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A NEW PROOF OF THE SMOOTHNESS OF 4-POINT DESLAURIERS-DUBUC SCHEME

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ABSTRACT. It is well-known that the smoothness of 4-point interpolatory Deslauriers-Dubuc(DD) subdivision scheme is C^1. N. Dyn[3] proved that 4-point interpolatory subdivision scheme is C^1 by means of eigenanalysis. In this paper we take advantage of Laurent polynomial method to get the same result, and give new way of strict proof on Laurent polynomial method.

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1. Introduction

The stationary binary subdivision scheme is a process which recursively defines a sequence of control points $P^k = \{p_i^k : i \in Z\}$ by a rule of the form with a mask $\mathbf{a} = \{a_i : i \in Z\}$

$$p_i^{k+1} = \sum_{j \in Z} a_{i-2j} p_j^k, \quad k \in Z_+,$$

which is denoted formally by $P^{k+1} = SP^k$. A point of P^{k+1} is defined by a finite linear combination of points in P^k with two different rules.

Since each component of vector valued functions in R^3 is a scalar function generated by the same subdivision scheme, the analysis of a binary subdivision scheme can be reduced to the scalar case to initial sets of control points. Therefore, starting with given control points $f^0 = \{f_i : i \in Z\}$, we consider scalar sets

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of control points $f^k = \{f_i^k : i \in Z\}$ generated by the relation

$$f_i^{k+1} = \sum_{j \in Z} a_{i-2j} f_j^k, \quad k \in Z_+.$$

We denote the function f by $S^\infty f^0$, and call it a limit function of S or a function generated by S. Interpolatory subdivision scheme retains the point of stage k as a subset of point of stage $k+1$, that is $a_{2i} = \delta_{0,i}, i \in Z$. Thus the general form of an interpolatory subdivision scheme is

$$\begin{aligned} f_{2i}^{k+1} &= f_i^k, \\ f_{2i+1}^{k+1} &= \sum_{j \in Z} a_{1+2j} f_{i-j}^k. \end{aligned}$$

Such schemes were introduced and analyzed in [1], [2] and [4]. The 4-point interpolatory subdivision scheme was introduced in [3] and is defined as follows:

$$\begin{aligned} f_{2i}^{k+1} &= f_i^k, \\ f_{2i+1}^{k+1} &= \Big(\frac{1}{2} + w\Big)\Big(f_i^k + f_{i+1}^k\Big) - w\Big(f_{i-1}^k + f_{i+2}^k\Big), \end{aligned}$$

where $\{f_i^0\}$ is a set of initial control points. The parameter w serves as a tension parameter in the sense that decreasing its value to zero is equivalent to tightening the limit curve toward piecewise linear curve between the initial control points. It is well-known that this N.Dyn 4-point scheme generates continuous limit functions for $|w| < \frac{1}{4}$, and C^1 limit functions for $0 < w < \frac{1}{8}$. Deslauriers and Dubuc in [1] suggested an interpolatory binary subdivision scheme using polynomial reproducing property. The idea behind Deslauriers and Dubuc scheme(DD scheme) is that if the original control points fall on a polynomial of degree $2n+1$, then the next level's control points must also lie on the same polynomial. With the special choice $w = \frac{1}{16}$, this 4-point DD scheme is exact for cubic polynomial.

N. Dyn[7] stated we can construct N. Dyn 4-point and Weissman 6-point schemes by taking a convex combination of the two DD schemes. And B. G. Lee, et. al[5] introduced the mask of interpolatory symmetric subdivision schemes(4-point and 6-point interpolatory schemes, ternary 4-point interpolatory scheme, butterfly scheme and modified butterfly scheme) using symmetry and necessary condition for smoothness. It is well-known that the smoothness of 4-point interpolatory DD subdivision scheme is C^1 but not C^2. N. Dyn[3] proved that 4-point interpolatory subdivision scheme is C^1 by means of eigenanalysis. In this paper we take advantage of Laurent polynomial method to get the same result.

2. Preliminaries

A binary univariate subdivision scheme is defined in terms of a mask consisting of a finite set of non-zero coefficients $\mathbf{a} = \{a_i : i \in Z\}$. The scheme is given by

$$f_i^{k+1} = \sum_{j \in Z} a_{i-2j} f_j^k, \quad i \in Z.$$

Interpolatory subdivision schemes retain the points of stage k as a subset of the points of stage $k+1$. Thus the general form of an interpolatory subdivision scheme is

$$\begin{aligned} f_{2i}^{k+1} &= f_i^k, \\ f_{2i+1}^{k+1} &= \sum_{j \in Z} a_{1+2j} f_{i-j}^k. \end{aligned}$$

We can now state, in a formal definition, the notion of convergence of subdivision scheme.

Definition 1. A subdivision scheme S is *uniformly convergent* if for any initial data $f^0 = \{f_i^0 : i \in Z\}$, there exist a continuous function f, such that for any closed interval $I \subset R$, satisfies

$$\lim_{k \to \infty} \sup_{i \in 2^k I} \left| f_i^k - f\left(2^{-k} i\right) \right| = 0.$$

Obviously $f = S^\infty f^0$. For each scheme S with mask $\mathbf{a}$, we define the symbol

$$a(z) = \sum_{i \in Z} a_i z^i.$$

Since the schemes we consider have masks of finite support, the corresponding symbols are Laurent polynomials, namely polynomials in positive and negative powers of the variables.

Theorem 1 ([Dyn])**.** *Let S be a convergent subdivision scheme with a mask* $\mathbf{a}$. *Then*

$$\sum_{j \in Z} a_{2j} = \sum_{j \in Z} a_{2j+1} = 1. \tag{1}$$

It follows from Theorem 1 that the symbol of a convergent subdivision scheme satisfies

$$a(-1) = 0 \text{ and } a(1) = 2.$$

This condition guarantees the existence of a related subdivision scheme for the divided differences of the original control points and the existence of Laurent polynomial

$$a_1(z) = \frac{2z}{(1+z)} a(z).$$

The subdivision S_1 with symbol $a_1(z)$ is related to S with symbol $a(z)$ by the following theorem.

Theorem 2 ([Dyn])**.** *Let S denote a subdivision scheme with symbol $a(z)$ satisfying (1). Then there exist a subdivision scheme S_1 with the property*

$$df^k = S_1 df^{k-1},$$

where $f^k = S^k f^0$ and $df^k = \left\{ \left(df^k \right)_i = 2^k \left(f^k_{i+1} - f^k_i \right) : i \in Z \right\}$.

We can now determine the convergence of S by analyzing the subdivision scheme $\frac{1}{2}S_1$.

Theorem 3 ([Dyn])**.** *S is a uniformly convergent subdivision scheme if and only if $\frac{1}{2}S_1$ converges uniformly to the zero function for all initial data f^0, that is*

$$\lim_{k->\infty} \left(\frac{1}{2} S_1 \right)^k f^0 = 0. \tag{2}$$

A scheme S_1 satisfying (2) for all initial data f^0 is termed 'contractive'. By Theorem 3, the check of the convergence of S is equivalent to checking whether S_1 is contractive, which is equivalent to checking whether $||(\frac{1}{2}S_1)^L||_\infty < 1$, for some $L \in Z_+$.

Since there are two rules for computing the values at the next refinement level, one with the even coefficients of the mask and one with odd coefficients of the mask, we define the norm

$$\|S\|_\infty = \max \left\{ \sum_i |a_{2i}|, \sum_i |a_{2i+1}| \right\},$$

and

$$\left\| \left(\frac{1}{2} S \right)^L \right\|_\infty = \max \left\{ \sum_\beta \left| a^L_{\gamma + 2^L \beta} \right| : \gamma = 0, 1, \ldots, 2^L - 1 \right\},$$

where

$$a^L(z) = \prod_{j=0}^{L-1} a\left(z^{2^j} \right).$$

Theorem 4 ([Dyn])**.** *Let $a(z) = \frac{(1+z)^m}{2^m} b(z)$. If S_b is convergent, then $S_a^\infty f^0 \in C^m(R)$ for any initial data f^0.*

3. The smoothness of Deslauriers-Dubuc scheme

The 4-point Deslauriers-Dubuc(DD) scheme is:

$$\begin{aligned} f_{2i}^{k+1} &= f_i^k, \\ f_{2i+1}^{k+1} &= \frac{9}{16}\Big(f_i^k + f_{i+1}^k\Big) - \frac{1}{16}\Big(f_{i-1}^k + f_{i+2}^k\Big). \end{aligned}$$

From the mask of this scheme, we get the Laurent polynomial $a(z)$

$$a(z) = -\frac{1}{16}z^{-3} + \frac{9}{16}z^{-1} + 1 + \frac{9}{16}z - \frac{1}{16}z^3.$$

We can easily prove that the smoothness of this scheme is C^1 by Laurent polynomial method.

We set

$$b^{[m,L]}(z) = \frac{1}{2^L}a_m^{[L]}(z), \quad m = 1, 2, \ldots, L$$

where

$$a_m(z) = \frac{2z}{1+z}a_{m-1}(z) = \left(\frac{2z}{1+z}\right)^m a(z),$$

and

$$a_m^{[L]}(z) = \prod_{j=0}^{L-1} a_m(z^{2^j}).$$

From the Laurent polynomial, we have

$$b^{[1,1]}(z) = \frac{1}{2}a_1(z) = \frac{z}{1+z}a(z) = -\frac{1}{16}z^{-2} + \frac{1}{16}z^{-1} + \frac{1}{2} + \frac{1}{2}z + \frac{1}{16}z^2 - \frac{1}{16}z^3.$$

And we measure the norm of subdivision $\frac{1}{2}S_1$.

$$\begin{aligned} \Big\|\frac{1}{2}S_1\Big\|_\infty &= \max\Big\{\sum_\beta \Big|b_{\gamma+2\beta}^{[1,1]}\Big| : \gamma = 0, 1\Big\} \\ &= \max\{5/8, 5/8\} = \frac{5}{8} < 1. \end{aligned}$$

Therefore $\frac{1}{2}S_1$ is contractive, by Theorem 3, and so S is convergent.

To prove that this 4-point DD scheme is C^1, from the Laurent polynomial we have

$$b^{[2,1]}(z) = \frac{1}{2}a_2(z) = \frac{z}{1+z}a_1(z) = -\frac{1}{8}z^{-1} + \frac{1}{4} + \frac{3}{4}z + \frac{1}{4}z^2 - \frac{1}{8}z^3.$$

And we find the norm of subdivision $\frac{1}{2}S_2$.

$$\begin{aligned} \Big\|\frac{1}{2}S_2\Big\|_\infty &= \max\Big\{\sum_\beta \Big|b_{\gamma+2\beta}^{[2,1]}\Big| : \gamma = 0, 1\Big\} \\ &= \max\{1/2, 1\} = 1. \end{aligned}$$

But for $L = 2$, we get

$$b^{[2,2]}(z) = \frac{1}{4}a_2^{[2]}(z) = \Big(\frac{1}{2}a_2(z)\Big)\Big(\frac{1}{2}a_2(z^2)\Big)$$

and

$$\begin{aligned}\left\|\left(\frac{1}{2}S_2\right)^2\right\|_\infty &= \max\left\{\sum_\beta\left|b^{[2,2]}_{\gamma+4\beta}\right| : \gamma=0,1,2,3\right\}\\ &= \max\{5/16, 1/4, 5/16, 3/4\} < 1.\end{aligned}$$

Since subdivision scheme $\frac{1}{2}S_2$ is contractive, by Theorem 4, S_1 is convergent and $S \in C^1$.

To prove that this 4-point DD scheme is not C^2, we should show that there exist $L \in Z^+$ such that $\|(\frac{1}{2}a_3)^L\|_\infty < 1$. From Laurent polynomial, we get $b^{[3,1]}(z)$

$$b^{[3,1]}(z) = \frac{1}{2}a_3(z) = \frac{z}{1+z}a_2(z) = -\frac{1}{4} + \frac{3}{4}z + \frac{3}{4}z^2 - \frac{1}{4}z^3.$$

We empirically calculate the norm of $\frac{1}{2}S_3$ for $L = 1, 2, 3$.

$$\begin{aligned}\left\|\frac{1}{2}S_3\right\|_\infty &= \max\{1,1\},\\ \left\|\left(\frac{1}{2}S_3\right)^2\right\|_\infty &= \max\left\{\sum_\beta\left|b^{[3,2]}_{\gamma+4\beta}\right| : \gamma=0,1,2,3\right\} = \max\{5/8, 5/8, 1, 1\},\\ \left\|\left(\frac{1}{2}S_3\right)^3\right\|_\infty &= \max\left\{\sum_\beta\left|b^{[3,3]}_{\gamma+8\beta}\right| : \gamma=0,1,\ldots,7\right\}\\ &= \max\{5/8, 7/32, 7/16, 7/16, 7/32, 5/8, 1, 1\}.\end{aligned}$$

Here we have the following question: Can we find some $L \in Z^+$ satisfying

$$\left\|\left(\frac{1}{2}S_3\right)^L\right\|_\infty < 1.$$

In the following we will prove that there exist no $L \in Z^+$ such that

$$\left\|\left(\frac{1}{2}S_3\right)^L\right\|_\infty < 1.$$

This means that the smoothness of 4-point DD scheme is not C^2.

If we denote Laurent polynomial $b^{[3,1]}(z)$

$$\begin{aligned}b^{[3,1]}(z) &= -\frac{1}{4} + \frac{3}{4}z + \frac{3}{4}z^2 - \frac{1}{4}z^3\\ &= -\alpha + \beta z + \beta z^2 - \alpha z^3\\ &= b^{[3,1]}_0 + b^{[3,1]}_1 z + b^{[3,1]}_2 z^2 + b^{[3,1]}_3 z^3,\end{aligned}$$

then we can get the following equations

$$\begin{aligned}
b^{[3,2]}(z) &= b^{[3,1]}(z)b^{[3,1]}(z^2).\\
b^{[3,3]}(z) &= b^{[3,1]}(z)b^{[3,1]}(z^2)b^{[3,1]}(z^4) = b^{[3,2]}(z)b^{[3,1]}(z^4).\\
&\cdots\cdots\cdots\\
b^{[3,L]}(z) &= b^{[3,1]}(z)b^{[3,1]}(z^2)\cdots b^{[3,1]}(z^{2^{L-1}}) = b^{[3,L-1]}(z)b^{[3,1]}(z^{2^L-1})\\
&= b_0^{[3,L]} + \cdots + b_{3(2^L-1)}^{[3,L]} z^{3(2^L-1)}.
\end{aligned}$$

Lemma 1. *For any $L \in Z^+$, we have*

$$B^{[3,L]} := \sum_\beta \left| b^{[3,L]}_{(2^L-1)+2^L\beta} \right| = \sum_{\beta=0}^{1} \left| b^{[3,L]}_{(2^L-1)+2^L\beta} \right| = \left| b^{[3,L]}_{(2^L-1)} \right| + \left| b^{[3,L]}_{2^{L+1}-1} \right|.$$

Namely, when $\beta \geq 2$, we have $b^{[3,L]}_{(2^L-1)+2^L\beta} = 0$.

Proof. The highest degree of Laurent polynomial $b^{[3,L]}(z)$ is $U := 3(2^L-1)$ since

$$z^3 \times z^{3\times 2} \times \cdots \times z^{3\times(2^{L-1})} = z^{3(2^L-1)}.$$

And when $\beta = 2$, the degree is $V := (2^L-1) + 2^L 2 = 3\cdot 2^L - 1$. So we get $U - V = -2 < 0$. This means that the highest degree of $b^{[3,L]}_{(2^L-1)+2^L\beta}$ is lower than that of $b^{[3,L]}_{(2^L-1)+2^L 2}$. Therefore for $\beta \geq 2$, we have $b^{[3,L]}_{(2^L-1)+2^L\beta} = 0$. □

Comparing the coefficients of z^{2^L-1} and $z^{2^{L+1}-1}$ in both hand sides, we obtain the following lemma.

Lemma 2. *For any $L \in Z^+$, we have*

$$\begin{aligned}
b^{[3,L]}_{(2^L-1)} &= b^{[3,L-1]}_{(2^{L-1}-1)} \times \beta + b^{[3,L-1]}_{(2^L-1)} \times (-\alpha),\\
b^{[3,L]}_{(2^{L+1}-1)} &= b^{[3,L-1]}_{(2^{L-1}-1)} \times (-\alpha) + b^{[3,L-1]}_{(2^L-1)} \times \beta.
\end{aligned}$$

Corollary 1. *For any $L \in Z^+$, we have*

$$\begin{aligned}
\left| b^{[3,L]}_{(2^L-1)} \right| &= b^{[3,L-1]}_{(2^{L-1}-1)} \times \beta + \left| b^{[3,L-1]}_{(2^L-1)} \right| \times \alpha,\\
\left| b^{[3,L]}_{(2^{L+1}-1)} \right| &= b^{[3,L-1]}_{(2^{L-1}-1)} \times \alpha + \left| b^{[3,L-1]}_{(2^L-1)} \right| \times \beta.
\end{aligned}$$

Proof. For $L=1$, we have

$$\begin{aligned} b_1^{[3,1]} &= \beta = \frac{3}{4} > 0, \\ b_3^{[3,1]} &= -\alpha = -\frac{1}{4} < 0. \end{aligned}$$

And for $L=2$, we have

$$\begin{aligned} b_3^{[3,2]} &= b_1^{[3,1]} \times \beta + b_3^{[3,1]} \times (-\alpha) > 0, \\ b_7^{[3,2]} &= b_1^{[3,1]} \times (-\alpha) + b_3^{[3,1]} \times \beta < 0. \end{aligned}$$

We assume that for $L-1$, it is true, that is

$$\begin{aligned} b_{2^{L-1}-1}^{[3,L-1]} &= b_{2^{L-2}-1}^{[3,L-2]} \times \beta + b_{2^{L-1}-1}^{[3,L-2]} \times (-\alpha) > 0, \\ b_{2^{L}-1}^{[3,L-1]} &= b_{2^{L-2}-1}^{[3,L-2]} \times (-\alpha) + b_{2^{L-1}-1}^{[3,L-2]} \times \beta < 0. \end{aligned}$$

We can easily check the following inequality

$$\begin{aligned} b_{2^{L}-1}^{[3,L]} &= b_{2^{L-1}-1}^{[3,L-1]} \times \beta + b_{2^{L}-1}^{[3,L-1]} \times (-\alpha) > 0, \\ b_{2^{L+1}-1}^{[3,L]} &= b_{2^{L-1}-1}^{[3,L-1]} \times (-\alpha) + b_{2^{L}-1}^{[3,L-1]} \times \beta < 0. \end{aligned}$$

So by mathematical induction, we prove

$$\begin{aligned} \left|b_{2^{L}-1}^{[3,L]}\right| &= b_{(2^{L-1}-1)}^{[3,L-1]} \times \beta + \left|b_{(2^{L}-1)}^{[3,L-1]}\right| \times \alpha, \\ \left|b_{2^{L+1}-1}^{[3,L]}\right| &= b_{(2^{L-1}-1)}^{[3,L-1]} \times \alpha + \left|b_{(2^{L}-1)}^{[3,L-1]}\right| \times \beta. \end{aligned}$$

□

From Lemma 1 and Corollary 1, we have the following lemma.

Lemma 3. *For any $L \in Z^+$, we have*

$$B^{[3,L]} = B^{[3,L-1]}.$$

Proof. By Lemma 1 and Corollary 1 and the fact that $\alpha+\beta=1$, we get

$$\begin{aligned} B^{[3,L]} &= \left|b_{2^{L}-1}^{[3,L]}\right| + \left|b_{2^{L+1}-1}^{[3,L]}\right| \\ &= \left(b_{2^{L-1}-1}^{[3,L-1]} \times \beta + \left|b_{2^{L}-1}^{[3,L-1]}\right| \times \alpha\right) + \left(b_{2^{L-1}-1}^{[3,L-1]} \times \alpha + \left|b_{2^{L}-1}^{[3,L-1]}\right| \times \beta\right) \\ &= b_{2^{L-1}-1}^{[3,L-1]} + \left|b_{2^{L}-1}^{[3,L-1]}\right| = B^{[3,L-1]}. \end{aligned}$$

□

Theorem 5. *For any $L \in Z^+$, we have*

$$\sum_{\beta=0}^{1} \left| b^{[3,L]}_{(2^L-1)+2^L\beta} \right| = 1.$$

Proof. For $L = 1$, we have

$$B^{[3,1]} = \left| b^{[3,1]}_1 | + | b^{[3,1]}_3 \right| = \beta + \alpha = 1.$$

By Lemma 3, we get

$$\sum_{\beta=0}^{1} \left| b^{[3,L]}_{(2^L-1)+2^L\beta} \right| = B^{[3,L]} = B^{[3,L-1]} = \cdots = B^{[3,1]} = 1.$$

□

By theorem 5, S_3 is not contractive and so S_2 is not convergent and S is not C^2.

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