mutually conflicting requirements. That is, the increase of the smoothness of a subdivision scheme results in that of the support size, which leads to an increase in computational effort. Our objective is to find an improved scheme derived from cubic polynomial interpolation, which has smaller support and a higher smoothness, comparing to binary 4-point and 6-point, ternary 3-point and 4-point schemes (see Table 2). The three subdivision rules of the proposed ternary 4-point approximating subdivision scheme are given by

$$F_{3i}^{k+1} = -\frac{55}{1296} F_{i-1}^k + \frac{385}{432} F_i^k + \frac{77}{432} F_{i+1}^k - \frac{35}{1296} F_{i+2}^k,$$
$$F_{3i+1}^{k+1} = -\frac{1}{16} F_{i-1}^k + \frac{9}{16} F_i^k + \frac{9}{16} F_{i+1}^k - \frac{1}{16} F_{i+2}^k,$$
$$F_{3i+2}^{k+1} = -\frac{35}{1296} F_{i-1}^k + \frac{77}{432} F_i^k + \frac{385}{432} F_{i+1}^k - \frac{55}{1296} F_{i+2}^k.$$

This ternary 4-point scheme generates C^2 curve and its approximation order is 4. And the support of the basic limit function is $\left[-\frac{11}{4}, \frac{11}{4}\right]$. The mask comes from the derive of cubic polynomial interpolation property: First, interpolating the initial data $(i+j, f_{i+j}^0)$, $j = -1, 0, 1, 2$, by a cubic polynomial and then evaluating it at $i + \frac{2j+1}{6}$, $j = 0, 1, 2$, for the values f_{3i+j}^1, $j = 0, 1, 2$. We choose a ternary scheme because a way to get a smaller support is to raise arity. And we use polynomial reproduction property to get higher approximation order. The method is easily generalized to ternary $(2n+2)$-point approximation subdivision schemes, using Lagrange interpolation polynomials.

Table 1
Mask of the proposed ternary 4-point approximating scheme

i	$\cdots$	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	$\cdots$
a_i	0	$-\frac{35}{1296}$	$-\frac{1}{16}$	$-\frac{55}{1296}$	$\frac{77}{432}$	$\frac{9}{16}$	$\frac{385}{432}$	$\frac{385}{432}$	$\frac{9}{16}$	$\frac{77}{432}$	$-\frac{55}{1296}$	$-\frac{1}{16}$	$-\frac{35}{1296}$	0

Table 2
Comparison of the proposed scheme to binary 4-point and 6-point, and ternary 3-point and 4-point schemes

Scheme	Type	Approximation order	Support (size)	C^n
Binary 4-point	Interpolating	4	6	1
Binary 6-point	Interpolating	6	10	2
Binary 4-point	Approximating	4	7	2
Ternary 3-point	Interpolating	2	4	1
Ternary 4-point	Interpolating	3	5	2
Our scheme	Approximating	4	5.5	2

2. Construction of scheme

Our primary concern in the work is to construct a uniform subdivision scheme with higher approximation order and smaller support. To this end, first, we choose a ternary scheme instead of a binary scheme in order to construct a scheme with smaller support size. Secondly, the approximation order can be guaranteed by the polynomial reproducing property (see Theorem 4.2 below). To obtain a uniform subdivision scheme, we derive the mask of this scheme by evaluation at 1/6,1/2 and 5/6 on local cubic interpolation.

In our argument, the Lagrange polynomials play a crucial role. Let $\{L_i(x)\}_{i=-1}^{2}$ be the fundamental Lagrange polynomials to the node points $\{-1,0,1,2\}$ given by

$$L_{-1}(x) = -\frac{x(x-1)(x-2)}{6}, \quad L_0(x) = \frac{(x+1)(x-1)(x-2)}{2}$$

and

$$L_1(x) = -\frac{x(x+1)(x-2)}{2}, \quad L_2(x) = \frac{x(x+1)(x-1)}{6}.$$

The Lagrange polynomials reproduce any cubic polynomial p_3 in the way that

$$p_3(x) = \sum_{\alpha=-1}^{2} p_3(\alpha)L_\alpha(x). \tag{1}$$

Now we construct the desired ternary 4-point (approximating) subdivision scheme. We sample the data $(j,f_j), j = i-1, i, i+1, i+2$ from an arbitrarily given cubic polynomial p_3;

$$p_3(j) = f_j, \quad j = i-1, i, i+1, i+2,$$

and request

$$f^1_{3i} = p_3\left(i+\frac{1}{6}\right), \quad f^1_{1+3i} = p_3\left(i+\frac{1}{2}\right), \quad f^1_{2+3i} = p_3\left(i+\frac{5}{6}\right).$$

Since our scheme is stationary, uniform and the space of polynomials up to a fixed degree are shift invariant, it is sufficient to consider the case $k = 0$ and $i = 0$, that is, the cubic polynomial such that $p_3(j) = f_j$ for $j = -1,0,1,2$. Using the Lagrange interpolation property, we have

$$p_3(1/6) = L_{-1}(1/6)f_{-1} + L_0(1/6)f_0 + L_1(1/6)f_1 + L_2(1/6)f_2,$$
$$p_3(1/2) = L_{-1}(1/2)f_{-1} + L_0(1/2)f_0 + L_1(1/2)f_1 + L_2(1/2)f_2,$$
$$p_3(5/6) = L_{-1}(5/6)f_{-1} + L_0(5/6)f_0 + L_1(5/6)f_1 + L_2(5/6)f_2.$$

For the initial values $x^0_i = i,\ i \in \mathbb{Z}$, we can see that the values x^{k+1}_{3i+j} given by

$$x^{k+1}_{3i+j} := \frac{1}{4}\left(1-\frac{1}{3^{k+1}}\right) + \frac{3i+j}{3^{k+1}}, \quad j = 0,1,2, \tag{2}$$

are obtained recursively from the subdivision rule,

$$\sum_{\alpha=-1}^{2} L_\alpha\left(\frac{1+2j}{6}\right)\left[\frac{1}{4}\left(1-\frac{1}{3^k}\right)+\frac{\alpha+i}{3^k}\right]=\frac{1}{4}\left(1-\frac{1}{3^{k+1}}\right)+\frac{3i+j}{3^{k+1}}=x^{k+1}_{3i+j}.$$

Now, as an affine combination of 4 points $f^k_{i-1}, f^k_i, f^k_{i+1}, f^k_{i+2}$, we suppose the $(k+1)$st level points f^{k+1}_{3i+j} to be attached to the values x^{k+1}_{3i+j} instead of being attached to the points $\frac{3i+j}{3^{k+1}}$.

Using the Lagrange polynomials, we propose a ternary 4-point approximating subdivision scheme as

$$f^{k+1}_{3i+j}=\sum_{\alpha=-1}^{2} L_\alpha\left(\frac{1+2j}{6}\right) f^k_{i+\alpha}, \quad j=0,1,2. \tag{3}$$

Here, we present the desired ternary 4-point approximating subdivision scheme:

$$f^{k+1}_{3i}=-\frac{55}{1296}f^k_{i-1}+\frac{385}{432}f^k_i+\frac{77}{432}f^k_{i+1}-\frac{35}{1296}f^k_{i+2},$$

$$f^{k+1}_{3i+1}=-\frac{1}{16}f^k_{i-1}+\frac{9}{16}f^k_i+\frac{9}{16}f^k_{i+1}-\frac{1}{16}f^k_{i+2},$$

$$f^{k+1}_{3i+2}=-\frac{35}{1296}f^k_{i-1}+\frac{77}{432}f^k_i+\frac{385}{432}f^k_{i+1}-\frac{55}{1296}f^k_{i+2}.$$

To obtain the scheme, we borrowed the idea of the derive of the corner-cutting subdivision scheme. And as preparing this work, we became aware that using the similar idea, Dyn et al. [4] obtained the binary 4-point scheme reproducing all the cubic polynomials.

Now, we need to check if the proposed scheme reproduces all the cubic polynomials, indeed.

Lemma 2.1. *The subdivision scheme reproduces all the cubic polynomials.*

Proof. Let p_3 be a polynomials of degree $\leqslant 3$. Assume that the data f^k_ℓ are sampled from $p_3(x^k_\ell)$ for the given values x^k_ℓ as in (2). Using the Lagrange interpolation property (1), we obtain

$$\begin{aligned} f^{k+1}_{3i+j} &= \sum_{\alpha=-1}^{2} L_\alpha\left(\frac{1+2j}{6}\right) f^k_{i+\alpha}=\sum_{\alpha=-1}^{2} L_\alpha\left(\frac{1+2j}{6}\right) p_3\left(\frac{1}{4}\left(1-\frac{1}{3^k}\right)+\frac{i+\alpha}{3^k}\right) \\ &= p_3\left(\frac{1}{4}\left(1-\frac{1}{3^k}\right)+\frac{i}{3^k}+\frac{1}{3^k}\,\frac{1+2j}{6}\right)=p_3(x^{k+1}_{3i+j}), \end{aligned}$$

which shows the lemma. □

With the same way, we can obtain the mask of a ternary $(2n+2)$-point approximation schemes by local interpolating polynomial p_{2n+1} using Lagrange interpolation polynomials $\{L_k(x)\}^{n+1}_{k=-n}$ defined by

$$L_k(x)=\prod_{j\neq k, j=-n}^{n+1}\frac{x-j}{k-j}, \quad k=-n,\ldots,n+1, \tag{4}$$

for which

$$L_k(j)=\delta_{k,j}, \quad k,j=-n,\ldots,n+1, \tag{5}$$

and

$$\sum_{k=-n}^{n+1} p(k)L_k(x)=p(x), \quad p\in P_{2n+1}. \tag{6}$$

Here, P_{2n+1} denotes the space of all polynomials of degree $\leqslant 2n+1$ for a nonnegative integer n.

We can generalize the problem of finding a mask $\mathbf{a}=\{a_i\}$ reproducing polynomials of degree $\leqslant 2n+1$, that is, we can find a $(2n+2)$-point ternary scheme reproducing all polynomials p of degree $\leqslant 2n+1$ by solving the linear equations,

$$\sum_k a_{3k}p(k) = p\left(\frac{1}{6}\right), \quad k \in \mathbb{Z},$$

$$\sum_k a_{1+3k}p(k) = p\left(\frac{1}{2}\right), \quad k \in \mathbb{Z},$$

$$\sum_k a_{2+3k}p(k) = p\left(\frac{5}{6}\right), \quad k \in \mathbb{Z}.$$

3. Analysis of the subdivision scheme

In this section, we analyze the smoothness of the proposed 4-point scheme S with the mask $\mathbf{a}$ given in Table 1. As mentioned in the introduction, our refinement rule is defined for an initial data $f^0 = \{f_i^0 : i \in \mathbb{Z}\}$ by

$$\begin{pmatrix} x_i^k \\ f_i^k \end{pmatrix} = \sum_{j\in\mathbb{Z}} a_{i-3j} \begin{pmatrix} x_j^{k-1} \\ f_j^{k-1} \end{pmatrix}, \quad i \in \mathbb{Z}\ (x_i^0 = i) \tag{7}$$

and the control points f_i^k are attached to the parameter values x_i^k ((a) of Fig. 1), not to the values $\frac{i}{3^k}$, as an usual rule ((a) of Fig. 1).

In general schemes, unlike our scheme, the control points $f_i^k = (S^k f^0)_i$ are attached to the parameter values $\frac{i}{3^k}$. And the analysis of convergence and smoothness for a subdivision scheme has been developed in this setting. However, the following theorem shows that the convergence of any of the rules induces that of the other.

Theorem 3.1. *Let S be the proposed ternary 4-point subdivision scheme with the mask $\mathbf{a} = \{a_i : i \in \mathbb{Z}\}$ given in Table 1. For each $k \geqslant 0$, let $\{(S^k\delta)_i : i \in \mathbb{Z}\}$ be the kth level points given by*

$$(S^k\delta)_i = \sum_{j\in\mathbb{Z}} a_{i-3j}(S^{k-1}\delta)_j$$

for the initial control points $\delta = \{\delta_{i,0} : i \in \mathbb{Z}\}$ and let $\{x_i^k : i \in \mathbb{Z}\}$ be the parameter values given by

$$x_i^k = \sum_{j\in\mathbb{Z}} a_{i-3j}x_j^{k-1} = \frac{1}{4}\left(1 - \frac{1}{3^k}\right) + \frac{i}{3^k}, \quad i \in \mathbb{Z}\ (x_i^0 = i).$$

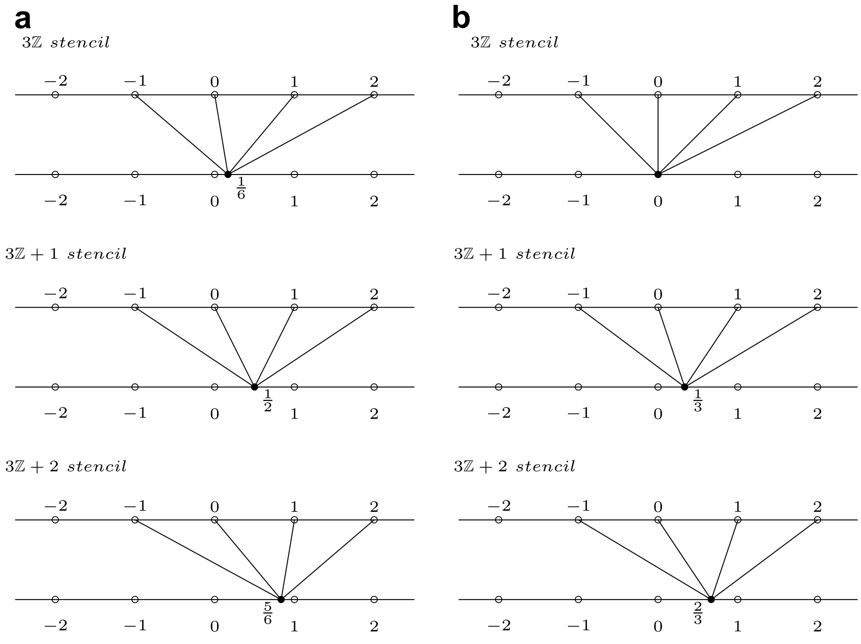

Fig. 1. (a) and (b) indicate the stencils of our scheme and a general ternary scheme.

Then the two statements are equivalent:

(i) *There is a continuous function* ϕ *on* $\mathbb{R}$ *such that*

$$\lim_{k\to\infty}\sup_{i\in\mathbb{Z}}|(S^k\delta)_i-\phi(x_i^k)|=0; \tag{8}$$

(ii) *There is a continuous function* ψ *on* $\mathbb{R}$ *such that*

$$\lim_{k\to\infty}\sup_{i\in\mathbb{Z}}\left|(S^k\delta)_i-\psi\left(\frac{i}{3^k}\right)\right|=0.$$

In this case, $\psi=\phi\left(\cdot+\frac{1}{4}\right)$ *and* ϕ *has a compact support set.*

Proof. The equivalence is straightforward and we show only the implement of (i) to (ii). We assume that the subdivision scheme S converges uniformly. Then there is a continuous function ϕ satisfying (8). From Theorem 4.3 in Section 4, ϕ has a compact support. Let $\psi=\phi\left(\cdot+\frac{1}{4}\right)$. Then for an arbitrarily given $\epsilon>0$, the assumption implies the existence of an integer $N_\epsilon>0$ such that for any $k\geqslant N_\epsilon$,

$$\sup_{i\in\mathbb{Z}}\left|(S^k\delta)_i-\psi\left(\frac{i}{3^k}-\frac{1}{4\cdot 3^k}\right)\right|\leqslant\epsilon.$$

On the other hand, we can see that ψ is uniformly continuous on $\mathbb{R}$ for ψ has a compact support. Thus, there is an integer $N_\psi>0$ such that for any $k\geqslant N_\psi$,

$$\sup_{x\in\mathbb{R}}\left|\psi(x)-\psi\left(x-\frac{1}{4\cdot 3^k}\right)\right|\leqslant\epsilon.$$

Combining these two estimates, we have that for any $k\geqslant N:=\max(N_\epsilon,N_\psi)$,

$$\sup_{i\in\mathbb{Z}}\left|(S^k\delta)_i-\psi\left(\frac{i}{3^k}\right)\right|\leqslant 2\epsilon,$$

which shows the statement (ii). The rest argument follows directly from the uniqueness of the limit of a convergent sequence. □

Due to Theorem 3.1, we may use well-known sufficient conditions to analyze the convergence and smoothness of our scheme. For each scheme S with a mask **a**, we define the Laurent polynomial as the symbol of a mask **a**

$$a(z):=\sum_{i\in\mathbb{Z}}a_iz^i.$$

From the refinement rule of S (3 refinement rules),

$$f_i^{k+1}=\sum_{j\in\mathbb{Z}}a_{i-3j}f_j^k,\quad i\in\mathbb{Z},$$

we may regard S as a operator of $\ell^\infty(\mathbb{Z})$ into itself and we have an estimate

$$\|f_i^{k+1}\|_\infty\leqslant\left(\sum_j|a_{i-3j}|\right)\max_j\|f_j^k\|.$$

Then we can calculate the norm of S:

$$\|S\|_\infty=\max\left\{\sum_j|a_{3j}|,\sum_j|a_{1+3j}|,\sum_j|a_{2+3j}|\right\}.$$

We define the generating functions of control points f^k as

$$F^k(z)=\sum_i f_i^kz^i.$$

Since the coefficient of z^i in $F^{k+1}(z)$ is f_i^{k+1} and the coefficient of z^i in $a(z)F^k(z^3)$ is $\sum_j a_{i-3j} f_j^k$, we have

$$F^{k+1}(z) = a(z)F^k(z^3).$$

Let $a^{[L]}(z) = \prod_{j=0}^{L-1} a(z^{3^j}) = \sum_i a_i^{[L]} z^i$. From the relation $F^{k+L}(z) = a^{[L]}(z)F^k(z^{3^L})$, we have the 3^L refinement rules and the norm of S^L:

$$F_i^{k+L} = \sum_j a_{i-3^L j}^{[L]} F_j^k,$$

and the norm of S^L is given by

$$\|S^L\|_\infty = \max\left\{\sum_j |a_{i-3^L j}^{[L]}|, i = 0, 1, \ldots, 3^L - 1\right\}.$$

From Theorem 3.1, we have the following theorems which play essential roles to analyze the convergence and smoothness of a subdivision scheme.

Theorem 3.2. *Let S be a convergent ternary subdivision scheme, with a mask* **a**. *Then*

$$\sum_j a_{3j} = \sum_j a_{3j+1} = \sum_j a_{3j+2} = 1. \tag{9}$$

Proof. Combining Theorem 1 in [7] and Theorem 3.1, we have the theorem. □

Applying the polynomial reproduction property (1) to the subdivision rule (3), the mask of the proposed scheme satisfies the condition (9). The symbol of a convergent ternary subdivision scheme satisfies,

$$a(e^{2i\pi/3}) = a(e^{4i\pi/3}) = 0 \quad \text{and } a(1) = 3,$$

and there exists the Laurent polynomial $a_1(z)$ such that

$$a_1(z) = \frac{3z^2}{(1 + z + z^2)} a(z).$$

Then the subdivision S_1 with symbol $a_1(z)$ is related to S with symbol $a(z)$ by the following theorem.

Theorem 3.3 [6]. *Let S denote a ternary subdivision scheme with symbol* $a(z)$ *satisfying* (9). *Then there exists a subdivision scheme* S_1 *with the property*

$$\mathrm{d}f^k = S_1 \mathrm{d}f^{k-1},$$

where $f^k = S^k f^0 = \{f_i^k : i \in \mathbb{Z}\}$ *and* $\mathrm{d}f^k = \{(\mathrm{d}f^k)_i = 3^k(f_{i+1}^k - f_i^k) : i \in \mathbb{Z}\}$.

Using the subdivision scheme S_1, we can check the convergence of S as follows:

Theorem 3.4. *S is a uniformly convergent ternary subdivision scheme if and only if* $\frac{1}{3}S_1$ *converges uniformly to the zero function for all initial data* f^0.

$$\lim_{k\to\infty} \left(\frac{1}{3}S_1\right)^k f^0 = 0. \tag{10}$$

Proof. It follows from Theorem 4.2 in [6] and Theorem 3.1. □

A scheme S_1 satisfying (10) for all initial data f^0 is termed contractive. Theorem 3.4 indicates that checking of the convergence of S is equivalent to checking whether S_1 is contractive, which is equivalent to checking whether $\left\|\left(\frac{1}{3}S_1\right)^L\right\|_\infty < 1$, for some integer $L > 0$. After the convergence of S is determined, we need to check the smoothness of the limit functions generated by S. An condition of C^m continuity is expressed in the following theorem.

Theorem 3.5. *Let us consider a scheme S with Laurent polynomial $a(z)$. If there exists a polynomial $b(z)$ such that*

$$a(z) = \left(\frac{1+z+z^2}{3z^2}\right)^m b(z),$$

and such that the associated scheme $\frac{1}{3}S_b$ is contractive, then the limit function is C^m for any initial data.

Proof. It follows from Theorem 4.4 in [6] and Theorem 3.1. □

Now, we are ready to analyze the smoothness of the proposed scheme. For the given ternary mask:

$$\mathbf{a} = \frac{1}{1296}[-35, -81, -55, 231, 729, 1155, 1155, 729, 231, -55, -81, -35],$$

we have the mask of scheme S_1:

$$\mathbf{a_1} = \frac{3}{1296}[-35, -46, 26, 251, 452, 452, 251, 26, -46, -35],$$

where $a_1(z) = \frac{3z^2 a(z)}{1+z+z^2}$. It is easy to verify that $a(z)$ and $a_1(z)$ satisfy the necessary condition (9) for the convergence S and S_1. Since

$$\left\|\frac{1}{3}S_1\right\|_\infty = \max\left\{\frac{524}{1296}, \frac{572}{1296}\right\} < 1,$$

this scheme converges uniformly. We have the mask of S_2

$$\mathbf{a_2} = \frac{9}{1296}[-35, -11, 72, 190, 190, 72, -11, -35],$$

and

$$\left\|\frac{1}{3}S_2\right\|_\infty = \max\left\{\frac{708}{1296}, \frac{432}{1296}\right\} < 1.$$

Hence this scheme has $C^1(\mathbb{R})$. We can verify that $a_2(z)$ satisfies the necessary condition for $C^2(\mathbb{R})$. And the mask of S_3 is

$$\mathbf{a_3} = \frac{27}{1296}[-35, 24, 83, 83, 24, -35],$$

and we get

$$\left\|\frac{1}{3}S_3\right\|_\infty = \max\left\{\frac{1062}{1296}, \frac{432}{1296}\right\} < 1.$$

Hence this scheme is $C^2(\mathbb{R})$. The mask of S_4 is

$$\mathbf{a_4} = \frac{81}{1296}[-35, 59, 59, -35],$$

and we have

$$\left\|\frac{1}{3}S_4\right\|_\infty = \max\left\{\frac{1890}{1296}, \frac{1593}{1296}\right\} > 1.$$

Actually, there exists no integer $L > 0$ such that

$$\left\|\left(\frac{1}{3}S_4\right)^L\right\|_\infty < 1,$$

therefore this scheme can not generate $C^3(\mathbb{R})$ functions.

4. Approximation order and support

While the regularity of the limit function for the subdivision scheme is important, another an important issue of subdivision scheme is how to attain the original function as close as possible if a given initial data f^0 is sampled from an underlying function.

Definition 4.1. Let us consider the initial grid $X_0 = h\mathbb{Z}$ and initial data $f_i^0 = g(ih)$ sampled a enough smooth function g. Let us denote by f^∞ the limit function obtained through subdivision. The subdivision scheme has approximation order p if

$$|(g - f^\infty)(x)| \leqslant Ch^p, \quad x \in \mathbb{R},$$

where C is a real constant and independent of h.

As seen in Theorem 4.3 below, the approximation order of a subdivision scheme can be obtained from its precision set.

Theorem 4.2 [3]. *An convergent subdivision scheme that reproduces polynomial P_n has an approximation order of $n+1$.*

From Lemma 2.1 and Theorem 4.2, the proposed scheme has approximation order 4.

Next, we consider the support of the proposed scheme. This is the support of the basic limit function $\phi = S^\infty\delta$ generated by the given control point $f_i^0 = \delta_{i,0}$ as shown in Fig. 2.

Theorem 4.3. *Let S be the proposed ternary 4-point subdivision scheme with a mask* **a** *given in Table 1. Then we have*

$$\mathrm{supp}(\phi) = \mathrm{supp}(S^\infty\delta) = \left[-\frac{11}{4}, \frac{11}{4}\right].$$

Proof. Choose $f^0 = \{f_\alpha^0 : f_\alpha^0 = \delta_{\alpha,0}, \alpha \in \mathbb{Z}\}$, and let $S^\infty\delta = \phi$. From the subdivision rule

$$(S^k\delta)_i = \sum_{j\in\mathbb{Z}} a_{i-3j}(S^{k-1}\delta)_j,$$

we have that $\mathrm{supp}(S\delta) = \mathrm{supp}(\mathbf{a}) = [-6, 5]$ and for each $k = 2, 3, \ldots,$

$$\mathrm{supp}(S^k\delta) = \{i \in \mathbb{Z} : i - 3j \in \mathrm{supp}(\mathbf{a}), j \in \mathrm{supp}(S^{k-1}\delta)\} = \{i \in \mathbb{Z} : i \in \mathrm{supp}(\mathbf{a}) + 3\mathrm{supp}(S^{k-1}\delta)\}.$$

Thus, $\mathrm{supp}(S^k\delta) = \frac{3^k-1}{2}\mathrm{supp}(\mathbf{a})$. The values $S^k\delta$ are attached to the parameter values

$$\frac{1}{4}\left(1 - \frac{1}{3^k}\right) + \frac{1}{3^k}\mathrm{supp}(S^k\delta) = \frac{1}{4}\left(1 - \frac{1}{3^k}\right) + \frac{1-3^{-k}}{2}\mathrm{supp}(S^k\delta).$$

Hence, the support of the limit function ϕ is

$$\mathrm{supp}(\phi) = \mathrm{supp}(S^\infty\delta) = \left[-\frac{11}{4}, \frac{11}{4}\right],$$

which completes the proof. □

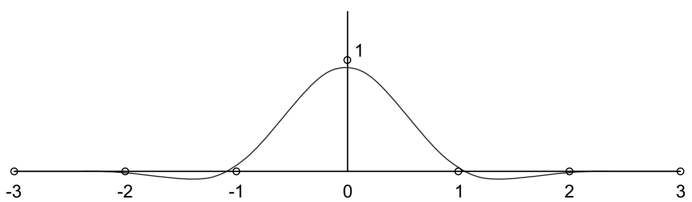

Fig. 2. The basic limit function of the proposed scheme.

The support of the proposed subdivision scheme is smaller than the support $[-3,3]$ of the binary 4-point DD subdivision scheme.

In Table 2, the mask of the classical binary 4-point [2] is given by

$$\frac{1}{16}[-1,0,9,16,9,0,-1].$$

In his thesis, Weissman [8] generated the 6-point interpolating subdivision scheme of the form

$$f_{2i}^{k+1} = f_i^k,$$
$$f_{2i+1}^{k+1} = \left(\frac{9}{16}+2\theta\right)(f_i^k + f_{i+1}^k) - \left(\frac{1}{16}+3\theta\right)(f_{i-1}^k + f_{i+2}^k) + \theta(f_{i-2}^k + f_{i+3}^k).$$

He proved that for $0 < \theta < 0.02$, this scheme creates C^2 limit functions. Dyn et al. [4] introduced the C^2 binary 4-point approximating scheme whose mask is given by

$$\frac{1}{128}[-5,-7,35,105,105,35,-7,-5].$$

And the ternary 3-point [5] and 4-point [6] schemes are related to tension parameters a and μ, respectively. Here we choose the tension parameters to generate the highest smoothness ($a = -\frac{1}{15}$ and $\mu = \frac{1}{11}$). In this case, the masks of the ternary 3-point and 4-point schemes are given by

$$\frac{1}{15}[-1,0,4,12,15,12,4,0,-1],$$

and

$$\frac{1}{99}[-4,-7,0,34,76,99,76,34,0,-7,-4],$$

respectively.

We illustrate the proposed scheme by applying to the control points forming the cross polygon in Fig. 3. In the figure, the curve interpolating the control points is generated by the 4-point DD scheme and the other is created by the proposed subdivision scheme.

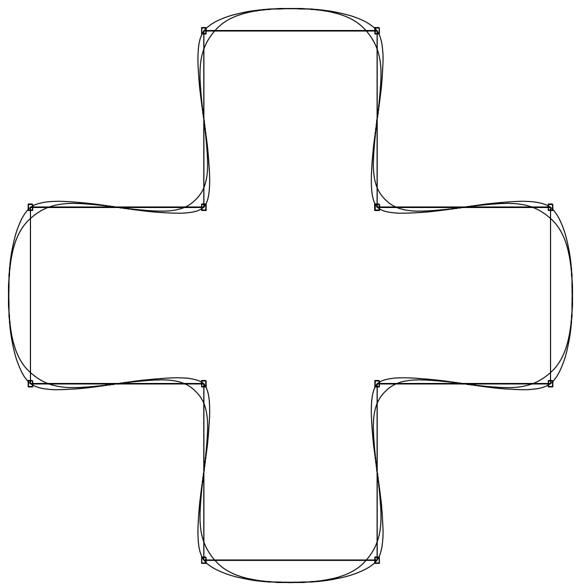

Fig. 3. Comparison of the binary 4-point interpolating DD scheme and the proposed ternary 4-point approximating scheme.

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