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A study on the mask of interpolatory symmetric subdivision schemes

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Abstract

In the work, we rebuild the masks of well-known interpolatory symmetric subdivision schemes-binary $2n$-point interpolatory schemes, the ternary 4-point interpolatory scheme using only the symmetry and the necessary condition for smoothness and the butterfly scheme, and the modified butterfly scheme using the factorization property.

Keywords: Mask; Interpolatory symmetric subdivision scheme; Ternary scheme; The butterfly scheme

1. Introduction

Subdivision schemes are a very efficient tool for the fast construction of smooth curves and surfaces from a set of control points through iterative refinements. In recent years, subdivision schemes became one of the most popular methods of creating curves and surfaces in computer aided geometric design (CAGD) and in the animation industry. Now, they have become a subject of study in their own right with a variety of application to computer graphics, geometric modeling and wavelets. Furthermore, schemes have been applied to solve problems in fluid flow. Their popularity is due to the facts that subdivision algorithms are simple to apprehend, suitable for computer applications, easy to implement. Also, the subject of subdivision schemes has an interdisciplinary relation to wavelet, which has brought on reciprocal improvement in both the areas.

Each subdivision scheme is associated with a mask $\mathbf{a} = \{a_\alpha \in \mathbb{R} : \alpha \in \mathbb{Z}^s\}$, where $s = 1$ in the curve case and $s = 2$ in the surface case. The (stationary) subdivision scheme is a process which recursively defines a sequence of control points $f^k = \{f^k_\alpha : \alpha \in \mathbb{Z}^s\}$ by a rule of the form with a mask $\mathbf{a} = \{a_i\}_{i\in\mathbb{Z}^s}$

$$f^{k+1}_\alpha = \sum_{\beta\in\mathbb{Z}^s} a_{\alpha-M\beta} f^k_\beta, \quad k \in \{0, 1, 2, \dots\},$$

which is denoted formally by $f^{k+1} = Sf^k$. Then a point of f^{k+1} is defined by a finite affine combination of points in f^k. Here M is an $s \times s$ integer matrix such that $\lim_{n\to\infty} M^{-n} = 0$. The matrix M is called a dilation

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matrix. Binary (or dyadic) and ternary subdivision schemes are schemes with the matrices $M = 2I$ and $M = 3I$, respectively, for the $s \times s$ identity matrix I.

Among subdivision schemes, an interpolatory subdivision is more intuitive because it preserves the data obtained at the former stage and refines new data by inserting values corresponding to intermediate points. The inserted values are determined by linear combinations of values at the neighboring points.

An interpolatory symmetric binary scheme is one of the most popular subdivision schemes in their applications. This results in its simplicity to use and the tendency that customers feel comfortable for symmetric objects. Despite its simplicity, an interpolatory binary subdivision scheme has the drawback that in order to create smoother curves or surfaces, it is necessary to enlarge the support of the mask. It is well-known, in general, that the creation of highly smooth curves or surfaces in a subdivision scheme and the shortness of the support size of its mask are two mutually conflicting requirements, in spite that designers in CAGD require subdivision schemes to have their masks with a possibly smaller support and to create good smooth curves or surfaces. For example, it was proved in [8] that there is no C^2 interpolatory binary subdivision scheme with a mask supported on $[-3,3]^s$.

There are two approaches to achieve a desired mask: one is to enlarge the size of the support so that we may allow free parameters in the mask; the other is to fix the size and find specific values of the mask. There is no unique method of obtaining a mask. Deslauriers and Dubuc [2] obtained the mask of $2n$-point interpolatory subdivision rule by polynomial reproducing property. Dyn [3] commented that we can get the mask of 4-point and 6-point schemes with a one parameter by taking a convex combination of the two Deslauriers and Dubuc schemes. In the work, we try to find smoother possible masks with the size fixed by applying repeatedly some smoothness conditions. Then, we rebuild the masks of interpolatory symmetric subdivision schemes-binary $2n$-point interpolatory schemes, ternary 4-point interpolatory scheme-in Sections 3 and 4. In Section 5, we study the butterfly scheme and we observe that in its stencil, the sums of even masks and odd masks along every vertical or horizontal line are the same, which induces a factorization of the corresponding Laurent polynomial (see Theorem 5). By applying the observation, we show that from the same structure of the stencil used in the butterfly scheme, we can obtain the mask of the scheme and the modified butterfly schemes, as well.

2. Preliminaries

Throughout the work, we consider schemes with a mask of finite support. That is, the set $\{i \in \mathbb{Z}^s : a_i \neq 0\}$ is finite. This property is useful in the practical implementation because changes in a control point affect only its local neighborhood.

In this section, we consider a subdivision scheme S with a mask $\mathbf{a}$ and a dilation matrix M.

We denote by ϕ_0 the tensor product of the symmetric hat function in $\mathbb{R}^s$ defined by

$$\phi_0(x) := \prod_{j=1}^{s} B_1(x_j) \quad \text{for } (x_1, \ldots, x_s) \in \mathbb{R}^s, \tag{1}$$

where B_1 is the B-spline of degree 1 (order 2) given as

$$B_1(x) = \begin{cases} 1 + x, & \text{for } x \in [-1, 0), \\ 1 - x, & \text{for } x \in [0, 1], \\ 0, & \text{for } x \in \mathbb{R} \setminus [-1, 1]. \end{cases}$$

The subdivision operator S associated with a mask $\mathbf{a}$ and a dilation matrix M is the linear operator on the space $\ell(\mathbb{Z}^s)$ of all sequences on $\mathbb{Z}^s$ defined by

$$Sv(\alpha) := \sum_{\beta \in \mathbb{Z}^s} a_{\alpha - M\beta} v(\beta), \quad v \in \ell(\mathbb{Z}^s).$$

Definition 1. A subdivision scheme S with a mask $\mathbf{a}$ and a dilation matrix M *converges uniformly* if for every sequence $f^0 = (f^0_\alpha)_{\alpha \in \mathbb{Z}^s}$ with compact support, there exists a function $f \in C(\mathbb{R}^s)$ such that

$$\lim_{n\to\infty}\left\|\sum_{\alpha\in\mathbb{Z}^s} S^n f^0_\alpha \phi_0(M^n\cdot-\alpha)-f\right\|_\infty = 0,$$

where ϕ_0 is the unit hat function given in (1). And for some sequence, the above function f is not identically zero. The function f is denoted by $S^\infty f^0$. A uniformly subdivision scheme is said to be C^m if for any initial data the limit function has continuous derivatives up to order m.

Let E be a complete set of representatives of the distinct cosets of $\mathbb{Z}^s/M\mathbb{Z}^s$. Then $\mathbb{Z}^s$ is the disjoint union of $\gamma+M\mathbb{Z}^s$, $\gamma\in E$. Also, it is not difficult to see that the set $\{M^{-n}\alpha:\alpha\in\mathbb{Z}^s,\ n\in\mathbb{N}\}$ is dense in $\mathbb{R}^s$.

Theorem 1. *Let S be a subdivision scheme with a mask* **a** *and a dilation matrix M. If S converges uniformly, then for every* $\gamma\in E$, *we have*

$$\sum_{\alpha\in\mathbb{Z}^s} a_{\gamma-M\alpha} = 1. \tag{2}$$

Proof. Let f^0 be an initial data such that $S^\infty f^0\neq 0$, and let $\gamma\in E$ be arbitrarily fixed. From (1) and Definition 1, we can see that

$$\lim_{n\to\infty}\sup_{\alpha\in\mathbb{Z}^s}|S^n f^0(\alpha)-f(M^{-n}\alpha)| = 0.$$

By the continuity of f, there are $\alpha\in\mathbb{Z}^s$ and $m\in\mathbb{N}$ such that $f(M^{-m}\alpha)\neq 0$. For $n\geqslant \max(m,N)$, we have that

$$S^n f^0(M^{n-m}\alpha+\gamma) = \sum_{\beta\in\mathbb{Z}^s} a_{M^{n-m}\alpha+\gamma-M\beta} S^{n-1}f^0(\beta) = \sum_{\beta\in\mathbb{Z}^s} a_{M\beta+\gamma} S^{n-1} f^0(M^{n-m-1}\alpha-\beta).$$

With the equation above, we induce that

$$\begin{aligned} &f(M^{-m}\alpha+M^{-n}\gamma)-\sum_{\beta\in\mathbb{Z}^s} a_{M\beta+\gamma} f(M^{-m}\alpha-M^{-n+1}\beta)\\ &\quad = f(M^{-m}\alpha+M^{-n}\gamma)-S^n f^0(M^{n-m}\alpha+\gamma)+\sum_{\beta\in\mathbb{Z}^s} a_{M\beta+\gamma}(S^{n-1}f^0(M^{n-m-1}\alpha-\beta)-f(M^{-m}\alpha-M^{-n+1}\beta)). \end{aligned}$$

On the other hand, since the support of the mask a is finite, we can exchange the limit and the summation as we take $n\to\infty$. Hence, by the uniform convergence of S and the continuity of f, we obtain that

$$f(M^{-m}\alpha) = \sum_{\beta\in\mathbb{Z}^s} a_{\gamma+M\beta} f(M^{-m}\alpha).$$

Since $f(M^{-m}\alpha)\neq 0$, we have, as a consequence, that

$$\sum_{\beta\in\mathbb{Z}^s} a_{\gamma+M\beta} = 1,\quad \gamma\in E. \qquad \square$$

For the case when M is two times the $s\times s$ identity matrix, Theorem 1 was proved by Cavaretta et al. [1], and Dyn [5]. Han and Jia [9] showed that the condition in (2) with $\sum_{\alpha\in\mathbb{Z}^s} a_\alpha = |\det(M)|$ is necessary for the subdivision scheme to converge in the L_p-norm.

3. Binary 2*n*-point interpolatory symmetric subdivision scheme

A binary univariate subdivision scheme is defined in terms of a mask consisting of a sequence of coefficients $\mathbf{a}=\{a_i : i\in\mathbb{Z}\}$, that is, a univariate scheme with a dilation matrix $M=2$. The scheme is given by

$$f_i^{k+1} = \sum_{j\in Z} a_{i-2j} f_j^k,\quad i\in\mathbb{Z}.$$

For each scheme S with a mask $\mathbf{a}$, we define the symbol

$$a(z) = \sum_{\alpha \in Z} a_\alpha z^\alpha.$$

The corresponding symbols, called Laurent polynomials, play a efficient role in analyzing the smoothness of a subdivision scheme.

Interpolatory subdivision schemes retain the points of stage k as a subset of the points of stage $k+1$. Thus, the general form of an interpolatory subdivision scheme is

$$f_{2i}^{k+1} = f_i^k,$$
$$f_{2i+1}^{k+1} = \sum_{j \in Z} a_{1+2j} f_{i-j}^k.$$

From Theorem 1, we can see that

$$\sum_j a_{2j} = \sum_j a_{2j+1} = 1 \tag{3}$$

and the Laurent polynomial of a convergent subdivision scheme satisfies

$$a(-1) = 0 \quad \text{and} \quad a(1) = 2. \tag{4}$$

This condition guarantees the existence of a related subdivision scheme for the divided differences of the original control points and the existence of associated Laurent polynomial $a_1(z)$ which can be defined as follows:

$$a_1(z) = \frac{2z}{(1+z)} a(z). \tag{5}$$

The subdivision S_1 with symbol $a_1(z)$ is related to S with $a(z)$ by the following theorem.

Theorem 2 [5]. *Let S denote a subdivision scheme with symbol $a(z)$ satisfying (3). Then there exist a subdivision scheme S_1 with the property*

$$\mathrm{d}f^k = S_1 \, \mathrm{d}f^{k-1},$$

where $f^k = S^k f^0$ and $\mathrm{d}f^k = \{(\mathrm{d}f^k)_i = 2^k (f_{i+1}^k - f_i^k) : i \in \mathbb{Z}\}$. Furthermore, S is a uniformly convergent subdivision scheme if and only if $\frac{1}{2}S_1$ converges uniformly to the zero function for all initial data f^0 in the sense that

$$\lim_{k \to \infty} \left(\frac{1}{2} S_1\right)^k f^0 = 0. \tag{6}$$

A scheme S_1 satisfying (6) for all initial data f^0 is said to be *contractive*. By Theorem 2, the check of the convergence of S is equivalent to checking whether S_1 is contractive, which is equivalent to checking whether there exists an integer $L > 0$ such that the operator of L iterations of S_1 satisfies $\|\left(\frac{1}{2}S_1\right)^L\|_\infty < 1$.

Here, how to compute the norm of $\|S^L\|_\infty$? From the refinement rule of S (two refinement rules)

$$f_i^{k+1} = \sum_j a_{i-2j} f_j^k,$$

we have

$$\|f_i^{k+1}\| \leqslant \left(\sum_j |a_{i-2j}|\right) \max_j \|f_j^k\|$$

and we can calculate the norm of S:

$$\|S\|_\infty = \max\left\{\sum_j |a_{2j}|, \sum_j |a_{1+2j}|\right\}.$$

We define generating function of control point f^k as

$$F^k(z) = \sum_i f_i^k z^i.$$

Since the coefficient of z^i in $F^{k+1}(z)$ is f_i^{k+1} and the coefficient of z^i in $a(z)F^k(z^2)$ is $\sum_j a_{i-2j}f_j^k$, we have

$$F^{k+1}(z) = a(z)F^k(z^2).$$

Let $a^{[L]}(z) = \prod_{j=0}^{L-1} a(z^{2^j}) = \sum_i a_i^{[L]} z^i$, $F^{k+L}(z) = \sum_i f_i^{k+L} z^i$ and well-known fact $F^{k+L}(z) = a^{[L]}(z)F^k(z^{2^L})$, we have the 2^L refinement rules

$$f_i^{k+L} = \sum_j a^{[L]}_{i-2^L j} f_j^k$$

and norm of S^L:

$$\|S^L\|_\infty = \max\left\{\sum_j |a^{[L]}_{i-2^L j}|, i = 0, 1, \ldots, 2^L - 1\right\}.$$

Theorem 3 [5]. *Let $a(z) = \frac{(1+z)^m}{2^m} b(z)$. If the subdivision S_b corresponding to $b(z)$ is convergent, then $S_a^\infty f^0 \in C^m(\mathbb{R})$ for any initial data f^0.*

Therefore, we get the fact that $a(-1) = 0$ is the necessary condition for the convergence of a subdivision scheme and $a_m(-1) = 0$ (where we define $a_m(z) = \frac{2z}{1+z} a(z), a_0(z) = a(z)$) is the necessary condition for C^m-smoothness. Now, we try to find the masks of binary interpolatory symmetric subdivision schemes using only the smoothness conditions.

3.1. 4-point interpolatory subdivision scheme

The insertion rule of a 4-point interpolatory symmetric subdivision scheme is

$$f_{2i}^{k+1} = f_i^k,$$
$$f_{2i+1}^{k+1} = a_3 f_{i-1}^k + a_1 f_i^k + a_{-1} f_{i+1}^k + a_{-3} f_{i+2}^k.$$

The Laurent polynomial of this scheme is given as

$$a(z) = a_{-3}z^{-3} + a_{-1}z^{-1} + 1 + a_1 z + a_3 z^3.$$

Here, $a_{-1} = a_1$ and $a_{-3} = a_3$ by the symmetric subdivision scheme condition. And in order for the scheme to be $C^0(\mathbb{R})$, we have necessarily $a(-1) = 0$, that is, $a_1 + a_3 = \frac{1}{2}$.

In this case, we obtain the Laurent polynomial of $\frac{1}{2}S_1$ as

$$\frac{1}{2}a_1(z) = \frac{z}{1+z}a(z) = a_3 z^{-2} - a_3 z^{-1} + \frac{1}{2} + \frac{1}{2}z - a_3 z^2 + a_3 z^3.$$

Applying the necessary condition for $C^1(\mathbb{R})$, we have $a_1(-1) = 0$. Set $a_3 = -w$, we find the mask of the Dyn 4-point interpolatory subdivision scheme:

$$\left[-w, \frac{1}{2} + w, \frac{1}{2} + w, -w\right].$$

Dyn [5] found a range of the parameter w for smoothness that this scheme generates C^1-functions for $0 < w < 1/8$.

3.2. 6-point interpolatory subdivision scheme

The general form of 6-point interploatory subdivision scheme is given by

$$f_{2i}^{k+1} = f_i^k,$$
$$f_{2i+1}^{k+1} = a_5 f_{i-2}^k + a_3 f_{i-1}^k + a_1 f_i^k + a_{-1} f_{i+1}^k + a_{-3} f_{i+2}^k + a_{-5} f_{i+3}^k.$$

The corresponding Laurent polynomial of this scheme is

$$a(z) = a_{-5}z^{-5} + a_{-3}z^{-3} + a_{-1}z^{-1} + 1 + a_1 z + a_3 z^3 + a_5 z^5.$$

The symmetric condition induces that $a_{-1} = a_1$, $a_{-3} = a_3$, $a_{-5} = a_5$, and the Laurent polynomial can be written by

$$a(z) = a_5 z^{-5} + a_3 z^{-3} + a_1 z^{-1} + 1 + a_1 z + a_3 z^3 + a_5 z^5.$$

In order for the scheme to converge uniformly, we need the condition $a(-1) = 0$, which induces the relation $a_1 + a_3 + a_5 = \frac{1}{2}$. The Laurent polynomial of $\frac{1}{2} S_1$ is

$$\begin{aligned}\frac{1}{2} a_1(z) &= \frac{z a(z)}{1+z} \\ &= a_5 z^{-4} - a_5 z^{-3} + (a_5 + a_3) z^{-2} - (a_5 + a_3) z^{-1} + \frac{1}{2} + \frac{1}{2} z - (a_3 + a_5) z^2 + (a_3 + a_5) z^3 - a_5 z^4 + a_5 z^5.\end{aligned}$$

To generate C^1-functions, the Laurent polynomial must satisfy $a_1(-1) = 0$. It is easily to see that this is always true in this case. And the Laurent polynomial of $\frac{1}{2} S_2$ becomes

$$\begin{aligned}\frac{1}{2} a_2(z) = \frac{z a_1(z)}{1+z} = 2\Big\{ & a_5 z^{-3} - 2a_5 z^{-2} + (3a_5 + a_3) z^{-1} - (4a_5 + 2a_3) + \left(\frac{1}{2} + 4a_5 + 2a_3\right) z \\ & - (4a_5 + 2a_3) z^2 + (3a_5 + a_3) z^3 - 2a_5 z^4 + a_5 z^5 \Big\}.\end{aligned}$$

The necessary condition for C^2-smoothness implies $a_2(-1) = 0$, i.e., $24a_5 + 8a_3 + \frac{1}{2} = 0$. Set $a_5 = w$, then $a_3 = -\frac{1}{16} - 3w$, $a_1 = \frac{9}{16} + 2w$, and we have found the mask of 6-point interpolatory subdivision scheme:

$$\begin{aligned} f_{2i}^{k+1} &= f_i^k, \\ f_{2i+1}^{k+1} &= w(f_{i-2}^k + f_{i+3}^k) - \left(\frac{1}{16} + 3w\right)(f_{i-1}^k + f_{i+2}^k) + \left(\frac{9}{16} + 2w\right)(f_i^k + f_{i+1}^k). \end{aligned}$$

We can easily obtain the mask of 4-point, 6-point, 8-point and 10-point interpolatory symmetric subdivision schemes (ISSS) by using the same process:

- 4-point scheme: $[a_1, a_3] = \left[w_2 + \frac{1}{2}, -w_2\right]$.
- 6-point scheme: $[a_1, a_3, a_5] = \left[2w_3 + \frac{9}{16}, -3w_3 - \frac{1}{16}, w_3\right]$.
- 8-point scheme: $[a_1, \ldots, a_7] = \left[5w_4 + \frac{75}{128}, -9w_4 + \frac{25}{256}, 5w_4 - \frac{3}{256} - w_4\right]$.
- 10-point scheme: $[a_1, \ldots, a_9] = \left[14w_5 + \frac{1225}{2048}, -28w_5 - \frac{245}{2048}, 20w_5 + \frac{49}{2048}, -7w_5 - \frac{5}{2048}, w_5\right]$.

From these masks, Ko et al. [11] found out a general formula for the mask of $(2n+4)$-point ISSS with two parameters which reproduces all polynomials of degree $\leqslant 2n+1$ and some relations between the mask of the $(2n+4)$-point ISSS and the $(2n+2)$-point Deslauriers and Dubuc schemes.

4. Ternary interpolatory symmetric subdivision scheme

A ternary subdivision scheme is a univariate one with a dilation matrix $M = 3$. The scheme is given by

$$f_i^{k+1} = \sum_{j \in Z} a_{i-3j} f_j^k, \quad i \in \mathbb{Z}. \tag{7}$$

And the general rule of a ternary interpolatory subdivision scheme is

$$\begin{aligned} f_{3i}^{k+1} &= f_i^k, \\ f_{3i+1}^{k+1} &= \sum_{j \in Z} a_{1+3j} f_{i-j}^k, \\ f_{3i+2}^{k+1} &= \sum_{j \in Z} a_{2+3j} f_{i-j}^k. \end{aligned}$$

Theorem 1 shows that the mask $\{a_i\}_{i\in\mathbb{Z}}$ of a convergent ternary subdivision scheme S satisfies

$$\sum_j a_{3j} = \sum_j a_{3j+1} = \sum_j a_{3j+2} = 1. \tag{8}$$

The Laurent polynomial of a convergent subdivision scheme satisfies

$$a(\mathrm{e}^{2\mathrm{i}\pi/3}) = a(\mathrm{e}^{4\mathrm{i}\pi/3}) = 0 \quad \text{and} \quad a(1) = 3$$

and there exists the Laurent polynomial $a_1(z)$ such that

$$a_1(z) = \frac{3z^2}{(1+z+z^2)}a(z).$$

The subdivision S_1 with symbol $a_1(z)$ is related to S with symbol $a(z)$ by the following theorem.

Theorem 4 [10]. *Let S denote a subdivision scheme with symbol $a(z)$ satisfying (8). Then there exists a subdivision scheme S_1 with the property*

$$\mathrm{d}f^k = S_1\,\mathrm{d}f^{k-1},$$

where $f^k = S^k f^0$ and $\mathrm{d}f^k = \{(\mathrm{d}f^k)_i = 3^k(f^k_{i+1} - f^k_i) : i \in Z\}$. And S is a uniformly convergent subdivision scheme if and only if $\frac{1}{3}S_1$ converges uniformly to the zero function for all initial data f^0

$$\lim_{k->\infty}\left(\frac{1}{3}S_1\right)^k f^0 = 0.$$

Furthermore, S generates C^m-limit functions provided that the subdivision scheme S_1 generates C^{m-1}-limit functions for some integer $m \geqslant 1$.

4.1. Ternary 4-point interpolatory subdivision scheme

We present a ternary 4-point interplatory subdivision scheme with three subdivision rules:

$$f^{k+1}_{3i} = f^k_i,$$
$$f^{k+1}_{3i+1} = a_4 f^k_{i-1} + a_1 f^k_i + a_{-2} f^k_{i+1} + a_{-5} f^k_{i+2},$$
$$f^{k+1}_{3i+2} = a_5 f^k_{i-1} + a_2 f^k_i + a_{-1} f^k_{i+1} + a_{-4} f^k_{i+2}.$$

The Laurent polynomial of this scheme is

$$a(z) = a_{-5}z^{-5} + a_{-4}z^{-4} + a_{-2}z^{-2} + a_{-1}z^{-1} + 1 + a_1 z + a_2 z^2 + a_4 z^4 + a_5 z^5.$$

With the symmetry condition, $a(z)$ can be written by

$$a(z) = a_5 z^{-5} + a_4 z^{-4} + a_2 z^{-2} + a_1 z^{-1} + 1 + a_1 z + a_2 z^2 + a_4 z^4 + a_5 z^5.$$

To generate C^0-functions, we require that $a(1) = 3$. And it implies $a_1 + a_2 + a_4 + a_5 = 1$. The Laurent polynomial of $\frac{1}{3}S_1$ is

$$\frac{1}{3}a_1(z) = \frac{z^2 a(z)}{1+z+z^2} = a_5 z^{-3} + (a_4 - a_5)z^{-2} - a_4 z^{-1} + (a_5 + a_2) + (1 - 2a_5 - 2a_2)z$$
$$+ (a_2 + a_5)z^2 - a_4 z^3 + (a_4 - a_5)z^4 + a_5 z^5.$$

From the necessary condition for C^1-smoothness, the mask corresponding to $a_1(z)$ satisfies the relation as in (8), i.e., $3a_2 - 3a_4 + 6a_5 = 1$. And the Laurent polynomial of $\frac{1}{3}S_2$ is

$$\frac{1}{3}a_2(z) = \frac{z^2 a_1(z)}{1+z+z^2} = 3a_5 z^{-1} + 3(a_4 - 2a_5) + 3(a_5 - 2a_4)z + 3\left(\frac{1}{3} + 2a_4\right)z^2$$
$$+ 3(a_5 - 2a_4)z^3 + 3(a_4 - 2a_5)z^4 + 3a_5 z^5.$$

To generate C^2-functions, we require the mask of S_2 to satisfy the condition (8), i.e., $a_4 + a_5 = -\frac{1}{9}$. Set $a_5 = -\frac{1}{18} + \frac{1}{6}\mu$, we can find the mask of ternary 4-point interpolatory subdivision scheme:

$$f_{3i}^{k+1} = f_i^k,$$
$$f_{3i+1}^{k+1} = -\left(\frac{1}{18}+\frac{1}{6}\mu\right)f_{i-1}^k + \left(\frac{13}{18}+\frac{1}{2}\mu\right)f_i^k + \left(\frac{7}{18}-\frac{1}{2}\mu\right)f_{i+1}^k - \left(\frac{1}{18}-\frac{1}{6}\mu\right)f_{i+2}^k,$$
$$f_{3i+2}^{k+1} = -\left(\frac{1}{18}-\frac{1}{6}\mu\right)f_{i-1}^k + \left(\frac{7}{18}-\frac{1}{2}\mu\right)f_i^k + \left(\frac{13}{18}+\frac{1}{2}\mu\right)f_{i+1}^k - \left(\frac{1}{18}+\frac{1}{6}\mu\right)f_{i+2}^k.$$

Hassan et al. [10] showed that the scheme is C^2 for $\frac{1}{15} < \mu < \frac{1}{9}$.

5. Bivariate schemes on regular meshes

In this section, we consider cases when $s = 2$ and M is two times the 2×2 identity matrix. In this case, we may assume that the complete set E of representatives of the distinct cosets of $\mathbb{Z}^2/2\mathbb{Z}^2$ consists of the vectors $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$.

For a quad-mesh, consider the refinement rule for a set of point $f_i \in \mathbb{R}^3$, $i \in \mathbb{Z}^2$

$$f_i^{k+1} = \sum_{j\in\mathbb{Z}^2} a_{i-2j} f_j^k, \quad i \in \mathbb{Z}^2. \tag{9}$$

In the bivariate case, there are 4 rules (even–even, even–odd, odd–even and odd-odd) depending on the parity of each component of the vector $i \in \mathbb{Z}^2$.

$$f_{(2i_1,2i_2)}^{k+1} = \sum_{j_1,j_2} a_{(2j_1,2j_2)} f_{(i_1-j_1,i_2-j_2)}^k,$$
$$f_{(1+2i_1,2i_2)}^{k+1} = \sum_{j_1,j_2} a_{(1+2j_1,2j_2)} f_{(i_1-j_1,i_2-j_2)}^k,$$
$$f_{(2i_1,1+2i_2)}^{k+1} = \sum_{j_1,j_2} a_{(2j_1,1+2j_2)} f_{(i_1-j_1,i_2-j_2)}^k,$$
$$f_{(1+2i_1,1+2i_2)}^{k+1} = \sum_{j_1,j_2} a_{(1+2j_1,1+2j_2)} f_{(i_1-j_1,i_2-j_2)}^k.$$

The easiest way to extend univariate to bivariate schemes is to consider tensor-product schemes. For example, using the mask of Chaikin's scheme, we can get the mask of the bivariate biquadratic scheme

$$a_{(2i,2j)}:\quad \frac{1}{16}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}, \qquad a_{(2i+1,2j)}:\quad \frac{1}{16}\begin{bmatrix} 3 & 9 \\ 1 & 3 \end{bmatrix},$$
$$a_{(2i,2j+1)}:\quad \frac{1}{16}\begin{bmatrix} 3 & 1 \\ 9 & 3 \end{bmatrix}, \qquad a_{(2i+1,2j+1)}:\quad \frac{1}{16}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}.$$

We can also obtain this mask set by convolution of single quadratic B-spline mask $\frac{1}{4}[1 \quad 3 \quad 3 \quad 1]$.

$$\frac{1}{4}[3 \quad 1] * \frac{1}{4}[3 \quad 1], \quad \frac{1}{4}[3 \quad 1] * \frac{1}{4}[1 \quad 3],$$
$$\frac{1}{4}[1 \quad 3] * \frac{1}{4}[3 \quad 1], \quad \frac{1}{4}[1 \quad 3] * \frac{1}{4}[1 \quad 3].$$

Doo–Sabin presented an algorithm which generalize the biquadratic B-spline subdivision rule to include arbitrary topology. Similarly, we can get the mask of bi-cubic refinement rule through the tensor product of single cubic B-spline mask $\frac{1}{8}[1 \quad 4 \quad 6 \quad 4 \quad 1]$.

$$a_{(2i,2j)}:\quad \frac{1}{64}\begin{bmatrix} 1 & 6 & 1 \\ 6 & 36 & 6 \\ 1 & 6 & 1 \end{bmatrix}, \qquad a_{(2i+1,2j)}:\quad \frac{1}{64}\begin{bmatrix} 4 & 4 \\ 24 & 24 \\ 4 & 4 \end{bmatrix},$$
$$a_{(2i,2j+1)}:\quad \frac{1}{64}\begin{bmatrix} 4 & 24 & 4 \\ 4 & 24 & 4 \end{bmatrix}, \qquad a_{(2i+1,2j+1)}:\quad \frac{1}{64}\begin{bmatrix} 16 & 16 \\ 16 & 16 \end{bmatrix}.$$

Let us consider the topology of a regular triangulation. We regard the subdivision scheme as operating on the three-directional grid, since the three-directional grid can be regarded also as $\mathbb{Z}^2$.

Let $a(\mathbf{z}) = a(z_1, z_2) = \sum_{i,j} a_{ij} z_1^i z_2^j$ be the symbol of a bivariate subdivision scheme S which is defined on regular triangulations.

From Theorem 1 the mask of a bivariate convergent subdivision scheme S satisfies

$$\sum_{\beta \in Z^2} a_{\alpha - 2\beta} = 1, \quad \alpha \in \{(0,0),(0,1),(1,0),(1,1)\}. \tag{10}$$

It follows that the symbol of a bivariate convergent subdivision scheme satisfies:

$$a(-1,1) = a(1,-1) = a(-1,-1) = 0 \quad \text{and} \quad a(1,1) = 4. \tag{11}$$

In contrast to the univariate case, the necessary condition (10) and the derived condition (11) on $a(z)$ do not imply a factorization of the corresponding symbol in the bivariate case. However, we have the following:

Theorem 5. *Let S be a bivariate subdivision scheme with a compactly supported mask corresponding to its Laurent polynomial $a(z_1, z_2) = \sum_{i,j \in \mathbb{Z}} a_{ij} z_1^i z_2^j$. Then we have*

(i) *for $i = 1, 2$, $a(z_1, z_2)$ has $1 + z_i$ as a factor if and only if*

$$a(z_1, z_2)|_{z_i = -1} = 0; \tag{12}$$

(ii) *$a(z_1, z_2)$ has $1 + z_1 z_2$ as a factor if and only if*

$$a(z_1, t/z_1)|_{t=-1} = 0 \quad \textit{equivalently or} \quad a(z_2, t/z_2)|_{t=-1} = 0. \tag{13}$$

Proof. The proof is straightforward and we show only that $a(-1, z_2) = 0$ if and only if $a(z_1, z_2)$ has $z_1 + 1$ as a factor. We can expand $a(z_1, z_2)$ with respect to z_2 as

$$a(z_1, z_2) = \sum_{i \in \mathbb{Z}} a_i(z_1) z_2^i$$

for some polynomials a_i in one variable. Then it is easy to see that $a(-1, z_2) = 0$ if and only if for every $i \in \mathbb{Z}$, $a_i(-1) = 0$, which implies that $a(z_1, z_2)$ has $z_1 + 1$ as a factor. The remains are shown in the same argument, which completes the proof. □

As we can see in the matrix form of the butterfly scheme below, Theorem 5 means in the geometrical point of view that when we plot the masks a_{ij} at the point (i,j) in $\mathbb{Z}^2$-plane, the condition $a(-1, z_2) = 0$ if and only if the sums of even masks and of odd masks along each horizontal line are the same, that is to say, for every $k \in \mathbb{Z}$,

$$\sum_{i \in \mathbb{Z}} (-1)^i a_{i,k} = 0.$$

And we can see that $a(z_1, t/z_1)|_{t=-1} = 0$ if and only if

$$\sum_{i \in \mathbb{Z}} (-1)^i a_{i,i+k} = 0, \quad k \in \mathbb{Z}.$$

For integers k and ℓ, expanding $a(z_1, z_2)$ as

$$a(z_1, z_2) = \sum_{i \in \mathbb{Z}} b_i(z_1^k z_2^\ell) z_1^i$$

with polynomials b_i in one variable, we have, in general, that $a(z_1, z_2)$ has $1 + z_1^k z_2^\ell$ as a factor if and only if the mask $\{a_{ij}\}$ satisfies

$$\sum_{i \in \mathbb{Z}} (-1)^i a_{ik, i\ell + j} = 0 \quad \text{for every } j \in \mathbb{Z}.$$

Comparing Theorem 2 in the univariate case, Dyn [4] found the following criterions for the verification of the convergence and smoothness of a bivariate subdivision scheme.

Theorem 6 [4]. *Let S be a bivariate subdivision scheme with its symbol $a(z_1,z_2)$ having $(1+z_1)(1+z_2)(1+z_1z_2)$ as a factor. Then, S is convergent if and only if the schemes corresponding to the symbols*

$$a_{1,0}(z_1,z_2)=\frac{a(z_1,z_2)}{1+z_1},\quad a_{0,1}(z_1,z_2)=\frac{a(z_1,z_2)}{1+z_2},\quad a_{1,1}(z_1,z_2)=\frac{a(z_1,z_2)}{1+z_1z_2} \tag{14}$$

are contractive. If any two of these schemes are contractive then the third is also contractive.

Theorem 7 [4]. *Let S be a bivariate subdivision scheme with its symbol $a(z_1,z_2)$ having $(1+z_1)(1+z_2)(1+z_1z_2)$ as a factor. Then S generates C^1 limit functions if the schemes with the symbols $2a_{1,0}(z_1,z_2)$, $2a_{0,1}(z_1,z_2)$ and $2a_{1,1}(z_1,z_2)$ in (14) are convergent. If any of two of these schemes are convergent then the third is also convergent. Moreover,*

$$\begin{aligned}
&\frac{\partial}{\partial z_1}S^\infty f^0=S_{1,0}\Delta_{1,0}f^0,\\
&\frac{\partial}{\partial z_2}S^\infty f^0=S_{0,1}\Delta_{0,1}f^0,\\
&\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}\right)S^\infty f^0=S_{1,1}\Delta_{1,1}f^0.
\end{aligned}$$

To check if a scheme generates smooth limit functions with the aid of Theorems 6 and 7, we have to assume that the symbol of a subdivision scheme is factorizable:

$$a(z_1,z_2)=(1+z_1)(1+z_2)(1+z_1z_2)b(z_1,z_2). \tag{15}$$

In the following three sections, we study symmetric subdivision schemes with factorizable symbols $a(z_1,z_2)$ as in (15). At first, we recall the most popular subdivision scheme in this case, the butterfly scheme, and then we show that the butterfly scheme and the modified butterfly scheme are rebuilt only using the factorization property and their stencil structures. Note that to verify the smoothness of a symmetric subdivision scheme, based on Theorems 6 and 7, we have only to check the contractivity of one of the two schemes $a_{1,0}(z_1,z_2)$ and $a_{0,1}(z_1,z_2)$ in (14), regarding the symmetry property.

5.1. The Dyn butterfly subdivision scheme

Dyn et al. [6] introduced the butterfly scheme. The butterfly scheme is an extension of the 4-point interpolatory subdivision scheme to the bivariate case with topology of regular triangulation, which is an interpolatory triangular subdivision scheme with stencil of small support (see Fig. 1).

The mask of the butterfly scheme is symmetric $(a_{i,j}=a_{j,i}, i,j\in\mathbb{Z})$ and given as

$$\begin{aligned}
&a_{0,0}=1,\\
&a_{1,0}=a_{-1,0}=a_{-1,-1}=a_{1,1}=1/2,\\
&a_{1,-1}=a_{-1,-2}=a_{1,2}=2w,\\
&a_{1,-2}=a_{-3,-2}=a_{-1,2}=a_{3,2}=a_{-1,-3}=a_{1,3}=-w
\end{aligned}$$

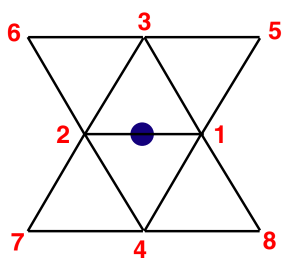
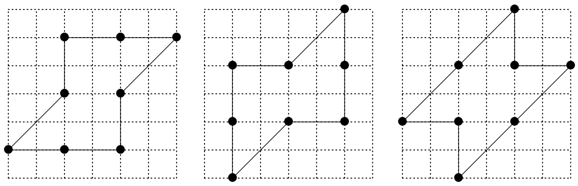


Fig. 1. Stencil of the Dyn butterfly scheme: $\left(1,2=\frac{1}{2}\right)$, $(3,4=2w)$, $(5,6,7,8=-w)$.



(a) Odd-Even masks (b) Even-Odd masks (c) Odd-Odd masks

Fig. 2. The mask maps of the butterfly scheme.

and zero otherwise. There are three kinds of refinement rules as in (9):

$$f^{k+1}_{2i+1,2j}=\frac{1}{2}\left(f^k_{i,j}+f^k_{i+1,j}\right)+2w\left(f^k_{i,j-1}+f^k_{i+1,j+1}\right)-w\left(f^k_{i-1,j-1}+f^k_{i+1,j-1}+f^k_{i,j+1}+f^k_{i+2,j+1}\right),$$
$$f^{k+1}_{2i,2j+1}=\frac{1}{2}\left(f^k_{i,j}+f^k_{i,j+1}\right)+2w\left(f^k_{i-1,j}+f^k_{i+1,j+1}\right)-w\left(f^k_{i-1,j-1}+f^k_{i-1,j+1}+f^k_{i+1,j}+f^k_{i+1,j+2}\right),$$
$$f^{k+1}_{2i+1,2j+1}=\frac{1}{2}\left(f^k_{i,j}+f^k_{i+1,j+1}\right)+2w\left(f^k_{i+1,j}+f^k_{i,j+1}\right)-w\left(f^k_{i,j-1}+f^k_{i-1,j}+f^k_{i+2,j+1}+f^k_{i+1,j+2}\right).$$

In Fig. 2, we plot the index of non-zero masks according to the refinement rules as follows. The mask maps in Fig. 2 suggest why the scheme is called the butterfly subdivision scheme. We express the mask of the butterfly scheme in the matrix form:

$$A=(a_{ij})=\begin{array}{rc}
 & \begin{array}{ccccccc} j\ \ 3 & 2 & 1 & 0 & -1 & -2 & -3 \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \end{array} \\
\begin{array}{r} i \\ -3\ \Rightarrow \\ -2\ \Rightarrow \\ -1\ \Rightarrow \\ 0\ \Rightarrow \\ 1\ \Rightarrow \\ 2\ \Rightarrow \\ 3\ \Rightarrow \end{array} &
\begin{array}{c} \\ \left[\begin{array}{ccccccc}
\cdot & \cdot & \cdot & \cdot & -w & -w & \cdot \\
\cdot & \cdot & -w & \cdot & 2w & \cdot & -w \\
\cdot & -w & 2w & \frac{1}{2} & \frac{1}{2} & 2w & -w \\
\cdot & \cdot & \frac{1}{2} & 1 & \frac{1}{2} & \cdot & \cdot \\
-w & 2w & \frac{1}{2} & \frac{1}{2} & 2w & -w & \cdot \\
-w & \cdot & 2w & \cdot & -w & \cdot & \cdot \\
\cdot & -w & -w & \cdot & \cdot & \cdot & \cdot
\end{array}\right]\end{array}
\end{array}.$$

From the matrix, we see that the mask satisfies the conditions that for any $k\in\mathbb{Z}$,

$$\sum_{i\in\mathbb{Z}}(-1)^i a_{i,k}=\sum_{i\in\mathbb{Z}}(-1)^i a_{k,i}=\sum_{i\in\mathbb{Z}}(-1)^i a_{i,i+k}=0$$

and Theorem 5 implies that the symbol of the butterfly scheme is factorizable. The symbol is written as

$$a(z_1,z_2)=\sum_{i=-3}^{3}\sum_{j=-3}^{3}a_{i,j}z_1^iz_2^j=\frac{1}{2}(1+z_1)(1+z_2)(1+z_1z_2)(1-wc(z_1,z_2))(z_1z_2)^{-1}, \tag{16}$$

where

$$c(z_1,z_2)=2z_1^{-2}z_2^{-1}+2z_1^{-1}z_2^{-2}-4z_1^{-1}z_2^{-1}-4z_1^{-1}-4z_2^{-1}+2z_1^{-1}z_2+2z_1z_2^{-1}$$
$$+12-4z_1-4z_2-4z_1z_2+2z_1^2z_2+2z_1z_2^2.$$

Gregory [7] computed an explicit shape parameter value $w_0>1/16$ such that for $0<w<w_0$, the butterfly scheme generates C^1 limit functions on regular triangulations. It is well-known that the scheme reproduces cubic polynomials for $w=1/16$, otherwise linear polynomials for $w\neq 1/16$.

5.2. Symmetric 8-point butterfly subdivision scheme

In this section, we consider an 8-point symmetric interpolatory bivariate subdivision scheme defined on a regular triangulation mesh.

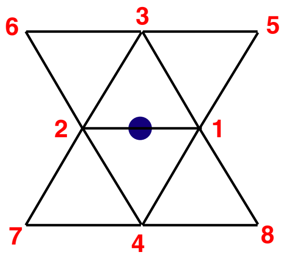


Fig. 3. Stencil of 8-point butterfly scheme: $(1,2=\alpha)$, $(3,4=\beta)$, $(5,6,7,8=\gamma)$.

If we set the stencil of the symmetric 8-point scheme such as Fig. 3.
The mask of such a scheme is given in the matrix form as follows:

$$A=(a_{ij})=\begin{array}{cc} j & \begin{array}{ccccccc} 3 & 2 & 1 & 0 & -1 & -2 & -3 \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \end{array} \\ \begin{array}{c} i \\ -3 \Rightarrow \\ -2 \Rightarrow \\ -1 \Rightarrow \\ 0 \Rightarrow \\ 1 \Rightarrow \\ 2 \Rightarrow \\ 3 \Rightarrow \end{array} & \begin{array}{c} \\ \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \gamma & \gamma & \cdot \\ \cdot & \cdot & \gamma & \cdot & \beta & \cdot & \gamma \\ \cdot & \gamma & \beta & \alpha & \alpha & \beta & \gamma \\ \cdot & \cdot & \alpha & 1 & \alpha & \cdot & \cdot \\ \gamma & \beta & \alpha & \alpha & \beta & \gamma & \cdot \\ \gamma & \cdot & \beta & \cdot & \gamma & \cdot & \cdot \\ \cdot & \gamma & \gamma & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \end{array} \end{array}.$$

The bivariate Laurent polynomial of this scheme is assumed to be factorizable:

$$a(z_1,z_2)=\sum_{i=-3}^{3}\sum_{j=-3}^{3}a_{i,j}z_1^iz_2^j=(1+z_1)(1+z_2)(1+z_1z_2)b(z_1,z_2).$$

By Theorem 5, the factorization implies that for each $k\in\mathbb{Z}$, we have

$$\sum_{i\in\mathbb{Z}}(-1)^ia_{i,k}=\sum_{i\in\mathbb{Z}}(-1)^ia_{k,i}=\sum_{i\in\mathbb{Z}}(-1)^ia_{i,i+k}=0.$$

In this case, we get $2\gamma+\beta=-2\alpha+1=0$. If we set $\gamma=-w$, we obtain the same mask of the butterfly scheme:

$$\alpha=\frac{1}{2},\quad \beta=2w,\quad \gamma=-w.$$

5.3. Symmetric 10-point butterfly subdivision scheme

As shown in the previous section, the butterfly scheme generates C^1 surfaces in the topology of regular setting. The smoothness of surface in geometric modeling is required to be up to C^2. To obtain a subdivision

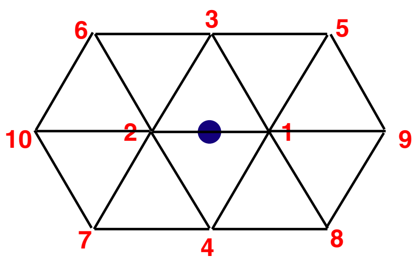


Fig. 4. Stencil of the symmetric 10-point butterfly scheme: $(1,2=\alpha)$, $(3,4=\beta)$, $(5,6,7,8=\gamma)$, $(9,10=w)$.

scheme retaining the simplicity of the butterfly scheme and creating C^2 limit surfaces, we need to enlarge the support of the butterfly scheme, as mentioned in the introduction. We try it by taking two more points into account to calculate a new control points as shown in Fig. 4.

Comparing the stencil of the butterfly scheme in Fig. 3, we consider two nearby points (points 9 and 10 in Fig. 4) with different mask value (weight) from those of the other eight points. The mask of 10-point butterfly scheme can is written in the matrix form:

$$A := \begin{bmatrix} & j & 3 & 2 & 1 & 0 & -1 & -2 & -3 \\ i & & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ -3 & \Rightarrow & \cdot & \cdot & \cdot & w & \gamma & \gamma & w \\ -2 & \Rightarrow & \cdot & \cdot & \gamma & \cdot & \beta & \cdot & \gamma \\ -1 & \Rightarrow & \cdot & \gamma & \beta & \alpha & \alpha & \beta & \gamma \\ 0 & \Rightarrow & w & \cdot & \alpha & 1 & \alpha & \cdot & w \\ 1 & \Rightarrow & \gamma & \beta & \alpha & \alpha & \beta & \gamma & \cdot \\ 2 & \Rightarrow & \gamma & \cdot & \beta & \cdot & \gamma & \cdot & \cdot \\ 3 & \Rightarrow & w & \gamma & \gamma & w & \cdot & \cdot & \cdot \end{bmatrix}.$$

We assume a factorization of the Laurent polynomial of this scheme:

$$a(z_1, z_2) = (1 + z_1)(1 + z_2)(1 + z_1 z_2)c(z_1, z_2).$$

From the factorization, we get

$$2\gamma + \beta = 0, \quad -2\alpha - 2w + 1 = 0.$$

Therefore, we find the mask of 10-point butterfly scheme:

$$\alpha = \frac{1}{2} - w, \quad \beta = 2\gamma, \quad 5, 6, 7, 8 = -\gamma, \quad 9, 10 = w.$$

This mask is exact with a modified butterfly scheme when $\gamma = 1/16 + w$. Zorin et al. [12] examined that the butterfly scheme exhibits undesirable artifacts in the case of an irregular topology, and derived an improved scheme (a modified butterfly scheme), which retains the simplicity of the butterfly scheme. The modified butterfly scheme is interpolating and results in smoother surfaces.

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