A Study on Subdivision Scheme-Draft

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Chapter 1

Introduction

Although initially studied in the late 1940s by G. de Rham, subdivision schemes had to wait the development of computer graphics, roughly the 1970s, to start being actively studied and improved. During the last two decades, the rise of multi-resolution analysis(MRA) gave birth to significant advances in a wide range of domains. Wavelet decomposition of signals or images, which is one of the most obvious and vastly used applications of MRA, is a valuable tool for building efficient algorithms dedicated to 3D models represented by discrete polygonal surfaces, along with the growth of computing power and the increase of network applications make discrete surfaces an attractive filed of study.

Modeling the geometry of surfaces of arbitrary topology is an important area of research in computer graphics and approximation theory. A powerful paradigm for the construction of such surfaces is **subdivision**. Beginning with a input mesh a sequence of meshes is defined new vertices are inserted as, preferably, simple local affine combinations of neighboring vertices. An attractive feature of these schemes is that they are local, i.e., no global system of equations needs to be solved. The mathematical analysis of the surfaces resulting from subdivision is not always straightforward. However, the simplicity of the algorithm and associated data structures make them attractive and interactive applications where speed is of the essence.

In the field of CAGD, the de-facto standard for shape modeling is at present non-uniform rational B-splines(NURBS). NURBS representation, however, uses a rigid rectangular grid of control points and has limitation in manipulating shapes of general topology. Subdivision is a method for generating smooth curves/surfaces, which first appeared an extension of splines to arbitrary topology control nets. Efficiency of subdivision algorithms, their flexibility and simplicity make them suitable for many interactive computer graphics applications. Although subdivision was introduced as a generalization of knot insertion algorithms for splines, it is much more general and allows considerable freedom in the choice of subdivision rules.

Subdivision in its pure form is useful for generating smooth curves/surfaces. However, applications such as special effects and animation require creation and manipulation of complex geometric models, which, like real world geometry, carry detail at many scales. Manipulating such as fine meshes can be difficult, especially when they are to be edited or animated. Interactively, which is crucial in these cases, is challenging to achieve.

Computer Aided Geometric Design(CAGD) is a branch of applied Mathematics concerned with algorithms for the design of smooth curves/surfaces and for their efficient mathematical representation. The representation is used for the computation of the curves/surfaces, as well as geometrical quantities of importance such as curvatures, intersection curves between two surfaces and offset surfaces. One common approach to the design of curves/surfaces which related to CAGD is the subdivision schemes. Subdivision schemes have become important in recent years because they provide a precise and efficient way to describe smooth curves/surfaces. It is an algorithmic technique to generate smooth surfaces as a sequence of successively refined polyhedral meshes. Their beauty lies in the elegant mathematical formulation and simple implementation.

Classical schemes:

- De Rahm(1947)
- Chaikin(1974)- An algorithm for high speed curve generation.
- Riesenfeld(1975)- On Chaikins algorithm.
- Catmull-Clark (1978)- Recursively generated B-spline surfaces on arbitrary topological meshes.
- Doo-Sabin (1978)- Behavior of recursive division surfaces near extraordinary points.
- Ball-Storry(1986)- A matrix approach to the analysis of recursively generated B-spline surfaces.
- Loop (1987)- Smooth subdivision surfaces based on triangles (new domain, eigen analysis).
- Dyn,Levin,Gregory(1987)- A 4-point interpolating subdivision scheme for curve design.
- (1990)- A butterfly subdivision scheme for surface interpolation with tension control.
- Reif(1995)- A unified approach to subdivision algorithms near extraordinary vertices.

New Schemes:

- Kobbelt(1996)- Interpolating subdivision on open quadrilateral nets with arbitrary topology.
- Kobbelt(2000)- $\sqrt{3}$ subdivision.
- Velho(2000)- Quasi 4-8 subdivision.
- Velho-Zorin(2000)- 4-8 subdivision.
- Dodgson,Ivrissimtzis,Sabin(2003)- Characteristic of dual triangular $\sqrt{3}$ subdivision.
- (2004)- $\sqrt{5}$ subdivision.

Non-uniform schemes:

- Warren(1995)- Binary subdivision schemes for functions of irregular knot sequences.
- Gregory,Qu(1996)- Non-uniform corner cutting.
- Sederberg,Sewell,Sabin(1998)- Non-uniform recursive subdivision surfaces.
- Dyn(1999)- Using Laurent polynomial representation for the analysis of non-uniform binary SS.

Non-stationary schemes:

- Dyn,Levin(1995)- Analysis of asymptotically equivalent binary subdivision schemes.
- Morin,Warren,Weimer(2001)- A subdivision scheme for surfaces of revolution.
- Dyn,Levin,Luzzatto(2003)- Non-stationary interpolating SS reproducing spaces of exponential polynomials.

Chapter 2

Subdivision of Univariate Data

Subdivision schemes define a smooth curve or surface as the limit of a sequence of successive refinements. By these methods at each refinement step, new inserted points on a finer grid are computed by linear combination (affine combination) of the already existing points. In the limit of the recursive process, data are defined on a dense set of points.

2.1 Definitions and Basic Results

Definition 1 *A n-ary subdivision scheme is* **linear** *if it consists of linear combination of control points from level k to generate level $k+1$, i.e., for all k and i, there exist sets of real number-called subdivision* **masks** $\mathbf{a}^k = \{a^k\}$ *such that*

$$p_i^{k+1} = \sum_j a_{i-nj}^k p_j^k.$$

In the case of a binary scheme, we have $n = 2$. Let $\sigma(\mathbf{a}^k) = \{j | a_j^k \neq 0\}$ be the **support** *of the mask $\mathbf{a}^k$. When $\sigma(\mathbf{a}^k) \subset K$ for some compact set K, the subdivision scheme is said to have a finite support. If the mask $\mathbf{a}^k$ does not depend on k, i.e., $\mathbf{a}^k = \mathbf{a}$, the scheme is called* **stationary**. *Otherwise, it is called* **non-stationary**. *Similarly, if the mask does not depend on i, i.e, each refinement rule operates in the same way at all locations, the scheme is termed* **uniform**.

The binary stationary subdivision scheme is a process which recursively defines a sequence of control points $P^{k+1} = \{p_i^{k+1} : i \in \mathbb{Z}\}$ by a finite linear combination of control points P^k with a mask $\mathbf{a} = \{a_i : i \in \mathbb{Z}\}$

$$p_i^{k+1} = \sum_{j \in \mathbb{Z}} a_{i-2j} p_j^k, \quad k = 0, 1, 2, \ldots,$$

which is denoted formally by $P^{k+1} = SP^k$. Since each component of vector valued functions in $\mathbb{R}^3$ is a scalar function generated by the same subdivision scheme, the analysis of a binary subdivision scheme can be reduced to the scalar case to initial sets of control points. Therefore, starting with given control points $f^0 = \{f_i : i \in \mathbb{Z}\}$, we consider scalar sets of control points $f^k = \{f_i^k : i \in \mathbb{Z}\}$ generated by the relation

$$f_i^{k+1} = \sum_{j \in \mathbb{Z}} a_{i-2j} f_j^k, \quad k = 0, 1, 2, \dots.$$

We denote the function f by $S^\infty f^0$, and call it a **limit function** of subdivision scheme S. If subdivision scheme retains the point of level k as a subset of point of level $k+1$, it is called **interpolating** scheme. Otherwise, it is termed **approximating**. In general, interpolating scheme have a larger support than approximating scheme for a given continuity. The general form of an interpolating subdivision scheme is

$$\begin{aligned} f_{2i}^{k+1} &= f_i^k, \\ f_{2i+1}^{k+1} &= \sum_{j \in \mathbb{Z}} a_{1+2j} f_{i-j}^k. \end{aligned}$$

We can easily see that mask satisfies $a_{2i} = \delta_i$ for an interpolating scheme.

Definition 2 *The sequence of piecewise linear functions f^k* **converges uniformly** *on compact set K if there exists a function f such that*

$$\lim_{k \to \infty} \sup_{x \in K} |f^k(x) - f(x)| = 0.$$

The following sufficient conditions is useful to prove uniform convergence.

Theorem 1 *If there exists $0 < \alpha < 1$ and $\beta > 0$ such that for all $i \in \mathbb{Z}_+$*

$$\|f_i - f_{i-1}\|_\infty \le \beta \alpha^{i-1},$$

then the sequence converges uniformly toward a limit function.

While the smoothness of the limit function is important, another criterion is worth being considered, namely the **approximation order**. The underlying concept is intuitive: if we consider an initial grid obtained by sampling a sufficiently regular function, the error between the original function and the limit function obtained through subdivision should decrease along with the sampling step.

Definition 3 *Let us consider the initial grid $X_0 = h\mathbb{Z}$ and initial data $f_i^0 = g(ih)$ sampled a function $g \in C^k$. Let us denote by f^∞ the limit function obtained through subdivision. The subdivision scheme has approximation order p if*

$$|(g - f^\infty)(x)| \le Ch^p, \quad x \in \mathbb{R}$$

where C is a real constant and independent of h.

2.2 Examples of Subdivision Schemes

In this section, some examples of linear stationary subdivision schemes are presented.

2.2.1 Corner Cutting Scheme

The idea of the earliest subdivision scheme is corner cutting. In 1947, Georges de Rham formulated subdivision rule based on the idea of corner cutting. Rham considered a general rule

$$\begin{aligned} f_{2i}^{k+1} &= (1-\beta_1)f_i^k + \beta_1 f_{i+1}^k, \\ f_{2i+1}^{k+1} &= \beta_2 f_i^k + (1-\beta_2) f_{i+1}^k, \end{aligned}$$

where β_1 and β_2 are two positive real numbers such that $0 < \beta_1 + \beta_2 < 1$. In 1974, G.Chaikin introduced a similar algorithm to generate what turned out to be quadratic B-spline. Chaikin's corner cutting rule followed with $\beta_1 = \beta_2 = \frac{1}{4}$. Using the classification criteria, corner cutting refinement rules are linear, finite-support, and stationary subdivision schemes. Moreover, they are not interpolating, but approximating. Higher order version of Chaikin's rule(B-spline) can be derived from the subdivision schemes. All coefficients for the scheme of order n can be derived from the $(n+1)$th row of Pascal's triangle. The mask of the scheme of order n is given by

$$a_i = 2^{-n}\binom{n+1}{i}, \quad i = 0, 1, \cdots, n+1.$$

As an example, the cubic spline algorithm is defined by

$$\begin{aligned} f_{2i}^{k+1} &= \frac{1}{2} f_i^k + \frac{1}{2} f_{i+1}^k, \\ f_{2i+1}^{k+1} &= \frac{1}{8} f_i^k + \frac{6}{8} f_{i+1}^k + \frac{1}{8} f_{i+2}^k. \end{aligned}$$

2.2.2 Four-point Interpolating Scheme

Although the approximating schemes give interesting results, for example, they have better regularity than interpolating schemes have, it is often required for particular applications to have interpolating schemes. The first works on linear interpolating schemes deals with dyadic uniform grids were done by Serge Dubuc [8]. The principle is to assimilate the initial values as samples from a cubic polynomial p_3, i.e., $f_i^0 = p_3(i)$. At the following level, such an assertion leads to $f_{2i+1}^1 = p_3(i + \frac{1}{2})$. The approach used by Dubuc was to use the cubic Lagrange interpolation polynomial. Computing $p_3(i + \frac{1}{2})$ leads to the following interpolation rule, called Deslauriers-Dubuc four-point scheme (4-point DD scheme)

$$\begin{aligned} f_{2i}^{k+1} &= f_i^k, \\ f_{2i+1}^{k+1} &= \frac{9}{16}(f_i^k + f_{i+1}^k) - \frac{1}{16}(f_{i-1}^k + f_{i+2}^k). \end{aligned}$$

Independently, Nira Dyn, John A. Gregory and David Levin proposed 4-point interpolating scheme with one parameter. The idea was to use the following interpolation rule

$$\begin{aligned} f_{2i}^{k+1} &= f_i^k, \\ f_{2i+1}^{k+1} &= \left(\frac{1}{2} + w\right)(f_i^k + f_{i+1}^k) - w(f_{i-1}^k + f_{i+2}^k). \end{aligned}$$

In this rule, w stands for a tension parameter. It is well-known that this Dyn 4-point scheme generates continuous limit functions for $|w| < \frac{1}{4}$, and C^1 limit functions for $0 < w < \frac{1}{8}$.

2.2.3 Four-point Ternary Scheme

Although binary subdivision schemes are the most common class of subdivision, an interesting ternary stationary interpolating scheme can be found [14, 15, 16]. Hassan, Ivrissimtzis, Dodgson and Sabin [16] proposed ternary 4-point interpolating subdivision scheme. The considered ternary scheme uses the following interpolating rule

$$\begin{aligned} f_{3i}^{k+1} &= f_i^k, \\ f_{3i+1}^{k+1} &= af_{i-1}^k + bf_i^k + cf_{i+1}^k + df_{i+2}^k, \\ f_{3i+2}^{k+1} &= df_{i-1}^k + cf_i^k + bf_{i+1}^k + af_{i+2}^k, \end{aligned}$$

where the masks are given by

$$a = -\frac{1}{18} - \frac{1}{6}w, b = \frac{13}{18} + \frac{1}{2}w, c = \frac{7}{18} - \frac{1}{2}w, d = -\frac{1}{18} + \frac{1}{6}w.$$

They [16] proved that this scheme generates C^2-limit functions for $\frac{1}{15} < w < \frac{1}{9}$.

2.3 Analysis of Subdivision Scheme

Most of the papers on new subdivision schemes gave a specific proof of the convergence and smoothness of the proposed schemes. In this section, we summarize the main analysis tool-Laurent polynomial method-to prove the convergence and regularity of the schemes.

Theorem 2 ([1]) *Let us consider a n-ary stationary scheme, with a mask* $\mathbf{a}$. *If the scheme is convergent, then*

$$\sum_{i\in\mathbb{Z}} a_{ni+k} = 1, \quad \forall k \in \{0, 1, \dots, n-1\}.$$

In the case of a binary scheme, this yields

$$\sum_{i\in\mathbb{Z}} a_{2i} = \sum_{i\in\mathbb{Z}} a_{2i+1} = 1. \tag{2.1}$$

A few additional properties can be derived in the case of a binary uniform interpolating scheme. The space of all polynomials of degree $\leq n$ will be denoted by π_n. The property of polynomial reproducing and the smoothness of the limit function are connected through the following theorem.

Theorem 3 ([9]) *An interpolating subdivision scheme that generates C^m limit function, only if it is exact for polynomial of degree not exceeding m.*

Proof. Let the interpolating subdivision scheme S be given by the rule

$$f_{1+2i}^{k+1} = \sum_j \alpha_j f_{i-j}^k,$$

and denote by $f = S^\infty f^0$ the limit function. Consider the n-th order divided differences of f

$$\delta_\epsilon^n f(x) = [x+\epsilon, x+2^{-1}\epsilon, \cdots, x+2^{-n}\epsilon]f = \epsilon^{-n}\sum_{i=0}^{n} b_i f(x+2^{-i}\epsilon),$$

where $b_i^{-1} = \prod_{j=0, j\neq i}^{n}(2^{-i}-2^{-j}), i = 0,1,\ldots,n$. For fixed $x \in 2^{-k}\mathbb{Z}$, $\epsilon = 2^{-l}, l > k$, we get after substituting $f(x+2^{-i-l}) = f^{i+l}_{2^{i+l}x+1}$, by its expression in terms of values at level $l+i-1$,

$$\begin{aligned}
\delta_{2^{-l}}^n f(x) &= 2^{ln}\sum_{i=0}^{n} b_i \sum_j \alpha_j f^{i+l-1}_{2^{l+i-1}x-j} \\
&= 2^{ln}\sum_{i=0}^{n} b_i \sum_j \alpha_j f(x - j2^{-l-i+1}) \\
&= 2^n \sum_j \alpha_j (-j)^n \delta^n_{j2^{-l+1}} f(x).
\end{aligned}$$

Taking the limit and by the assumption that $f \in C^m$, we have for $n \leq m$,

$$f^{(n)}(x) = 2^n \sum_j \alpha_j (-j)^n f^{(n)}(x).$$

Since this equation holds for all $x \in 2^{-k}\mathbb{Z}$, which is a dense set in $\mathbb{R}$, and since $f^{(n)}$ cannot be identically zero for all initial data. So we obtain

$$\left(\frac{1}{2}\right)^n = \sum_j \alpha_j(-j)^n, \quad n \leq m$$

which proves that the scheme is exact polynomials of degree not exceeding m. ♣

It is sufficient to check the polynomial reproducing property for monomial x^n, since the scheme is linear. *The converse of this theorem is not true.* As an example, the four-point Deslauriers-Dubuc scheme is exact for cubic polynomials, but this scheme only produces C^1 limit function. The following theorem connects the polynomial reproducing property with the approximation order of the scheme.

Exercise 1 *Prove that DD scheme is exact for cubic polynomials.*

Theorem 4 ([12]) *An convergent subdivision scheme that reproduces polynomial π_n has an approximation order of $n+1$.*

Proof. Consider $G = F - T^n_{F;x}$, where $T^n_{F;x}$ is the Taylor polynomial of F of degree n at the point x. Then

$$G^{(j)}(x) = 0, \quad j = 0,1,\ldots,n, \quad G^{(n+1)} = F^{(n+1)}.$$

Since S is exact for π_n, it follows that $T^n_{F;x} = S^\infty(f^0 - g^0)$, where $g^0_i = G(ih), i \in \mathbb{Z}$. Therefore, F and G have the same error,

$$F - S^\infty f^0 = F - T^n_{F;x} + S^\infty(f^0 - g^0) - S^\infty f^0 = G - S^\infty g^0.$$

The stationarity of S implies that

$$(S^\infty g^0)(x) = \sum_i g^0_i \phi_S\left(\frac{x - ih}{h}\right),$$

where ϕ_S is the basic limit function of the scheme. By the property of partition of unity of ϕ_S, we have

$$(G - S^\infty g^0)(x) = \sum_i \phi_S(x/h - i)(G(x) - g^0_i) = \sum_{i \in I_h(x)} (G(x) - G(ih))\phi_S(x/h - i),$$

where $I_h(x) = \{i : \phi_S(x/h - i) \neq 0\}$. Since the support of ϕ_S is finite, the number of elements in $I_h(x)$ is bounded by a constant independent of x, h. Denote this constant by N_S, and denote the support of ϕ_S by $M_S = [K_1, K_2]$. Let $\|\phi_S\| = \max_{x \in M_S} |\phi_S(x)|$, and $\Omega_{x,h} = [x - hK_1, x + hK_2]$. Then we get

$$|(G - S^\infty g^0)(x)| = \|\phi_S\| \sum_{i \in I_h(X)} |G(x) - G(ih)| \leq N_S \|\phi_S\| \max_{y \in \Omega_{x,h}} |G(x) - G(y)|,$$

and

$$\max_{y \in \Omega_{x,h}} |G(x) - G(y)| \leq \max |F^{(n+1)}(y)| [K_2 - K_1]^{n+1} h^{n+1},$$

which completes the proof. ♣

Although the above criteria are useful to observe the behavior of the scheme, regarding its convergence and smoothness of the limit function, they may not be sufficient. The representation of schemes using Laurent polynomial is much easier to handle the sufficient condition for the convergence and smoothness of subdivision scheme. Let us consider subdivision scheme with a mask **a**. The symbol of a mask **a** is the Laurent polynomial

$$a(z) = \sum_{i \in \mathbb{Z}} a_i z^i, \quad z \in \mathbb{C}.$$

We shall say that a subdivision scheme S has the mth degree polynomial reproducing property (PRP) if it holds that

$$\sum_k a_{j-2k} p(k) = p(\frac{j}{2}), \quad j \in \mathbb{Z}, \quad p \in \pi_m.$$

We define the integers set $\mathbb{Z}_k := \{0, 1, \cdots, k\}$.

Theorem 5 ([29]) *Suppose the mask symbol $a(z)$ is such that interpolatory condition $a(z) + a(-z) = 2$, i.e., $a_{2j} = \delta_j$ holds. Then $(2n - 1)$th degree polynomial reproducing property is satisfied if and only if*

$$a^{(j)}(-1) = 0, \quad j \in \mathbb{Z}_{2n-1}.$$

Proof. Since we have $a_{2j} = \delta_j$, PRP is equivalent to the condition

$$\sum_k a_{2k+1} p(j-k) = p(j+\frac{1}{2}), \quad j \in \mathbb{Z}, \quad p \in \pi_{2n-1}.$$

It will therefore suffice to prove that equation above if and only if $a^{(j)}(-1) = 0, \quad j \in \mathbb{Z}_{2n-1}$. Since $a_{2j} = \delta_j$, we see that $a(z) = 1 + \sum_k a_{2k+1} z^{2k+1}$ and thus

$$a^{(j)}(-1) = \delta_j + (-1)^{j+1} \sum_k q_j(2k+1) a_{2k+1}, \quad j \in \mathbb{Z}_{2n-1}$$

where, for $x \in \mathbb{R}$,

$$q_0(x) = 1, \quad q_j(x) = \prod_{l \in \mathbb{Z}_{j-1}} (x - l), \quad j = 1, 2, \cdots, 2n-1.$$

Observe that $q_j \in \pi_j, \quad j \in \mathbb{Z}_{2n-1}$. Hence if we define

$$p_j = q_j(-2 \cdot +1 + 2j), \quad j \in \mathbb{Z}_{2n-1}$$

then also $p_j \in \pi_j$. Moreover,

$$a^{(j)}(-1) = \delta_j + (-1)^{j+1} \sum_k p_j(j-k) a_{2k+1}, \quad j \in \mathbb{Z}_{2n-1}$$

and

$$p_j(j + \frac{1}{2}) = q_j(0) = \delta_j, \quad j \in \mathbb{Z}_{2n-1}.$$

Suppose $\sum_k a_{2k+1} p(j-k) = p(j+\frac{1}{2}), \quad j \in \mathbb{Z}, p \in \pi_{2n-1}$ holds. Then for any integer $j \in \mathbb{Z}_{2n-1}$, we have

$$a^{(j)}(-1) = \delta_j + (-1)^{j+1} \sum_k p_j(j-k) a_{2k+1} = \delta_j + (-1)^{j+1} p_j(j + \frac{1}{2}) = 0.$$

Conversely, suppose $a^{(j)}(-1) = 0, \quad j \in \mathbb{Z}_{2n-1}$ holds. Then we have

$$\sum_k q_j(2k+1) a_{2k+1} = (-1)^j \delta_j, \quad j \in \mathbb{Z}_{2n-1}.$$

Suppose now $p \in \pi_{2n-1}$. We see that $\{q_l : l \in \mathbb{Z}_{2n-1}\}$ is a basis for π_{2n-1}. Hence we deduce that $\{p_l(\cdot - j + l) : l \in \mathbb{Z}_{2n-1}\}$ is a basis for π_{2n-1}. Thus there exists a unique coefficients sequence $\{\alpha_{j,l} : l \in \mathbb{Z}_{2n-1}\}$ such that $p = \sum_{l \in \mathbb{Z}_{2n-1}} \alpha_{j,l} p_l(\cdot - j + l)$. Using the assumption, we obtain

$$\begin{aligned} \sum_k a_{2k+1} p(j-k) &= \sum_k a_{2k+1} \sum_{l \in \mathbb{Z}_{2n-1}} \alpha_{j,l} p_l(-k+l) \\ &= \sum_{l \in \mathbb{Z}_{2n-1}} \alpha_{j,l} \sum_k a_{2k+1} q_l(2k+1) = \alpha_{j,0}. \end{aligned}$$

Also, we have
$$p(j+\frac{1}{2})=\sum_{l\in\mathbb{Z}_{2n-1}}\alpha_{j,l}p_l(l+\frac{1}{2})=\alpha_{j,0}.$$
Which completes the proof. ♣

If we define the generating function of the control point f_i^k as $F^k(z)=\sum_{i\in\mathbb{Z}}f_i^k z^i$, then $F^{k+1}(z)$ satisfies the following relation
$$F^{k+1}(z)=a(z)F^k(z^n).$$

Exercise 2 *Verify the last statement for binary case* $(n=2)$*.*

Binary scheme can be expressed in the generating function formalism with symbol $a(z)$:
$$F^{k+1}(z)=a(z)F^k(z^2).$$
We can see immediately that
$$F^k(z)=a^k(z)F^0(z^{2^k}),$$
where
$$\begin{aligned} a^1(z) &= a(z),\\ a^{k+1}(z) &= a(z)a^k(z^2). \end{aligned}$$

This Laurent polynomial has interesting properties, which are stated in the following theorem.

Theorem 6 ([1]) *Let us consider a n-ary convergent scheme with a mask* $\mathbf{a}$*. The Laurent polynomial $a(z)$ satisfies the following properties*
$$\begin{aligned} a(1) &= n,\\ a(z_n^p) &= 0, \quad z_n^p=e^{\frac{2ip\pi}{n}}, \quad p\in\{1,2,\ldots,n-1\}, i=\sqrt{-1}. \end{aligned}$$

Corollary 1 ([1]) *There exists a polynomial $b(z)$ such that*
$$a(z)=b(z)\frac{1}{n}\frac{z^n-1}{z-1}=b(z)\frac{1}{n}\prod_{p=1}^{n-1}(z-z_n^p),$$
with $b(1)=n$.

The application of these properties to a Laurent polynomial associated to a convergent binary scheme yields

$$a(-1) = 0, \quad a(1) = 2,$$

which implies that there exists a polynomial $b(z)$ such that

$$a(z) = \frac{1+z}{2} b(z),$$

with $b(1) = 2$. For example, in the case of Dyn four-point scheme, we have

$$a(z) = 1 + \left(\frac{1}{2} + w\right)(z + z^{-1}) - w(z^3 + z^{-3}),$$

which yields

$$b(z) = -2wz^{-3} + 2wz^{-2} + z^{-1} + 1 + 2wz - 2wz^2.$$

The Laurent polynomial $b(z)$ can be seen as another subdivision scheme, related to the initial scheme through the following theorem.

Theorem 7 ([1],[9]) *Let us denote respectively by S_a and S_b the schemes having $a(z)$ and $b(z)$ as associated Laurent polynomials. Let us also define the difference operator Δ as follows*

$$\Delta f_i^k = \{f_i^k - f_{i-1}^k\}.$$

If S_a is a convergent scheme, then

$$\Delta(S_a f) = \frac{1}{n} S_b \Delta f.$$

If the scheme is convergent, it is clear that $\lim_{k\to\infty} \Delta f^k = 0$. Although less intuitive, the converse is also true, which provides a necessary and sufficient condition for a scheme to converge.

Theorem 8 ([1],[9]) *The subdivision scheme S_a is convergent if and only if $\frac{1}{n}S_b$ is contractive, that is,*

$$\lim_{k\to\infty} \left(\frac{1}{n} S_b\right)^k f^0 = 0,$$

for any initial data f^0.

According to this theorem, checking if a scheme S_a converges is equivalent to check whether we have $\|(\frac{1}{n}S_b)^L\|_\infty < 1$ for some $L \in \mathbb{Z}_+$. How to calculate the norm of $\|(\frac{1}{n}S_b)^L\|_\infty$? **See Appendix**. This theorem is also the key to prove higher order regularity of the limit function. Since the condition on the norm of S_b guarantee the uniform convergence of the sequence of functions, it also guarantees the continuity of the limit function. The condition of C^m continuity is expressed in the following theorem.

Theorem 9 ([1]) *Let us consider a scheme S_a with Laurent polynomial $a(z)$. If there exists a polynomial $b(z)$ such that*

$$a(z) = \left(\frac{1}{n}\frac{z^n - 1}{z - 1}\right)^m b(z),$$

and such that the associated scheme $\frac{1}{n}S_b$ is contractive, the limit function is C^m for any initial data.

Chapter 3

The Mask of Subdivision Scheme

In this chapter we obtain the mask of well-known interpolating symmetric subdivision schemes, binary 4-point and 6-point interpolating schemes, ternary 4-point interpolating scheme and Butterfly type scheme, by using symmetry and necessary condition for the smoothness. Using this method we can get the mask of $2n$-point interpolating scheme with one parameter.

3.1 Binary Subdivision Scheme

A binary subdivision scheme is defined in terms of a mask consisting of a finite set of non-zero coefficients $\mathbf{a} = \{a_i : i \in \mathbb{Z}\}$. The scheme is given by

$$f_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-2j} f_j^k, \quad i \in \mathbb{Z}.$$

This scheme can be written by two rules (even and odd rules)

$$\begin{aligned} f_{2i}^{k+1} &= \sum_{j\in\mathbb{Z}} a_{2i-2j} f_j^k = \sum_{j\in\mathbb{Z}} a_{2j} f_{i-j}^k, \\ f_{2i+1}^{k+1} &= \sum_{j\in\mathbb{Z}} a_{2i+1-2j} f_j^k = \sum_{j\in\mathbb{Z}} a_{2j+1} f_{i-j}^k. \end{aligned}$$

For each scheme S with a mask $\mathbf{a}$, we define the symbol

$$a(z) = \sum_{i\in\mathbb{Z}} a_i z^i.$$

Since the schemes we consider have masks of finite support, the corresponding symbols are Laurent polynomials.

Theorem 10 ([9]) *Let S be a convergent subdivision scheme with a mask $\mathbf{a}$. Then*

$$\sum_j a_{2j} = \sum_j a_{2j+1} = 1. \tag{3.1}$$

From (3.1), we can see that the symbol of a convergent subdivision scheme satisfies,

$$a(-1) = 0 \text{ and } a(1) = 2.$$

This condition guarantees the existence of a related subdivision scheme for the divided differences of the original control points and the existence of associated Laurent polynomial $a_1(z)$ which can be derived as follows:

$$a_1(z) = \frac{2z}{(1+z)} a(z).$$

The subdivision S_1 with symbol $a_1(z)$ is related to S with symbol $a(z)$ by the following theorem.

Theorem 11 ([9]) *Let S denote a subdivision scheme with symbol $a(z)$ satisfying (3.1). Then there exist a subdivision scheme S_1 with the property*

$$df^k = S_1 df^{k-1},$$

where $f^k = S^k f^0 = \{f_i^k : i \in \mathbb{Z}\}$ and $df^k = \{(df^k)_i = 2^k(f_{i+1}^k - f_i^k) : i \in \mathbb{Z}\}$.

Theorem 12 ([9]) *S is a uniformly convergent subdivision scheme, if and only if $\frac{1}{2}S_1$ converges uniformly to the zero function for all initial data f^0.*

$$\lim_{k\to\infty} \left(\frac{1}{2}S_1\right)^k f^0 = 0. \tag{3.2}$$

A scheme S_1 satisfying (3.2) for all initial data f^0 is termed **contractive**. The convergence of S is equivalent to checking whether S_1 is contractive, which is equivalent to checking whether $||(\frac{1}{2}S_1)^L||_\infty < 1$, for some $L \in \mathbb{Z}_+$. If L is large, the convergence is slow and the scheme may not be useful from a practical point of view.

The algorithm for verifying convergence of the binary scheme S.

1. If $a(-1) \neq 0$ or $a(1) \neq 2$, the scheme does not converge. stop!
2. Compute $b^{[1]} = \frac{2z}{1+z}a(z) = \sum_j b_j^{[1]} z^j$.
3. For $l = 1, 2, \ldots, M_0$.
4. Compute $N_l = \max_{0 \leq i < 2^l} \sum_j |b_{i-2^l j}^{[l]}|$.
5. If $N_l < 1$, the scheme is convergent. Stop!
6. If $N_l > 1$, compute $b^{[l+1]}(z) = b^{[l]}(z)b^{[l]}(z^2)$.
7. End loop.
8. S_b is not contractive after M_0 iterations. Stop!

Since $2z[a_i(z)] = [a_{i+1}(z)](1+z)$, we have useful coefficients relation between level i and next level $i+1$

$$2(a_i)_{k-1} = (a_{i+1})_{k-1} + (a_{i+1})_k.$$

We can notice that $a(-1) = 0$ is the necessary condition for convergence and $a_m(-1) = 0$ is the necessary condition for C^m smoothness of subdivision scheme. Now we will obtain the mask of 4-point and 6-point binary interpolating subdivision schemes using symmetry and necessary condition for smoothness.

3.1.1 4-point Interpolating Subdivision Scheme

The insertion rule of the 4-point interpolating subdivision scheme is given by

$$\begin{aligned} f_{2i}^{k+1} &= f_i^k, \\ f_{2i+1}^{k+1} &= a_3 f_{i-1}^k + a_1 f_i^k + a_{-1} f_{i+1}^k + a_{-3} f_{i+2}^k. \end{aligned}$$

The Laurent polynomial of this scheme is

$$a(z) = a_{-3}z^{-3} + a_{-1}z^{-1} + 1 + a_1 z^1 + a_3 z^3.$$

We get $a_{-1} = a_1, a_{-3} = a_3$ by symmetric condition. From the necessary condition for C^0, that is, $a(-1) = 0$, we have $a_1 + a_3 = \frac{1}{2}$. The Laurent polynomial of $\frac{1}{2}S_1$ is

$$\frac{1}{2}a_1(z) = \frac{z}{1+z}a(z) = a_3 z^{-2} - a_3 z^{-1} + \frac{1}{2} + \frac{1}{2}z - a_3 z^2 + a_3 z^3.$$

We find that $a_1(z)$ always satisfies $a_1(-1) = 0$. Let $a_3 = -w$, we can find the mask of Dyn 4-point interpolating subdivision scheme for $C^1(\mathbb{R})$

$$\left[-w, \frac{1}{2} + w, \frac{1}{2} + w, -w \right].$$

It is well-known that this scheme generates $C^1(\mathbb{R})-$functions for $0 < w < 0.17$.

3.1.2 6-point Interpolating Subdivision Scheme

The general form of the 6-point interpolating subdivision scheme is given by

$$\begin{aligned} f_{2i}^{k+1} &= f_i^k, \\ f_{2i+1}^{k+1} &= a_5 f_{i-2}^k + a_3 f_{i-1}^k + a_1 f_i^k + a_{-1} f_{i+1}^k + a_{-3} f_{i+2}^k + a_{-5} f_{i+3}^k. \end{aligned}$$

The Laurent polynomial of this scheme is

$$a(z) = a_{-5}z^{-5} + a_{-3}z^{-3} + a_{-1}z^{-1} + 1 + a_1 z^1 + a_3 z^3 + a_5 z^5.$$

From the symmetric condition, that is, $a_{-1} = a_1, a_{-3} = a_3, a_{-5} = a_5$, the Laurent polynomial can be written by

$$a(z) = a_5 z^{-5} + a_3 z^{-3} + a_1 z^{-1} + 1 + a_1 z + a_3 z^3 + a_5 z^5.$$

Since $a(-1) = 0$, we get $a_1 + a_3 + a_5 = \frac{1}{2}$. The Laurent polynomial of $\frac{1}{2}S_1$ is

$$\frac{1}{2}a_1(z) = a_5 z^{-4} - a_5 z^{-3} + (a_5 + a_3)z^{-2} - (a_5 + a_3)z^{-1} + \frac{1}{2}$$
$$+\frac{1}{2}z - (a_3 + a_5)z^2 + (a_3 + a_5)z^3 - a_5 z^4 + a_5 z^5.$$

It is easily to see that $a_1(-1) = 0$ is always true. And the Laurent polynomial of $\frac{1}{2}S_2$ is

$$\frac{1}{2}a_2(z) = \frac{za_1(z)}{1+z} = a_5 z^{-3} - 2a_5 z^{-2} + (3a_5 + a_3)z^{-1} - (4a_5 + 2a_3)$$
$$+\left(\frac{1}{2} + 4a_5 + 2a_3\right) z - (4a_5 + 2a_3)z^2 + (3a_5 + a_3)z^3 - 2a_5 z^4 + a_5 z^5.$$

From the necessary condition for $C^2(\mathbb{R})$, $a_2(-1) = 0$, we get $24a_5 + 8a_3 + \frac{1}{2} = 0$. Let $a_5 = w$, then $a_3 = -\frac{1}{16} - 3w$, $a_1 = \frac{9}{16} + 2w$ and we can find the mask of 6-point interpolating subdivision scheme

$$\left[w, -\frac{1}{16} - 3w, \frac{9}{16} + 2w, \frac{9}{16} + 2w, -\frac{1}{16} - 3w, w\right].$$

Weissman [30] proved that this scheme generates $C^2(\mathbb{R})-$functions for $0 < w < 0.0415$.

Exercise 3 *Find the mask of 8-point interpolating subdivision scheme.*

We can easily obtain the mask of 4-point, 6-point, 8-point and 10-point interpolating symmetric subdivision schemes (ISSS) by using the same process.

- 4-point scheme: $[a_1, a_3] = [w_2 + \frac{1}{2}, -w_2]$.

- 6-point scheme: $[a_1, a_3, a_5] = [2w_3 + \frac{9}{16}, -3w_3 - \frac{1}{16}, w_3]$.

- 8-point scheme: $[a_1, \ldots, a_7] = [5w_4 + \frac{75}{128}, -9w_4 + \frac{25}{256}, 5w_4 - \frac{3}{256}, -w_4]$.

- 10-point scheme: $[a_1, \ldots, a_9] = [14w_5 + \frac{1225}{2048}, -28w_5 - \frac{245}{2048}, 20w_5 + \frac{49}{2048}, -7w_5 - \frac{5}{2048}, w_5]$.

From these masks, Ko et al. [27] found out a general formula for the mask of $(2n+4)$-point ISSS with two parameters which reproduces all polynomials of degree $\leq 2n+1$ and some relations between the mask of the $(2n+4)$-point ISSS and the $(2n+2)$-point Deslauriers and Dubuc schemes.

3.2 Ternary Subdivision Scheme

A ternary subdivision scheme S is defined in terms of a mask consisting of a finite set of non-zero coefficients $\mathbf{a} = \{a_i : i \in \mathbb{Z}\}$. The scheme is given by

$$f_i^{k+1} = \sum_{j \in \mathbb{Z}} a_{i-3j} f_j^k, \quad i \in \mathbb{Z}. \tag{3.3}$$

The general form of a ternary interpolating subdivision scheme is

$$\begin{aligned} f_{3i}^{k+1} &= f_i^k, \\ f_{3i+1}^{k+1} &= \sum_{j \in \mathbb{Z}} a_{1+3j} f_{i-j}^k, \\ f_{3i+2}^{k+1} &= \sum_{j \in \mathbb{Z}} a_{2+3j} f_{i-j}^k. \end{aligned}$$

For each scheme S with a mask $\mathbf{a}$, we define the symbol

$$a(z) = \sum_{i \in \mathbb{Z}} a_i z^i.$$

Theorem 13 ([13]) *Let S be a convergent ternary subdivision scheme with a mask $\mathbf{a}$. Then*

$$\sum_j a_{3j} = \sum_j a_{3j+1} = \sum_j a_{3j+2} = 1. \tag{3.4}$$

The symbol of a convergent ternary subdivision scheme satisfies,

$$a(e^{2i\pi/3}) = a(e^{4i\pi/3}) = 0 \text{ and } a(1) = 3,$$

and there exist the Laurent polynomial $a_1(z)$ such that

$$a_1(z) = \frac{3z^2}{(1+z+z^2)} a(z).$$

The subdivision S_1 with symbol $a_1(z)$ is related to S with symbol $a(z)$ by the following theorem.

Theorem 14 ([13]) *Let S denote a ternary subdivision scheme with symbol $a(z)$ satisfying (3.4). Then there exist a subdivision scheme S_1 with the property*

$$df^k = S_1 df^{k-1},$$

where $f^k = S^k f^0 = \{f_i^k : i \in \mathbb{Z}\}$ and $df^k = \{(df^k)_i = 3^k(f_{i+1}^k - f_i^k) : i \in \mathbb{Z}\}$.

Theorem 15 ([13]) *S is a uniformly convergent ternary subdivision scheme, if and only if $\frac{1}{3}S_1$ converges uniformly to the zero function for all initial data f^0.*

$$\lim_{k\to\infty}\left(\frac{1}{3}S_1\right)^k f^0 = 0.$$

The algorithm for verifying convergence of the ternary scheme S.
For $i = 0, \ldots, d$

- If $a(e^{2i\pi/3}) = a(e^{4i\pi/3}) = 0$ and $a(1) = 3$, then the scheme can generates $C^i(\mathbb{R})$-functions else stop!
- Compute $a_{i+1}(z) = \frac{3z^2}{1+z+z^2}a_i(z)$.

Since $3z^2[a_i(z)] = [a_{i+1}(z)](1+z+z^2)$, we have the relation

$$3(a_i)_{k-2} = (a_{i+1})_{k-2} + (a_{i+1})_{k-1} + (a_{i+1})_k.$$

3.2.1 Ternary 4-point Interpolating Subdivision Scheme

We present a ternary 4-point interpolating subdivision scheme. The rule is given by

$$\begin{aligned}
f_{3i}^{k+1} &= f_i^k,\\
f_{3i+1}^{k+1} &= a_4 f_{i-1}^k + a_1 f_i^k + a_{-2} f_{i+1}^k + a_{-5} f_{i+2}^k,\\
f_{3i+2}^{k+1} &= a_5 f_{i-1}^k + a_2 f_i^k + a_{-1} f_{i+1}^k + a_{-4} f_{i+2}^k.
\end{aligned}$$

The Laurent polynomial of this scheme is

$$a(z) = a_{-5}z^{-5} + a_{-4}z^{-4} + a_{-2}z^{-2} + a_{-1}z^{-1} + 1 + a_1 z + a_2 z^2 + a_4 z^4 + a_5 z^5.$$

With the symmetry condition, $a(z)$ can be written by

$$a(z) = a_5 z^{-5} + a_4 z^{-4} + a_2 z^{-2} + a_1 z^{-1} + 1 + a_1 z + a_2 z^2 + a_4 z^4 + a_5 z^5.$$

To generate C^0-functions, we require that $a(1) = 3$. And it implies $a_1 + a_2 + a_4 + a_5 = 1$. The Laurent polynomial of $\frac{1}{3}S_1$ is

$$\begin{aligned}
\frac{1}{3}a_1(z) &= \frac{z^2 a(z)}{1+z+z^2} = a_5 z^{-3} + (a_4 - a_5)z^{-2} - a_4 z^{-1} + (a_5 + a_2)\\
&\quad +(1 - 2a_5 - 2a_2)z + (a_2 + a_5)z^2 - a_4 z^3 + (a_4 - a_5)z^4 + a_5 z^5.
\end{aligned}$$

From the necessary condition for C^1-smoothness, the mask corresponding to $a_1(z)$ satisfies the relation as in (3.4), i.e., $3a_2 - 3a_4 + 6a_5 = 1$. And the Laurent polynomial of $\frac{1}{3}S_2$ is

$$\begin{aligned}
\frac{1}{3}a_2(z) = \frac{z^2 a_1(z)}{1+z+z^2} &= 3a_5 z^{-1} + 3(a_4 - 2a_5) + 3(a_5 - 2a_4)z + 3(\frac{1}{3} + 2a_4)z^2\\
&\quad +3(a_5 - 2a_4)z^3 + 3(a_4 - 2a_5)z^4 + 3a_5 z^5.
\end{aligned}$$

To generate C^2-functions, we require the mask of S_2 to satisfy the condition (3.4), i.e., $a_4 + a_5 = -\frac{1}{9}$.
Set $a_5 = -\frac{1}{18} + \frac{1}{6}\mu$, we can find the mask of ternary 4-point interpolating subdivision scheme:

$$\begin{aligned}
f_{3i}^{k+1} &= f_i^k, \\
f_{3i+1}^{k+1} &= -\big(\frac{1}{18} + \frac{1}{6}\mu\big) f_{i-1}^k + \big(\frac{13}{18} + \frac{1}{2}\mu\big) f_i^k + \big(\frac{7}{18} - \frac{1}{2}\mu\big) f_{i+1}^k - \big(\frac{1}{18} - \frac{1}{6}\mu\big) f_{i+2}^k, \\
f_{3i+2}^{k+1} &= -\big(\frac{1}{18} - \frac{1}{6}\mu\big) f_{i-1}^k + \big(\frac{7}{18} - \frac{1}{2}\mu\big) f_i^k + \big(\frac{13}{18} + \frac{1}{2}\mu\big) f_{i+1}^k - \big(\frac{1}{18} + \frac{1}{6}\mu\big) f_{i+2}^k.
\end{aligned}$$

Hassan et al. [16] showed that the scheme is C^2 for $\frac{1}{15} < \mu < \frac{1}{9}$.

3.3 Bivariate Schemes on Regular Quad-Mesh

Let us consider a quad-mesh. The points $f_{i,j}, f_{i+1,j}, f_{i,j+1}, f_{i+1,j+1}$ are connected by edges and form a face (quadrilateral). Consider the refinement rule for a set of point $f_{i,j} \in \mathbb{R}^3, (i,j) \in \mathbb{Z}^2$

$$f_i^{k+1} = \sum_{j \in \mathbb{Z}^2} a_{i-2j} f_j^k, \quad i \in \mathbb{Z}^2.$$

In the bivariate case, there are 4 rules depending on the parity of each component of the vector $i \in \mathbb{Z}^2$.

$$\begin{aligned}
f_{(2i_1,2i_2)}^{k+1} &= \sum_{j_1,j_2} a_{(2j_1,2j_2)} f_{(i_1-j_1,i_2-j_2)}^k, \\
f_{(1+2i_1,2i_2)}^{k+1} &= \sum_{j_1,j_2} a_{(1+2j_1,2j_2)} f_{(i_1-j_1,i_2-j_2)}^k, \\
f_{(2i_1,1+2i_2)}^{k+1} &= \sum_{j_1,j_2} a_{(2j_1,1+2j_2)} f_{(i_1-j_1,i_2-j_2)}^k, \\
f_{(1+2i_1,1+2i_2)}^{k+1} &= \sum_{j_1,j_2} a_{(1+2j_1,1+2j_2)} f_{(i_1-j_1,i_2-j_2)}^k.
\end{aligned}$$

In this section, we will restrict ourselves to the binary case. The easiest way to extend univariate to bivariate schemes is to consider tensor-product schemes, that is, each mask $a_{i,j}$ of the refinement rule satisfies

$$a_{i,j} = b_i b_j,$$

where b_i, b_j is the mask of a univariate scheme.
Consider the biquadratic scheme. We can represent the refinement rule as matrices, for example

$$f_{2i,2j}^{k+1} : \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

which is equivalent to

$$f_{2i,2j}^{k+1} = \alpha f_{i,j}^k + \beta f_{i+1,j}^k + \gamma f_{i,j+1}^k + \delta f_{i+1,j+1}^k.$$

Using the mask of Chaikin's scheme, the bivariate biquadratic scheme can be written

$$f^{k+1}_{2i,2j} : \frac{1}{16}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}, \qquad f^{k+1}_{2i+1,2j} : \frac{1}{16}\begin{bmatrix} 3 & 9 \\ 1 & 3 \end{bmatrix},$$
$$f^{k+1}_{2i,2j+1} : \frac{1}{16}\begin{bmatrix} 3 & 1 \\ 9 & 3 \end{bmatrix}, \qquad f^{k+1}_{2i+1,2j+1} : \frac{1}{16}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}.$$

Similarly, we can get bi-cubic the refinement rule through the tensor product.

$$f^{k+1}_{2i,2j} : \frac{1}{64}\begin{bmatrix} 1 & 6 & 1 \\ 6 & 36 & 6 \\ 1 & 6 & 1 \end{bmatrix}, \qquad f^{k+1}_{2i+1,2j} : \frac{1}{64}\begin{bmatrix} 0 & 4 & 4 \\ 0 & 24 & 24 \\ 0 & 4 & 4 \end{bmatrix},$$
$$f^{k+1}_{2i,2j+1} : \frac{1}{64}\begin{bmatrix} 0 & 0 & 0 \\ 4 & 24 & 4 \\ 4 & 24 & 4 \end{bmatrix}, \qquad f^{k+1}_{2i+1,2j+1} : \frac{1}{64}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 16 & 16 \\ 0 & 16 & 16 \end{bmatrix}.$$

The rule for $f_{2i,2j}$ is called a vertex rule, the rules for $f_{2i+1,2j}, f_{2i,2j+1}$ are called edge rules and the rule for $f_{2i+1,2j+1}$ is termed face rule.

Theorem 16 ([12]) *Let $a(\mathbf{z}) = a(z_1, z_2) = \sum_{i,j} a_{i,j} z_1^i z_2^j$ be the symbol of a bivariate subdivision scheme S, which is defined on regular quad mesh. Then a necessary condition for the convergence of S is*

$$\sum_{\beta \in \mathbb{Z}^2} a_{\alpha - 2\beta} = 1 \ , \ \ \alpha \in \{(0,0), (0,1), (1,0), (1,1)\}.$$

Unlike the univariate case, this condition only implies that

$$a(-1,1) = a(1,-1) = a(-1,-1) = 0 \text{ and } a(1,1) = 4.$$

but does not imply that the associated Laurent polynomial can be factorized. If $a(z_1, z_2)$ has the form

$$a(z_1, z_2) = (1 + z_1)(1 + z_2)b(z_1, z_2),$$

the convergence of the scheme can be checked by verifying that $(1 + z_1)b(z_1, z_2)$ and $(1 + z_2)b(z_1, z_2)$ are contractive.

Theorem 17 ([12]) *If $a(z_1, z_2)$ can be factorized as*

$$a(z_1, z_2) = (1 + z_1)^n (1 + z_2)^n b(z_1, z_2),$$

and the schemes $a_{i,j}(z_1, z_2)$ defined by

$$a_{i,j}(z_1, z_2) = 2^{i+j} \frac{a(z_1, z_2)}{(1 + z_1)^i (1 + z_2)^j}, \quad 0 \le i, j \le n,$$

are contractive, then the scheme generates C^n limit function.

The analysis in the general case is more difficult. We refer to Appendix.

3.4 Bivariate Schemes on Regular Triangulations

For the topology of a regular triangulation, we regard the subdivision scheme as operating on the 3-directional grid. (The vertices of $\mathbb{Z}^2$ with edges in the directions (1,0),(0,1),(1,1).) The 3-directional grid can be regarded also as $\mathbb{Z}^2$.

Theorem 18 ([12]) *Let $a(\mathbf{z}) = a(z_1, z_2) = \sum_{i,j} a_{i,j} z_1^i z_2^j$ be the symbol of a bivariate subdivision scheme S, which is defined on regular triangulations. Then a necessary condition for the convergence of S is*

$$\sum_{\beta\in\mathbb{Z}^2} a_{\alpha-2\beta} = 1 \ , \ \alpha \in \{(0,0),(0,1),(1,0),(1,1)\}. \tag{3.5}$$

In the bivariate case, there are 4 rules (Even-Even, Even-Odd, Odd-Even and Odd-Odd) depending on the parity of each component of the vector α. The symbol of a convergent bivariate subdivision scheme satisfies,

$$a(-1,1) = a(1,-1) = a(-1,-1) = 0 \text{ and } a(1,1) = 4. \tag{3.6}$$

Exercise 4 *Verify (3.6).*

In contrast to the univariate case, in the bivariate case the necessary condition (3.5) and the derived condition on $a(z)$ do not imply a factorization of the mask.
The symbol of such a scheme, when being factorizable, has the form

$$a(z_1,z_2) = (1+z_1)(1+z_2)(1+z_1z_2)b(z_1,z_2), \tag{3.7}$$

where $b(z_1, z_2)$ is symmetric in the sense, $b(z_1, z_2) = b(z_2, z_1)$.

Theorem 19 *Let S be a bivariate subdivision scheme with a compactly supported mask corresponding to its Laurent polynomial $a(z_1, z_2) = \sum_{i,j\in\mathbb{Z}} a_{ij} z_1^i z_2^j$. Then we have*

(i) for $i = 1, 2$, $a(z_1, z_2)$ has $1 + z_i$ as a factor if and only if

$$a(z_1,z_2)\big|_{z_i=-1} = 0; \tag{3.8}$$

(ii) $a(z_1, z_2)$ has $1 + z_1z_2$ as a factor if and only if

$$a(z_1,t/z_1)\big|_{t=-1} = 0 \quad \textit{equivalently or} \quad a(z_2,t/z_2)\big|_{t=-1} = 0. \tag{3.9}$$

Proof. The proof is straightforward and we show only that $a(-1, z_2) = 0$ if and only if $a(z_1, z_2)$ has $z_1 + 1$ as a factor. We can expand $a(z_1, z_2)$ with respect to z_2 as

$$a(z_1,z_2) = \sum_{i\in\mathbb{Z}} a_i(z_1) z_2^i,$$

for some polynomials a_i in one variable. Then it is easy to see that $a(-1, z_2) = 0$ if and only if for every $i \in \mathbb{Z}$, $a_i(-1) = 0$, which implies that $a(z_1, z_2)$ has $z_1 + 1$ as a factor. The remains are shown in the same argument, which completes the proof. □

As we can see in the matrix form of the butterfly scheme below, Theorem 19 means in the geometrical point of view that when we plot the masks a_{ij} at the point (i, j) in $\mathbb{Z}^2$-plane, the condition $a(-1, z_2) = 0$ if and only if the sums of even masks and of odd masks along each horizontal line are the same, that is to say, for every $k \in \mathbb{Z}$,

$$\sum_{i \in \mathbb{Z}} (-1)^i a_{i,k} = 0.$$

And we can see that $a(z_1, t/z_1)\big|_{t=-1} = 0$ if and only if

$$\sum_{i \in \mathbb{Z}} (-1)^i a_{i,i+k} = 0, \quad k \in \mathbb{Z}.$$

For integers k and ℓ, expanding $a(z_1, z_2)$ as

$$a(z_1, z_2) = \sum_{i \in \mathbb{Z}} b_i(z_1^k z_2^\ell) z_1^i,$$

with polynomials b_i in one variable, we have, in general, that $a(z_1, z_2)$ has $1 + z_1^k z_2^\ell$ as a factor if and only if the mask $\{a_{ij}\}$ satisfies

$$\sum_{i \in \mathbb{Z}} (-1)^i a_{ik, i\ell+j} = 0, \quad \text{for every } j \in \mathbb{Z}.$$

Comparing with the univariate case, N. Dyn found the following criterions for the verification of the convergence and smoothness of a bivariate subdivision scheme.

Theorem 20 ([12]) *Let S have the symbol $a(z_1, z_2)$ is convergent if and only if the schemes with symbols*

$$a_{1,0}(z_1, z_2) = \frac{a(z_1, z_2)}{1 + z_1}, a_{0,1}(z_1, z_2) = \frac{a(z_1, z_2)}{1 + z_2}, a_{1,1}(z_1, z_2) = \frac{a(z_1, z_2)}{1 + z_1 z_2} \tag{3.10}$$

are contractive. If any two of these schemes are contractive then the third is also contractive.

Theorem 21 ([12]) *Let S have the symbol (3.7). Then S generates C^1 limit functions if the schemes with the symbols $2a_{1,0}(z_1, z_2)$, $2a_{0,1}(z_1, z_2)$ and $2a_{1,1}(z_1, z_2)$ are convergent. If any of two of these schemes are convergent then the third is also convergent. Moreover,*

$$\frac{\partial}{\partial z_1} S_a^\infty f^0 = S_{1,0} \Delta_{1,0} f^0,$$

$$\frac{\partial}{\partial z_2} S_a^\infty f^0 = S_{0,1} \Delta_{0,1} f^0,$$

$$\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \right) S_a^\infty f^0 = S_{1,1} \Delta_{1,1} f^0.$$

The verification that the scheme S with symbol (3.7) generates C^1 limit functions requires checking the contractivity of the three schemes with symbols,

$$2(1+z_1)b(z_1,z_2), \quad 2(1+z_2)b(z_1,z_2), \quad 2(1+z_1z_2)b(z_1,z_2).$$

If these three schemes are contractive, then S generates C^1 limit functions.

Exercise 5 *Verify the last statement*

3.4.1 Butterfly Subdivision Scheme

Dyn, Gregory, and Levin introduced the butterfly scheme. The butterfly scheme is an extension of the 4-point interpolatory subdivision scheme to the bivariate case with topology of regular triangulation, which is an interpolatory triangular subdivision scheme with stencil of small support.
The mask of butterfly scheme is symmetric ($a_{i,j} = a_{j,i}, i, j \in \mathbb{Z}$) and given as

$$\begin{aligned}
&a_{0,0} = 1,\\
&a_{1,0} = a_{-1,0} = a_{-1,-1} = a_{1,1} = 1/2,\\
&a_{1,-1} = a_{-1,-2} = a_{1,2} = 2w,\\
&a_{1,-2} = a_{-3,-2} = a_{-1,2} = a_{3,2} = a_{-1,-3} = a_{1,3} = -w,
\end{aligned}$$

and zero otherwise.
There are three kind of refinement rules:

$$\begin{aligned}
f^{k+1}_{2i+1,2j} &= \frac{1}{2}(f^k_{i,j} + f^k_{i+1,j}) + 2w(f^k_{i,j-1} + f^k_{i+1,j+1})\\
&\quad -w(f^k_{i-1,j-1} + f^k_{i+1,j-1} + f^k_{i,j+1} + f^k_{i+2,j+1}),\\
f^{k+1}_{2i,2j+1} &= \frac{1}{2}(f^k_{i,j} + f^k_{i,j+1}) + 2w(f^k_{i-1,j} + f^k_{i+1,j+1})\\
&\quad -w(f^k_{i-1,j-1} + f^k_{i-1,j+1} + f^k_{i+1,j} + f^k_{i+1,j+2}),\\
f^{k+1}_{2i+1,2j+1} &= \frac{1}{2}(f^k_{i,j} + f^k_{i+1,j+1}) + 2w(f^k_{i+1,j} + f^k_{i,j+1})\\
&\quad -w(f^k_{i,j-1} + f^k_{i-1,j} + f^k_{i+2,j+1} + f^k_{i+1,j+2}).
\end{aligned}$$

The mask maps suggest why the scheme is called the butterfly subdivision scheme.
We express the mask of the butterfly scheme in the matrix form

$$A := \begin{bmatrix}
 & j & 3 & 2 & 1 & 0 & -1 & -2 & -3\\
i & & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow\\
-3 & \Rightarrow & . & . & . & . & -w & -w & .\\
-2 & \Rightarrow & . & . & -w & . & 2w & . & -w\\
-1 & \Rightarrow & . & -w & 2w & \frac{1}{2} & \frac{1}{2} & 2w & -w\\
0 & \Rightarrow & . & . & \frac{1}{2} & 1 & \frac{1}{2} & . & .\\
1 & \Rightarrow & -w & 2w & \frac{1}{2} & \frac{1}{2} & 2w & -w & .\\
2 & \Rightarrow & -w & . & 2w & . & -w & . & .\\
3 & \Rightarrow & . & -w & -w & . & . & . & .
\end{bmatrix}.$$

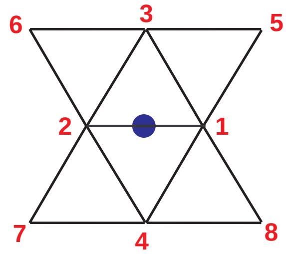

Figure 3.1: Dyn butterfly scheme : $(1, 2 = \frac{1}{2})$, $(3, 4 = 2w)$, $(5, 6, 7, 8 = -w)$

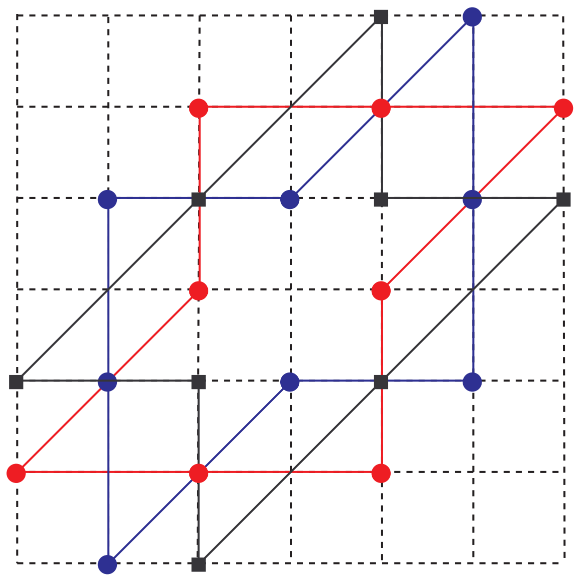

Figure 3.2: The mask map of the butterfly scheme

From the matrix, we see that the mask satisfies the conditions that for any $k \in \mathbb{Z}$,

$$\sum_{i\in\mathbb{Z}}(-1)^i a_{i,k} = \sum_{i\in\mathbb{Z}}(-1)^i a_{k,i} = \sum_{i\in\mathbb{Z}}(-1)^i a_{i,i+k} = 0,$$

and Theorem 19 implies that the symbol of the butterfly scheme is factorizable. The symbol is written as

$$\begin{aligned} a(z_1, z_2) &= \sum_{i=-3}^{3}\sum_{j=-3}^{3} a_{i,j} z_1^i z_2^j \\ &= \frac{1}{2}(1+z_1)(1+z_2)(1+z_1z_2)(1-wc(z_1,z_2))(z_1z_2)^{-1}, \end{aligned}$$

where

$$c(z_1, z_2) = 2z_1^{-2}z_2^{-1} + 2z_1^{-1}z_2^{-2} - 4z_1^{-1}z_2^{-1} - 4z_1^{-1} - 4z_2^{-1}$$

$$+2z_1^{-1}z_2 + 2z_1z_2^{-1} + 12 - 4z_1 - 4z_2 - 4z_1z_2 + 2z_1^2z_2 + 2z_1z_2^2.$$

Gregory computed an explicit shape parameter value $w_0 > 1/16$ such that for $0 < w < w_0$, the butterfly scheme generates C^1 limit functions on regular triangulations. It is well-known that the scheme reproduces cubic polynomials for $w = 1/16$, otherwise linear polynomials for $w \neq 1/16$.

Exercise 6 *Verify the last statement for $w = 1/16$.*

3.4.2 Symmetric 8-point Butterfly Subdivision Scheme

In this section, we consider an 8-point symmetric interpolatory bivariate subdivision scheme defined on a regular triangulation mesh. If we set the mask of symmetric 8-point Butterfly scheme as follows, we get the mask in the matrix form

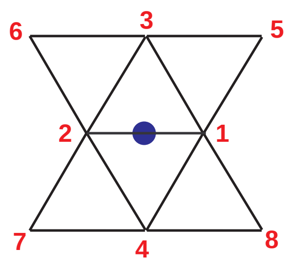

Figure 3.3: Symmetric 8-point butterfly scheme : $(1, 2 = \alpha)$, $(3, 4 = \beta)$, $(5, 6, 7, 8 = \gamma)$

The mask of such a scheme is given in the matrix form as follows:

$$A := \begin{array}{rc}
\begin{array}{r} j \\ i \end{array} & \begin{array}{ccccccc} 3 & 2 & 1 & 0 & -1 & -2 & -3 \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \end{array} \\
\begin{array}{r} -3 \Rightarrow \\ -2 \Rightarrow \\ -1 \Rightarrow \\ 0 \Rightarrow \\ 1 \Rightarrow \\ 2 \Rightarrow \\ 3 \Rightarrow \end{array} & \left[\begin{array}{ccccccc}
. & . & . & . & \gamma & \gamma & . \\
. & . & \gamma & . & \beta & . & \gamma \\
. & \gamma & \beta & \alpha & \alpha & \beta & \gamma \\
. & . & \alpha & 1 & \alpha & . & . \\
\gamma & \beta & \alpha & \alpha & \beta & \gamma & . \\
\gamma & . & \beta & . & \gamma & . & . \\
. & \gamma & \gamma & . & . & . & .
\end{array}\right]
\end{array}.$$

The bivariate Laurent polynomial of this scheme is assumed to be factorizable:

$$a(z_1, z_2) = \sum_{i=-3}^{3}\sum_{j=-3}^{3} a_{i,j} z_1^i z_2^j = (1 + z_1)(1 + z_2)(1 + z_1 z_2) b(z_1, z_2).$$

By Theorem 19, the factorization implies that for each $k \in \mathbb{Z}$, we have

$$\sum_{i\in\mathbb{Z}} (-1)^i a_{i,k} = \sum_{i\in\mathbb{Z}} (-1)^i a_{k,i} = \sum_{i\in\mathbb{Z}} (-1)^i a_{i,i+k} = 0.$$

In this case, we get $2\gamma + \beta = -2\alpha + 1 = 0$. If we set $\gamma = -w$, we obtain the same mask of the butterfly scheme:

$$\alpha = \frac{1}{2}, \quad \beta = 2w, \quad \gamma = -w.$$

3.4.3 Modified Butterfly Subdivision Scheme

As shown in the previous section, the butterfly scheme generates C^1 surfaces in the topology of regular setting. The smoothness of surface in geometric modeling is required to be up to C^2. To obtain a subdivision scheme retaining the simplicity of the butterfly scheme and creating C^2 limit surface, we need to enlarge the support of the butterfly scheme. We try it by taking two more points into account to calculate a new control points as shown in Figure below.

Comparing the stencil of the butterfly scheme, we consider two nearby points (points 9 and 10) with different weight from those of the other 8 points.
The mask of 10-point Butterfly scheme can be expressed by

$$A := \begin{array}{rc}
\begin{array}{r} j \\ i \end{array} & \begin{array}{ccccccc} 3 & 2 & 1 & 0 & -1 & -2 & -3 \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \end{array} \\
\begin{array}{r} -3 \Rightarrow \\ -2 \Rightarrow \\ -1 \Rightarrow \\ 0 \Rightarrow \\ 1 \Rightarrow \\ 2 \Rightarrow \\ 3 \Rightarrow \end{array} & \left[\begin{array}{ccccccc}
. & . & . & w & \gamma & \gamma & w \\
. & . & \gamma & . & \beta & . & \gamma \\
. & \gamma & \beta & \alpha & \alpha & \beta & \gamma \\
w & . & \alpha & 1 & \alpha & . & w \\
\gamma & \beta & \alpha & \alpha & \beta & \gamma & . \\
\gamma & . & \beta & . & \gamma & . & . \\
w & \gamma & \gamma & w & . & . & .
\end{array}\right]
\end{array}.$$

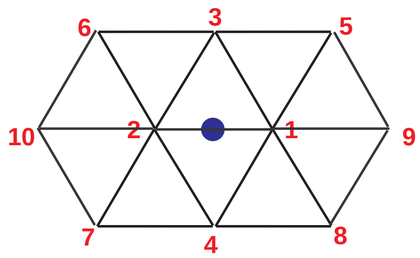

Figure 3.4: Symmetric 10-point butterfly scheme : $(1, 2 = \alpha)$, $(3, 4 = \beta)$, $(5, 6, 7, 8 = \gamma)$, $(9, 10 = w)$

We assume a factorization of the Laurent polynomial of this scheme;

$$a(z_1, z_2) = (1 + z_1)(1 + z_2)(1 + z_1 z_2)c(z_1, z_2).$$

From the factorization, we get

$$2\gamma + \beta = 0, \quad -2\alpha - 2w + 1 = 0.$$

Therefore, we find the mask of 10-point butterfly scheme :

$$\alpha = \frac{1}{2} - w, \quad \beta = 2\gamma, \quad 5, 6, 7, 8 = -\gamma, \quad 9, 10 = w.$$

This mask is exact with a modified butterfly scheme when $\gamma = 1/16 + w$.

Zorin et al. [31] examined that the butterfly scheme exhibits undesirable artifacts in the case of an irregular topology, and derived an improved scheme (a modified Butterfly scheme), which retains the simplicity of the butterfly scheme. The modified butterfly scheme is interpolating and results in smoother surfaces.

3.5 Analysis of Bivariate Schemes on Regular Grids

Let us consider a bivariate convergent subdivision scheme

$$f^{m+1}_{j_1,j_2} = \sum_{(i_1,i_2)\in\mathbb{Z}^2} a_{(j_1-2i_1, j_2-2i_2)} f^m_{i_1,i_2},$$

with the symbol

$$a(z_1, z_2) = \sum_{(i_1,i_2)} a_{(i_1,i_2)} z_1^{i_1} z_2^{i_2}.$$

The generating functions

$$F^m(z_1, z_2) = \sum_{(i_1,i_2)} f^m_{i_1,i_2} z_1^{i_1} z_2^{i_2}$$

satisfy

$$F^{m+1}(z_1, z_2) = a(z_1, z_2)F^m(z_1^2, z_2^2).$$

We have four subdivision rules, according to the parity of (j_1, j_2) and the symbol $a(z_1, z_2)$ should satisfy

$$a(1, 1) = 4; \quad a(1, -1) = a(-1, 1) = a(-1, -1) = 0.$$

For an example, consider the scheme with the symbol

$$a(z_1, z_2) = \frac{1}{2}(1 + \frac{1}{2}z_1 + \frac{1}{2}z_2)^2(1 + z_1z_2).$$

This scheme satisfies the necessary conditions for convergence. Does it converge to a continuous limit? How to check it?

Let us try to find a difference scheme:

$$(1 - z_1)F^{m+1}(z_1, z_2) = \frac{a(z_1, z_2)}{1 + z_1}(1 - z_1^2)F^m(z_1^2, z_2^2).$$

If the symbol has the factor $(1 + z_1)$, we have a difference scheme transforming in one direction from one level to the next. To verify convergence to a continuous limit we have to show that differences in two independent directions tend to zero. In the example above, we do have a nice difference scheme for differences in the diagonal direction.

$$(1 - z_1z_2)F^{m+1}(z_1, z_2) = \frac{a(z_1, z_2)}{1 + z_1z_2}(1 - z_1^2z_2^2)F^m(z_1^2, z_2^2).$$

Hence, $b(z_1, z_2) = \frac{1}{2}(1+\frac{1}{2}z_1+\frac{1}{2}z_2)^2$ is the symbol of the scheme taking differences $\{f_{i,j}^m - f_{i-1,j-1}^m\}$ into differences of the next level $\{f_{i,j}^{m+1} - f_{i-1,j-1}^{m+1}\}$. From the symbol

$$b(z_1, z_2) = \frac{1}{2} + \frac{1}{8}z_1^2 + \frac{1}{8}z_2^2 + \frac{1}{2}z_1 + \frac{1}{2}z_2 + \frac{1}{4}z_1z_2,$$

we have

$$\sum_{i,j} |b_{2i,2j}| = \frac{3}{4}, \sum_{i,j} |b_{2i+1,2j}| = \frac{1}{2}, \sum_{i,j} |b_{2i,2j+1}| = \frac{1}{2}, \sum_{i,j} |b_{2i+1,2j+1}| = \frac{1}{4}.$$

Therefore, differences in the diagonal direction tend to zero.

Chapter 4

Subdivision for Curves

4.1 Corner Cutting Algorithm

For $0 \le s < r \le 1, i \in \mathbb{Z}$, consider the rule

$$\begin{aligned} f_{2i}^{k+1} &= r f_i^k + (1-r) f_{i+1}^k, \\ f_{2i+1}^{k+1} &= s f_i^k + (1-s) f_{i+1}^k. \end{aligned}$$

Table 4.1: Masks of corner cutting scheme

i	$\cdots$	-2	-1	0	1	$\cdots$
a_i	0	$1-r$	$1-s$	r	s	0

This scheme satisfies (2.1) for all r, s. The Laurent polynomial of this scheme is

$$\begin{aligned} a(z) &= (1-r)z^{-2} + (1-s)z^{-1} + r + sz \\ &= z^{-2}(1+z)[(1-r) + (r-s)z + sz^2]. \end{aligned}$$

Hence the Laurent polynomial of S_1 is

$$a_1(z) = \frac{2za(z)}{1+z} = 2(1-r)z^{-1} + 2(r-s) + 2sz,$$

and the scheme $\frac{1}{2}S_1$ for Δf^k is given by

$$\begin{aligned} g_{2i}^{k+1} &= (r-s) g_i^k, \\ g_{2i+1}^{k+1} &= s g_i^k + (1-r) g_{i+1}^k, \end{aligned}$$

where we denote by $g_i^k = \Delta f_i^k$. Since $\|\frac{1}{2}S_1\|_\infty = \max\{r-s, 1-(r-s)\} < 1$, the corner cutting algorithm converges uniformly to a continuous limit function.

Observe that $a_1(z) = 2(1-r)z^{-1} + 2(r-s) + 2sz$ satisfies the necessary condition for uniform convergence (2.1) if and only if $r - s = \frac{1}{2}$. Under the additional condition i.e., $s < \frac{1}{2}$, $a_1(z) = (1-2s)z^{-1} + 1 + 2sz$, and there exists $\frac{1}{2}S_2$ with Laurent polynomial

$$\frac{1}{2}a_2(z) = \frac{z}{1+z}a_1(z) = 1 - 2s + 2sz.$$

and norm $\|\frac{1}{2}S_2\|_\infty = \max\{1-2s, 2s\} < 1$. Therefore S_1 converges uniformly and $S^\infty f^0 \in C^1(\mathbb{R})$, for all initial control point f^0.

The scheme S_2 satisfies (2.1) if and only if $s = \frac{1}{4}, r = \frac{3}{4}$, namely the case of the Chaikin's algorithm. For these values of s and r, we have

$$a_2(z) = 1 + z, \quad \frac{1}{2}a_3(z) = z$$

and

$$\left\|\frac{1}{2}S_3\right\|_\infty = 1.$$

In fact $\|(\frac{1}{2}S_3)^k\|_\infty = 1$ for all k. As is expected, the Chaikin's algorithm does not produce $C^2(\mathbb{R})$ functions.

Note. We can get Chaikin's algorithm by interpolating the data $(2^{-k}(i+j), f^k_{i+j}), j = 0, 1$ by a linear polynomial and evaluating it at $2^{-k}(i + 1/4)$ and $2^{-k}(i + 3/4)$ for the values f^{k+1}_{2i} and f^{k+1}_{2i+1} respectively. It is sufficient to consider p_1, the linear polynomial such that $p_1(j) = f_j$ for $j = 0, 1$. Since

$$p_1(t) = \sum_{j=0}^{1} L_j(t) f_j, \quad L_j(t) = \prod_{k=0, k\neq j}^{1} \frac{t-k}{j-k},$$

and predict

$$f^{k+1}_{2i} = p_1\left(i + \frac{1}{4}\right), \quad f^{k+1}_{1+2i} = p_1\left(i + \frac{3}{4}\right),$$

we find

$$\begin{aligned} f^1_0 = p_1(1/4) &= L_0(1/4)f_0 + L_1(1/4)f_1 = \frac{3}{4}f_0 + \frac{1}{4}f_1, \\ f^1_1 = p_1(3/4) &= L_0(3/4)f_0 + L_1(3/4)f_1 = \frac{3}{4}f_0 + \frac{1}{4}f_1. \end{aligned}$$

4.2 Cubic Spline Algorithm

Consider the rule

$$\begin{aligned} f^{k+1}_{2i} &= \frac{1}{2}f^k_i + \frac{1}{2}f^k_{i+1}, \\ f^{k+1}_{2i+1} &= \frac{1}{8}f^k_i + \frac{6}{8}f^k_{i+1} + \frac{1}{8}f^k_{i+2}. \end{aligned}$$

It is very easy to show that $S^\infty f^0 \in C^2(\mathbb{R})$.

Exercise 7 • *Find the Laurent polynomial of S.*

- *Calculate the $a_3(z)$ and prove that $\|\frac{1}{2}S_3\|_\infty = \frac{1}{2}$. This estimate of norm shows that $S^\infty f^0 \in C^2(\mathbb{R})$.*

4.3 Binary Subdivision of B-Spline

The masks of the subdivision scheme for B-spline B^m of degree m are given by

$$a_m = 2^{-m}\binom{m+1}{j}, \quad j = 0, 1, \ldots, m+1.$$

The Laurent polynomial of the subdivision scheme for B-spline B^m curves has the form

$$a(z) = \sum_m a_m z^m = 2^{-m}(1+z)^{m+1},$$

and

$$\frac{1}{2}a_1(z) = \left(\frac{1}{2}\right)^m z(1+z)^m.$$

Exercise 8 *Prove that the smoothness of B^m is C^{m-1}.*

- *Prove that the uniform convergence of the subdivision scheme for B^m, i.e., prove that $\|\frac{1}{2}S_1\|_\infty = \frac{1}{2}$.*
- *Calculate the $a_m(z)$ and prove that $\|\frac{1}{2}S_m\|_\infty = \frac{1}{2}$.*

A degree r, piecewise polynomial B-spline curve $S^r(u)$ is defined by

$$S^r(u) = \sum_i d_i^r B_i^r(u-i). \tag{4.1}$$

The vector valued coefficients d_i^r form the de Boor polygon. The $B_i^r(u)$ are normalized B-spline of degree r defined over a sequence of knot i.

$B_i^r(u)$ **has the following properties:**

- Partition of unity
- Positivity
- Local support
- Continuity
- Recursion

The idea of a binary subdivision scheme is to rewrite the curve (4.1), as a curve over a refined knot sequence. Binary refinement of the sequence $\mathbb{Z}$ results in the sequence

$$\mathbb{Z}/2 = \{\cdots, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \cdots\}.$$

From this, (4.1) becomes

$$S^r(u) = \sum_{j \in \mathbb{Z}/2} \hat{d}^r_j B^r(2(u-j)).$$

It is possible to determine the $\hat{d}^r_j$ by considering the subdivision of a single B-spline. A single B-spline may be decomposed into similar B-spline of half the support. This result in

$$B^r(u) = \sum_{j \in \mathbb{Z}/2} c^r_j B^r(2(u-j)),$$

where c^r_j can be written by

$$c^r_j = 2^{-r} \binom{r+1}{2j}.$$

The process of subdividing an entire curve follows. The translated B-spline $B^r(u-i)$ are subdivided

$$B^r(u-i) = \sum_{j \in \mathbb{Z}/2} c^r_{j-i} B^r(2(u-j)).$$

This is substituted into (4.1) to give

$$S^r(u) = \sum_{i \in \mathbb{Z}} d^r_i \sum_{j \in \mathbb{Z}/2} c^r_{j-i} B^r(2(u-j)).$$

Rearranging the order of summation gives

$$S^r(u) = \sum_{j \in \mathbb{Z}/2} \sum_{i \in \mathbb{Z}} c^r_{j-i} d^r_i B^r(2(u-j)).$$

It then follows that

$$\hat{d}^r_j = \sum_{i \in \mathbb{Z}} c^r_{j-i} d^r_i.$$

For example, if $r = 2$, then we have

$$\hat{d}^2_0 = \sum_{i \in \mathbb{Z}} c^2_{-i} d^2_i = \cdots + (0)d^2_{-2} + \left(\frac{3}{4}\right) d^2_{-1} + \left(\frac{1}{4}\right) d^2_0 + (0)d^2_1 + \cdots.$$

and similarly

$$\hat{d}^2_{1/2} = \cdots + (0)d^2_{-2} + \left(\frac{1}{4}\right) d^2_{-1} + \left(\frac{3}{4}\right) d^2_0 + (0)d^2_1 + \cdots.$$

The mask and support size for B-spline of degree 1,2 and 3.

degree	mask	support size
1	$\frac{1}{2}[1,2,1]$	2
2	$\frac{1}{4}[1,3,3,1]$	3
3	$\frac{1}{8}[1,4,6,4,1]$	4

Exercise 9 *For $r=3$, calculate $\hat{d}_0^3, \hat{d}_{1/2}^3$.*

Corner cutting=split + average. In the case of B-spline of degree k, one needs to perform one doubling operation and follow it by k averaging steps.

Example 1 *Consider for B-spline of degree 1. Let the control points be*

$$P_1, P_2, \dots, P_n.$$

Splitting operator yields

$$P_1, P_1, P_2, P_2, \dots, P_n, P_n.$$

As a result of averaging, we have

$$P_1, \frac{P_1+P_2}{2}, P_2, \dots, \frac{P_{n-1}+P_n}{2}, P_n.$$

At each step it adds all the mid-points. And after a few iteration, we get the piecewise linear curve (B-spline of degree 1). We can get cubic B-spline by one splitting operation and follow by 3-times averaging operations. As a result of 2-times averaging, we have Chaikin's algorithm

$$\frac{3P_1+P_2}{4}, \frac{P_1+3P_2}{4}, \frac{3P_2+P_3}{4}, \dots, \frac{3P_{n-1}+P_n}{4}, \frac{P_{n-1}+3P_n}{4},$$

and we get cubic B-spline finally

$$\frac{P_1+P_2}{2}, \frac{P_1+6P_2+P_3}{8}, \frac{P_2+P_3}{2}, \dots, \frac{P_{n-2}+6P_{n-1}+P_n}{8}, \frac{P_{n-1}+P_n}{2}.$$

The Lane-Riesenfeld algorithm

- split $1\times$ average piecewise linear C^0.
- split $2\times$ average piecewise quadratic C^1.
- split $3\times$ average piecewise cubic C^2.
- split $n\times$ average piecewise degree n C^{n-1}.

A split followed by n averaging steps results in a subdivision scheme generating uniform splines of degree n. The basic limit function is the corresponding B-spline.
We can explain split and averaging operator by generating function. For example consider the given control points

$$P = [2, 10, 14, 6],$$

and generating function

$$F(z) = 2 + 10z + 14z^2 + 6z^3.$$

We can get the doubling control points from $F(z)$

$$\begin{aligned} G(z) &:= (1+z)F(z^2) \\ &= 2 + 2z + 10z^2 + 10z^3 + 14z^4 + 14z^5 + 6z^6 + 6z^7, \end{aligned}$$

and

$$Q := [2, 2, 10, 10, 14, 14, 6, 6].$$

The split operator is given by

$$\begin{aligned} Q &= SP, \\ G(z) &= (1+z)F(z^2). \end{aligned}$$

Similarly, we can get the averaging control points from $F(z)$

$$G(z) := \frac{(1+z)}{2}F(z) = 1 + 6z + 12z^2 + 10z^3 + 3z^4,$$

and

$$Q := [1, 6, 12, 10, 3].$$

The averaging operator is given by

$$\begin{aligned} Q &= AP, \\ G(z) &= \frac{(1+z)}{2}F(z). \end{aligned}$$

The Lane-Riesenfeld subdivision operator

$$\begin{aligned} Q &= A^n SP, \\ G(z) &= \left(\frac{1+z}{2}\right)^n (1+z)F(z^2) \end{aligned}$$

has the symbol

$$a(z) = \frac{(1+z)^{n+1}}{2^n}.$$

It generates spline of degree n and smoothness C^{n-1}.

Exercise 10 *Determine the mask of the subdivision scheme that generates quartic and quintic B-spines.*

4.4 Deslauries-Dubuc Subdivision Scheme

The construction principle of the Deslauries-Dubuc scheme can be generalized as follows: A new odd point is obtained by interpolating its old $2n$ neighborhood at equidistant knots by a polynomial of degree $2n-1$ and evaluation at the center. The masks can be computed easily using Lagrange interpolation polynomials, but smoothness is not optimal. It is well-known that the smoothness of 4-point interpolating Deslauriers-Dubuc(DD) subdivision scheme is C^1. Dyn proved that 4-point interpolating subdivision scheme with one parameter is C^1 by means of eigenanalysis. In this section we take advantage of Laurent polynomial method to prove the regularity of scheme.

The binary stationary subdivision scheme is a process which recursively defines a sequence of control points $f^k = \{f_i^k : i \in \mathbb{Z}\}$ by a rule of the form with a mask $\mathbf{a} = \{a_i : i \in \mathbb{Z}\}$

$$f_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-2j} f_j^k, \quad k = 0, 1, 2, \ldots.$$

The 4-point Deslauriers-Dubuc(DD) scheme is given by

$$\begin{aligned} f_{2i}^{k+1} &= f_i^k, \\ f_{2i+1}^{k+1} &= \frac{9}{16}(f_i^k + f_{i+1}^k) - \frac{1}{16}(f_{i-1}^k + f_{i+2}^k). \end{aligned}$$

Table 4.2: Masks of 4-point DD scheme

i	$\cdots$	-3	-2	-1	0	1	2	3	$\cdots$
a_i	0	$-\frac{1}{16}$	0	$\frac{9}{16}$	1	$\frac{9}{16}$	0	$-\frac{1}{16}$	0

We can construct the mask of this scheme by interpolating the data $(j, f_j^k), j = i-1, i, i+1, i+2$, by a cubic polynomial p_3, satisfying

$$p_3(j) = f_j^k, \quad j = i-1, i, i+1, i+2,$$

and then predict

$$f_{2i+1}^{k+1} = p_3\left(i + \frac{1}{2}\right).$$

It is sufficient to consider p_3, the cubic polynomial such that $p_3(j) = f_j$ for $j = -1, 0, 1, 2$. Since

$$p_3(t) = \sum_{j=-1}^{2} L_j(t) f_j, \quad L_j(t) = \prod_{k=-1, k\neq j}^{2} \frac{t-k}{j-k},$$

we find

$$\begin{aligned} p_3\left(\frac{1}{2}\right) &= L_{-1}(1/2)f_{-1} + L_0(1/2)f_0 + L_1(1/2)f_1 + L_2(1/2)f_2 \\ &= -\frac{1}{16}f_{-1} + \frac{9}{16}f_0 + \frac{9}{16}f_1 - \frac{1}{16}f_2. \end{aligned}$$

There is another way to get 4-point DD scheme: suppose that we have 4-point P_1, P_2, P_3, P_4 lying on unit circle with center O, such that

$$\angle P_1OP_2 = \angle P_2OP_3 = \angle P_3OP_4 = 2\theta.$$

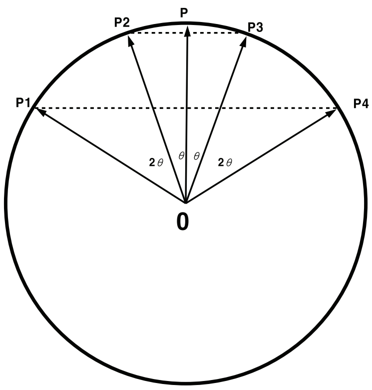

Figure 4.1: Configuration of 4-point scheme.

To calculate the midpoint P of the arc P_2P_3 as a linear combination with sum of weights 1 of these points, we start with the linear combinations

$$L_1 = \frac{1}{2}(\overrightarrow{OP_2} + \overrightarrow{OP_3}), \quad L_2 = \frac{1}{2}(\overrightarrow{OP_1} + \overrightarrow{OP_4}).$$

From the following condition

$$L_1 + \alpha(L_1 - L_2) = \frac{1}{2}(\overrightarrow{OP_2} + \overrightarrow{OP_3}) + \alpha\frac{1}{2}(\overrightarrow{OP_2} + \overrightarrow{OP_3} - \overrightarrow{OP_1} - \overrightarrow{OP_4}) = \overrightarrow{OP},$$

we have

$$\cos\theta + \alpha(\cos\theta - \cos 3\theta) = 1,$$

giving

$$\alpha = \frac{1}{4\cos\theta(1+\cos\theta)}.$$

Taking the limit, we get

$$\lim_{\theta\to 0} \alpha = \frac{1}{8},$$

and so, in the limit, the mask of the scheme is

$$\overrightarrow{OP} = \frac{9}{16}(\overrightarrow{OP_2} + \overrightarrow{OP_3}) - \frac{1}{16}(\overrightarrow{OP_1} + \overrightarrow{OP_4}).$$

This is the mask of the 4-point DD scheme.

Exercise 11 *Find the mask of 6-point DD scheme.*

From the mask of DD scheme, we get the Laurent polynomial $a(z)$

$$a(z) = -\frac{1}{16}z^{-3} + \frac{9}{16}z^{-1} + 1 + \frac{9}{16}z - \frac{1}{16}z^3.$$

We can easily prove that the smoothness of this scheme is C^1 by Laurent polynomial method. We set

$$b^{[m,L]}(z) = \frac{1}{2^L}a_m^{[L]}(z), \quad m = 1, 2, \ldots, L$$

where

$$a_m(z) = \frac{2z}{1+z}a_{m-1}(z) = \left(\frac{2z}{1+z}\right)^m a(z),$$

and

$$a_m^{[L]}(z) = \prod_{j=0}^{L-1} a_m(z^{2^j}).$$

From the Laurent polynomial, we have

$$b^{[1,1]}(z) = \frac{1}{2}a_1(z) = \frac{z}{1+z}a(z) = -\frac{1}{16}z^{-2} + \frac{1}{16}z^{-1} + \frac{1}{2} + \frac{1}{2}z + \frac{1}{16}z^2 - \frac{1}{16}z^3.$$

And we measure the norm of subdivision $\frac{1}{2}S_1$. Refer to Appendix A.6.

$$\begin{aligned} \left\|\frac{1}{2}S_1\right\|_\infty &= \max\{\sum_\beta |b^{[1,1]}_{\gamma+2\beta}| : \gamma = 0, 1\} \\ &= \max\{5/8, 5/8\} = \frac{5}{8} < 1. \end{aligned}$$

Therefore $\frac{1}{2}S_1$ is contractive, we have S is convergent.

To prove this 4-point DD scheme is C^1, from the Laurent polynomial $a_1(z)$, we have

$$b^{[2,1]}(z) = \frac{1}{2}a_2(z) = \frac{z}{1+z}a_1(z) = -\frac{1}{8}z^{-1} + \frac{1}{4} + \frac{3}{4}z + \frac{1}{4}z^2 - \frac{1}{8}z^3.$$

And we find the norm of subdivision $\frac{1}{2}S_2$.

$$\begin{aligned} \left\|\frac{1}{2}S_2\right\|_\infty &= \max\{\sum_\beta |b^{[2,1]}_{\gamma+2\beta}| : \gamma = 0, 1\} \\ &= \max\{1/2, 1\} = 1. \end{aligned}$$

But for $L = 2$, we get

$$b^{[2,2]}(z) = \frac{1}{4}a_2^{[2]}(z) = \left(\frac{1}{2}a_2(z)\right)\left(\frac{1}{2}a_2(z^2)\right),$$

and

$$\left\|\left(\frac{1}{2}S_2\right)^2\right\|_\infty = \max\{\sum_\beta |b^{[2,2]}_{\gamma+4\beta}| : \gamma = 0,1,2,3\}$$
$$= \max\{5/16, 1/4, 5/16, 3/4\} < 1.$$

Since subdivision scheme $\frac{1}{2}S_2$ is contractive, we have S_1 is convergent and $S \in C^1$.

Exercise 12

- *Construct the DD subdivision scheme, based on 6 points and 2n point.*
- *What is the maximal possible smoothness of the limit functions generated by 4-point DD scheme?*
- *Determine the approximation order of the 4-point DD scheme.*
- *Prove that this 4-point DD scheme is not C^2, i.e., Show that for all $L \in \mathbb{Z}_+$, we have $\|(\frac{1}{2}S_3)^L\|_\infty \geq 1$. See [26].*

Question: Let C be the curve generated by DD scheme S. Can we refine S, inserting k equidistant new points between any two consecutive point of S, so that the curve generated by DD scheme is again C?
Note: A refinement for spline is always possible in the form of knot insertion.

Let $F(t)$ be the basis function of the DD scheme. We will compare F with function H_k obtained from the refined initial control point set $\{(t, F(t))\}$,

$$t = \cdots, -\frac{2}{k}, -\frac{1}{k}, 0, \frac{1}{k}, \frac{2}{k}, \cdots, \quad k \geq 3.$$

The zeros of F inside the interval $(-3, 3)$ are

$$F\left(-3 + \frac{1}{2^n}\right) = F\left(3 - \frac{1}{2^n}\right) = 0, \quad n = -1, 0, 1, 2, \ldots.$$

For the first subdivision step, we see that

$$F(0) = 1, F(1/2) = 9/16, F(1) = 0, F(3/2) = -1/16, F(2) = F(5/2) = \cdots = 0.$$

After the second step, we find

$$F(3/2) = -1/16, F(7/4) = -9/16^2, F(2) = 0, F(9/4) = 1/16^2, F(5/2) = F(11/4) = \cdots = 0.$$

We notice that F is self-similar in $(3/2, \infty)$ and $(9/4, \infty)$, as the node point are multiplied by $1/2$ and translated by $3/2$, while the ordinates are multiplied by $-1/16$. So there is only one zero in each of the intervals,

$$\left[\frac{3}{2}, \frac{9}{4}\right] \left[\frac{9}{4}, \frac{21}{8}\right] \cdots \left[3 - \frac{3}{2^n}, 3 - \frac{3}{2^{n+1}}\right] \cdots$$

and

$$3-\frac{3}{2^n}<3-\frac{1}{2^{n-1}}<3-\frac{3}{2^{n+1}}.$$

By symmetry, we obtain for $k=2^n$

$$F\left(-3+\frac{1}{k}\right)=F\left(3-\frac{1}{k}\right)=0,$$

and for $k\neq 2^n$

$$F\left(-3+\frac{1}{k}\right)=F\left(3-\frac{1}{k}\right)\neq 0.$$

Finally, after k-refinement of the initial control point, with $k\neq 2^n$, in the next step for the four consecutive control points

$$F\left(3-\frac{1}{k}\right)\neq 0, F(3)=0, F\left(3+\frac{1}{k}\right)=0, F\left(3+\frac{2}{k}\right)=0,$$

we have

$$H_k\left(3+\frac{1}{2k}\right)=-\frac{1}{16}\left(F\left(3-\frac{1}{k}\right)+F\left(3+\frac{2}{k}\right)\right)+\frac{9}{16}\left(F(3)+F\left(3+\frac{1}{k}\right)\right)\neq 0.$$

Hence H_k has lager support than F.

In [28], it was shown that, for a given $n\in\mathbb{N}$, the unique minimally supported symmetric subdivision mask **a** satisfying the polynomial reproducing property. The mask is given by $a=d_n$, where

$$\begin{cases} d_{n,2j}=\delta_j, & j\in\mathbb{Z}, \\ d_{n,1-2j}=l_{n,j}(\frac{1}{2}), & j=-n+1,-n+2,\cdots,n, \\ d_{n,j}=0, & |j|\geq 2n, \end{cases}$$

and with $\{l_{n,j}: j=-n+1,\cdots,n\}$ denoting the $(2n-1)$th degree Lagrange fundamental polynomials

$$l_{n,j}(x)=\prod_{j\neq k,k=-n+1}^{n}\frac{x-k}{j-k},\quad j=-n+1,\cdots,n,$$

for which

$$l_{n,j}(k)=\delta_{j,k},\quad k,j=-n+1,\cdots,n,$$

and

$$\sum_{j=-n+1}^{n}p(j)l_{n,j}(x)=p(x),\quad p\in\pi_{2n-1}.$$

The subdivision scheme S, which is known as the Dubuc-Deslauriers (DD) subdivision scheme of order n, as introduced and analyzed in [6], [8], can be seen to be interpoltory and satisfies, by construction, the polynomial reproducing property

$$\sum_k d_{n,j-2k}p(k)=p(\frac{j}{2}),\quad j\in\mathbb{Z},\quad p\in\pi_{2n-1}.$$

4.5 4-point Approximating Subdivision Scheme

We present 4-point approximating scheme that generate C^2 curves. The refinement rule is based on local cubic interpolation, followed by evaluation at $1/4$ an $3/4$ of the refined interval.

Suppose we are given data $f_i, i \in \mathbb{Z}$. We set $f_i^0 = f_i, i \in \mathbb{Z}$ and define for each $k = 0, 1, \ldots$ and $i \in \mathbb{Z}$,

$$\begin{aligned} f_{2i}^{k+1} &= -\frac{7}{128} f_{i-1}^k + \frac{105}{128} f_i^k + \frac{35}{128} f_{i+1}^k - \frac{5}{128} f_{i+2}^k, \\ f_{2i+1}^{k+1} &= -\frac{5}{128} f_{i-1}^k + \frac{35}{128} f_i^k + \frac{105}{128} f_{i+1}^k - \frac{7}{128} f_{i+2}^k. \end{aligned}$$

Table 4.3: Masks of 4-point approximating scheme.

i	$\cdots$	-4	-3	-2	-1	0	1	2	3	$\cdots$
a_i	0	$-\frac{5}{128}$	$-\frac{7}{128}$	$\frac{35}{128}$	$\frac{105}{128}$	$\frac{105}{128}$	$\frac{35}{128}$	$-\frac{7}{128}$	$-\frac{5}{128}$	

This scheme comes from interpolating the data $(2^{-k}(i+j), f_{i+j}^k), j = -1, 0, 1, 2$ by a cubic polynomial and evaluating it at $2^{-k}(i+1/4)$ and $2^{-k}(i+3/4)$ for the values f_{2i}^{k+1} and f_{2i+1}^{k+1} respectively. It is sufficient to consider p_3, the cubic polynomial such that $p_3(j) = f_j$ for $j = -1, 0, 1, 2$. Since

$$p_3(t) = \sum_{j=-1}^{2} L_j(t) f_j, \quad L_j(t) = \prod_{k=-1, k \neq j}^{2} \frac{t-k}{j-k},$$

we find

$$\begin{aligned} p_3(1/4) &= L_{-1}(1/4) f_{-1} + L_0(1/4) f_0 + L_1(1/4) f_1 + L_2(1/4) f_2, \\ p_3(3/4) &= L_{-1}(3/4) f_{-1} + L_0(3/4) f_0 + L_1(3/4) f_1 + L_2(3/4) f_2. \end{aligned}$$

The Laurent polynomial of S_a is

$$a(z) = \sum_i a_i z^i = \frac{1}{128}(-5z^3 - 7z^2 + 35z + 105 + 105z^{-1} + 35z^{-2} - 7z^{-3} - 5z^{-4}).$$

This can be written as

$$a(z) = \frac{(1+z)^2}{4} b(z),$$

where

$$b(z) = \frac{1}{32}(-5z + 8 + 26z^{-1} + 8z^{-2} - 5z^{-3}).$$

If S_b is contractive then S_a is C^2. Defining

$$\|S_b^{[l]}\|_\infty := \max\{\sum_{j \in \mathbb{Z}} |b_{i-2^l j}^{[l]}| : 0 \leq i \leq 2^l - 1\},$$

where

$$b^{[l]}(z) := b(z)b(z^2)\cdots b(z^{2^{l-1}}),$$

we find that

$$\|S_b^{[1]}\|_\infty = \frac{1}{32}\max\{5+26+5, 8+8\} = \frac{9}{8} > 1,$$

which does not show that S_b is contractive. However, it is easy to show that

$$b^{[2]}(z) = b(z)b(z^2), \quad \|S_b^{[2]}\|_\infty = \frac{117}{128} < 1, \tag{4.2}$$

which show that S_b is contractive.

Exercise 13 *Verify (4.2).*

We have the mask of S:

$$\mathbf{a} = \frac{1}{128}[-5, -7, 35, 105, 105, 35, -7, -5]$$

and the mask of S_1:

$$\mathbf{a_1} = \frac{1}{64}[-5, -2, 37, 68, 37, -2, -5].$$

Since

$$\left\|\frac{1}{2}S_1\right\|_\infty = \max\{\frac{84}{128}, \frac{72}{128}\} < 1,$$

this scheme converges to continuous limit function. It is easy to check that $a_1(z)$ satisfies the necessary condition for C^1. We have the mask of S_2:

$$\mathbf{a_2} = \frac{1}{32}[-5, 3, 34, 34, 3, -5]$$

and

$$\left\|\frac{1}{2}S_2\right\|_\infty = \frac{42}{64} < 1.$$

Hence this scheme has C^1. We can verify that $a_2(z)$ satisfies the necessary condition for C^2. And the mask of S_3 is

$$\mathbf{a_3} = \frac{1}{16}[-5, 8, 26, 8, -5]$$

and

$$\left\|\frac{1}{2}S_3\right\|_\infty = \max\{\frac{36}{32}, \frac{16}{32}\} > 1,$$

which does not show that $\frac{1}{2}S_3$ is contractive.
But for $L = 2$, we get

$$b^{[3,2]}(z) := \frac{1}{4}a_3^{[2]}(z) = \left(\frac{1}{2}a_3(z)\right)\left(\frac{1}{2}a_3(z^2)\right)$$

and

$$\left\| \left(\frac{1}{2} S_3 \right)^2 \right\|_\infty = \max\{\sum_\beta |b^{[3,2]}_{\gamma+4\beta}| : \gamma = 0, 1, 2, 3\} = \frac{936}{1024} < 1.$$

Hence we have S is C^2.

We can extend this construction to locally fit a Lagrange interpolation polynomial of degree $2n-1$ to the $2n$ points that are closest to the interval to be refined, and to evaluate it at $1/4$ and $3/4$ of the interval. Here n can be any fixed integer. For example, $n = 1$ gives Chaikin's algorithm and $n = 2$ gives this approximating scheme.

Consider the problem of interpolating the data $(2^{-k}(i+j), f^k_{i+j}), j = -N, -N+1, \ldots, N+1$ by polynomial of degree $2N+1$ and evaluating it at $2^{-k}(i+1/4)$ and $2^{-k}(i+3/4)$ for the values f^{k+1}_{2i} and f^{k+1}_{2i+1} respectively. It is sufficient to consider p_{2N+1}, polynomial such that $p_{2N+1}(j) = f_j$ for $j = -N, -N+1, \ldots, N, N+1$.

$$\begin{aligned} f^{k+1}_{2i} &= p_{2N+1}\left(i + \frac{1}{4}\right) = \sum_j a_{-2j} f^k_{i+j}, \\ f^{k+1}_{2i+1} &= p_{2N+1}\left(i + \frac{3}{4}\right) = \sum_j a_{1-2j} f^k_{i+j}. \end{aligned}$$

Since

$$p_{2N+1}(t) = \sum_{j=-N}^{N+1} L_j(t) f_j, \quad L_j(t) = \prod_{k=-N, k\neq j}^{N+1} \frac{t-k}{j-k},$$

we find

$$L_k(1/4) = \frac{(-1)^N (4N+3)(4N+1)\cdots 3\cdot 1}{(-1)^{N+1+k} 4^{2N+1}(4k-1)(N+k)!(N+1-k)!}.$$

It can be shown that these new $(2N+2)$-point approximating scheme reproduce polynomials up to degree $(2N+1)$ and have approximation order $(2N+2)$. It is open problem to find optimal smoothness of $(2N+2)$-point scheme.

Exercise 14
- *Find* $L_k(3/4)$*.*
- *Show that* $N = 0$ *gives Chaikin's algorithm and* $N = 1$ *gives 4-point approximating scheme.*

4.6 4-point Interpolating Scheme

Interpolating subdivision scheme retains the points of stage k as a subset of the points of stage $k+1$. The general form of an interpolating subdivision scheme is

$$\begin{aligned} f^{k+1}_{2i} &= f^k_i, \\ f^{k+1}_{2i+1} &= \sum_{j\in\mathbb{Z}} a_{1+2j} f^k_{i-j}. \end{aligned}$$

The example we consider is a one parameter family of schemes given by the non-zero mask:

$$a_{\pm 3} = -w, \quad a_0 = 1, \quad a_{\pm 1} = \frac{1}{2} + w.$$

Table 4.4: Masks of 4-point interpolating scheme.

i	$\cdots$	-3	-2	-1	0	1	2	3	$\cdots$
a_i	0	$-w$	0	$\frac{1}{2}+w$	1	$\frac{1}{2}+w$	0	$-w$	0

Note that for interpolating schemes, convergence implies uniform convergence, since the control points $\{f_i^k\}$ are on the limit function. The Laurent polynomial of this scheme is

$$\begin{aligned} a(z) &= -wz^{-3} + (\frac{1}{2} + w)z^{-1} + 1 + (\frac{1}{2} + w)z - wz^3 \\ &= z^{-3}(1+z)^2 \left(\frac{1}{2}z^2 - w(z-1)^2(1+z^2)\right), \end{aligned}$$

and

$$\frac{1}{2}a_1(z) = \frac{z}{1+z}a(z) = -wz^{-2} + wz^{-1} + \frac{1}{2} + \frac{1}{2}z + wz^2 - wz^3.$$

Hence

$$\left\| \frac{1}{2}S_1 \right\|_\infty = \frac{1}{2} + 2|w|,$$

and the range of w which guarantees uniform convergence to zero of $\frac{1}{2}S_1$ is $|w| < \frac{1}{4}$. But this range is not the best possible. By considering the scheme $(\frac{1}{2}S_1)^2$ with the Laurent polynomial

$$\frac{1}{4}a_1(z)a_1(z^2),$$

and

$$\begin{aligned} \left\| \left(\frac{1}{2}S_1\right)^2 \right\|_\infty = \max \quad & \{|\tfrac{1}{2} + w||w| + |\tfrac{1}{4} + w^2 - \tfrac{1}{2}w| + |w||1+w| + w^2, \\ & |w||w - \tfrac{1}{2}| + |\tfrac{1}{4} + \tfrac{1}{2}w - w^2| + 2w^2\} \end{aligned}$$

Thus we can conclude that the subdivision scheme converges for $-\frac{3}{8} < w < \frac{-1+\sqrt{13}}{8}$.

Exercise 15

- *Find* $\frac{1}{2}a_2(z)$.
- *Show that* $\|\frac{1}{2}S_2\|_\infty = 1$.
- *Show that this scheme is* C^1*, i.e.,* $\|(\frac{1}{2}S_2)^2\|_\infty < 1$ *for* $0 < w < \frac{-1+\sqrt{5}}{8}$
- *For* $w = 1/16$*, find* $\frac{1}{2}a_3(z)$ *and prove that this scheme is not* C^2*, i.e.,* $\|(\frac{1}{2}S_3)^k\|_\infty = 1$ *for all* $k \in \mathbb{Z}_+$*.*

4.7 Ternary 4-point Interpolating Scheme

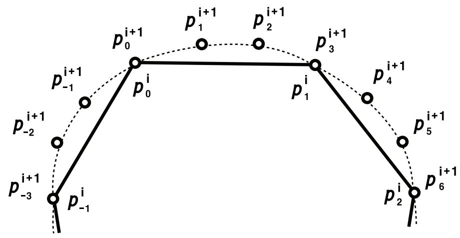

Figure 4.2: Ternary Scheme: A polygon P^i is mapped to a refined polygon P^{i+1}.

Here we present a ternary 4-point interpolating subdivision scheme.

$$\begin{aligned} f_{3i}^{k+1} &= f_i^k, \\ f_{3i+1}^{k+1} &= a_0 f_{i-1}^k + a_1 f_i^k + a_2 f_{i+1}^k + a_3 f_{i+2}^k, \\ f_{3i+2}^{k+1} &= a_3 f_{i-1}^k + a_2 f_i^k + a_1 f_{i+1}^k + a_0 f_{i+2}^k, \end{aligned}$$

where the weights are given by

$$\begin{aligned} a_0 = -\frac{1}{18} - \frac{1}{6}w, \quad a_1 = \frac{13}{18} + \frac{1}{2}w, \\ a_2 = \frac{7}{18} - \frac{1}{2}w, \quad a_3 = -\frac{1}{18} + \frac{1}{6}w. \end{aligned}$$

These weights were the solutions of a constraint problem derived from the constant, linear and quadratic precision conditions, which are necessary condition for C^2.

Exercise 16 *Verify the last statement.*

Table 4.5: Masks of ternary 4-point interpolating scheme.

i	$\cdots$	-5	-4	-2	-1	0	1	2	4	5	$\cdots$
a_i	0	a_3	a_0	a_2	a_1	1	a_1	a_2	a_0	a_3	0

Suppose the eigenvalues of subdivision matrix are $\{\lambda_i\}$, where $\lambda_0 = 1$ and $|\lambda_i| \geq |\lambda_{i+1}|$. We have the following necessary conditions for the corresponding properties:

- $|\lambda_1| = |\lambda_2| \quad \Leftarrow$ kink, i.e., not C^1.

- $\lambda_1^2 < \lambda_2 \quad \Leftarrow$ unbounded curvature.
- $\lambda_1^2 = |\lambda_2| = |\lambda_3| \quad \Leftarrow$ mildly diverging curvature.
- $\lambda_1^2 = |\lambda_2| > |\lambda_3| \quad \Leftarrow$ bounded curvature
- $\lambda_1^2 > |\lambda_2| \quad \Leftarrow$ zero curvature.

We are interested in the case $\lambda_1^2 = |\lambda_2| > |\lambda_3|$, i.e., the curvature of the limit function is bounded, which is a necessary condition for C^2.

The **mark points** are the points which are topologically invariant under the subdivision step. For this scheme, the mark points are the mid-point between two vertices and vertices themselves.

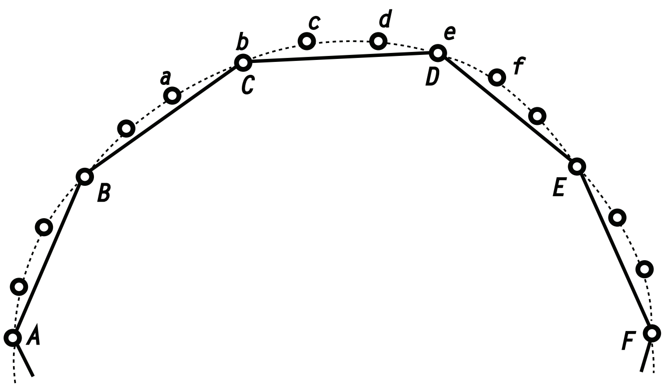

Figure 4.3: Configuration around mid-point.

We need only three vertices on either side of the mid-point, because the support tells us that the vertices lying further than this have no effect at the point we wish to analyze. We have from given ternary scheme

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix}.$$

The eigenvalues for this matrix are $1, \frac{1}{3}, \frac{1}{9}, w, -\frac{1}{18} + \frac{1}{6}w, -\frac{1}{18} + \frac{1}{6}w$.

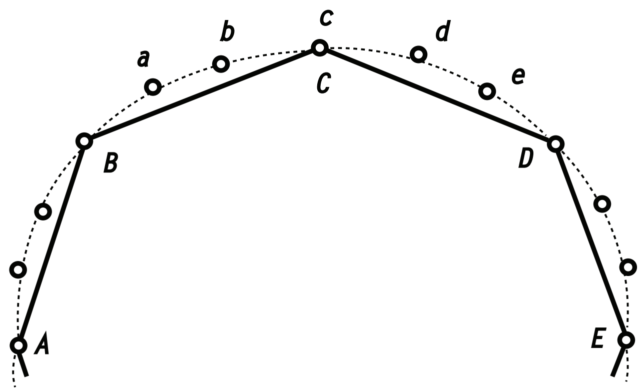

Figure 4.4: Configuration around vertex.

For the vertex subdivision matrix, we need only the two vertices on each side of the vertex. We have

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ 0 & a_3 & a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix}.$$

The eigenvalues are $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{18} - \frac{1}{2}w, \frac{1}{6} - \frac{5}{6}w$.
The mid-point subdivision matrix satisfies the necessary conditions for C^2 if and only if

$$|w| < \frac{1}{9}.$$

It is easy to show that for this range of w, we have $|\frac{1}{6} - \frac{5}{6}w| > |\frac{1}{18} - \frac{1}{2}w|$. And we see that the necessary conditions for C^2 are satisfied by the vertex subdivision if and only if

$$\left|\frac{1}{6} - \frac{5}{6}w\right| < \frac{1}{9}.$$

Both satisfied if and only if

$$\frac{1}{15} < w < \frac{1}{9}.$$

From the given ternary scheme we can see immediately that

$$\sum_{j\in\mathbb{Z}} a_{3j} = \sum_{j\in\mathbb{Z}} a_{3j+1} = \sum_{j\in\mathbb{Z}} a_{3j+2} = 1. \tag{4.3}$$

We have

$$\mathbf{a} = \frac{1}{18}[\quad \ldots, 0, 0, 3w-1, -3w-1, 0, -9w+7, 9w+13, 18,$$
$$9w+13, -9w+7, 0, -3w-1, 3w-1, 0, \ldots].$$

$$\mathbf{a_1} = \frac{1}{6}[\quad \ldots, 0, 0, 3w-1, -6w, 3w+1, -6w+6, 12w+6,$$
$$-6w+6, 3w+1, -6w, 3w-1, 0, \ldots].$$

$$\mathbf{a_2} = \frac{1}{2}[\quad \ldots, 0, 0, 3w-1, -9w+1, 9w+1, -6w+4,$$
$$9w+1, -9w+1, 3w-1, 0, \ldots].$$

$$\mathbf{a_3} = \frac{3}{2}[\ldots, 0, 0, 3w-1, -12w+2, 18w, -12w+2, 3w-1, 0, \ldots].$$

Exercise 17 *Verify the last equation.*

For any given ternary subdivision scheme S, we can prove $S^\infty P^0 \in C^k$ by deriving the mask of $\frac{1}{3}S_{k+1}$ and then computing $\|(\frac{1}{3}S_{k+1})^i\|_\infty$ for $i = 1, 2, \ldots, L$, where L is the first integer for which $\|(\frac{1}{3}S_{k+1})^L\|_\infty < 1$. If such an L exists, $S^\infty P^0 \in C^k$. It is easy to verify that for $\frac{1}{15} < w < \frac{1}{9}$, we have

$$\begin{aligned}\left\|\frac{1}{3}S_1\right\|_\infty &= \frac{4w+1}{3} < 1,\\ \left\|\frac{1}{3}S_2\right\|_\infty &= -2w+1 < 1,\end{aligned}$$

$$\left\|\frac{1}{3}S_3\right\|_\infty = \max\left\{9w, \frac{-15w+3}{2}\right\} < 1. \tag{4.4}$$

Hence all the sufficient conditions are satisfied for this scheme to be C^2.

Exercise 18

- *Verify the last statement.*
- *Verify $a(z), a_1(z) = \frac{3z^2 a(z)}{1+z+z^2}$ and $a_2(z)$ satisfy (4.3).*
- *Prove that this scheme is exact for quadratic not cubic polynomial.*
- *Find the approximation order.*
- *Read the paper [16]*

If we have 3 points, p_0, p_1 and p_2, we can get a quadratic passing through them, as follows:

$$P(t) = \frac{t}{2}(t-1)p_0 + (1-t^2)p_1 + \frac{t}{2}(t+1)p_2,$$

such that

$$P(-1) = p_0, \quad P(0) = p_1, \quad P(1) = p_2.$$

If we define

$$p_3 = P(2) = p_0 - 3p_1 + 3p_2$$

we can get the new vertices from this 4 vertices through the ternary 4-point interpolating scheme

$$\begin{aligned} p_1^1 &= \left(-\frac{1}{18}-\frac{1}{6}w\right)p_0+\left(\frac{13}{18}+\frac{1}{2}w\right)p_1+\left(\frac{7}{18}-\frac{1}{2}w\right)p_2+\left(-\frac{1}{18}+\frac{1}{6}w\right)p_3 \\ &= -\frac{1}{9}p_0+\frac{8}{9}p_1+\frac{2}{9}p_2 = P\left(\frac{1}{3}\right). \end{aligned}$$

$$\begin{aligned} p_2^1 &= \left(-\frac{1}{18}+\frac{1}{6}w\right)p_0+\left(\frac{7}{18}-\frac{1}{2}w\right)p_1+\left(\frac{13}{18}+\frac{1}{2}w\right)p_2+\left(-\frac{1}{18}-\frac{1}{6}w\right)p_3 \\ &= -\frac{1}{9}p_0+\frac{5}{9}p_1+\frac{5}{9}p_2 = P\left(\frac{2}{3}\right). \end{aligned}$$

The new vertices lie on the original quadratic. This means that if we define a set of vertices $\{p_j\}$, where $p_j = P(j), i \in \mathbb{Z}$, all the new vertices generated by this scheme also lie on this quadratic. Hence the precision set of this scheme is the quadratic. The approximation order of this scheme can be obtained directly from its precision. This scheme has approximation order 3.

We have presented 4-point C^2 ternary scheme. It is open problem to investigate whether we can keep increasing the number of new points introduced in each subdivision step to achieve even greater smoothness. i.e., **whether a quinary 4-point scheme can yield a C^3 curve**.

4.7.1 Hölder Continuity

First we selected 20 values for w equally distributed within the range for which the scheme is $C^2(\frac{1}{15} < w < \frac{1}{9})$. For each of these values we calculated ν_k such that

$$3^{-k\nu_k} = \left\| \left(\frac{1}{3}S_3\right)^k \right\|_\infty ,$$

for $k = 1, 2, \ldots, 20$. We found that

$$\left\| \left(\frac{1}{3}S_3\right)^k \right\|_\infty = \left(\|(\frac{1}{3}S_3)\|_\infty \right)^k ,$$

for all w and k. This yields

$$3^{-\nu_k} = \left\| \left(\frac{1}{3}S_3\right) \right\|_\infty .$$

Recalling (4.4), we can get the Hölder continuity (C^r) for this scheme

$$r = \begin{cases} 2-\log_3(\frac{1}{2}(3-15w)), & -\frac{1}{15} < w \leq \frac{1}{11} \\ 2-\log_3(9w), & \frac{1}{11} \leq w < \frac{1}{9}. \end{cases}$$

For $w = \frac{1}{11}$ the scheme is $C^{2.18}$.

Support: How far does the effect of one control point stretch?

This is answered simply and directly by looking at the sequence of influence coefficients. Suppose that the furthest non-zero coefficient is distance p spans from the center of the sequence, where a span is the distance from one original control point to the next. That point is obviously influenced. After second iteration we find that a point p/a further on (where a is arity) is also influenced, and successive iterations push the effect out by $p/a^2, p/a^3, \ldots$. This geometric sequence can be summed to give the half-width of the support region as $pa/(a-1)$.

Theorem 22 ([13]) *Let S be a uniformly convergent ternary subdivision scheme given by the compactly supported mask $\mathbf{a}$, with $supp(\mathbf{a}) = \{j : a \leq j \leq b, \quad j \in \mathbb{Z}\}$. Then $S^{\infty}\delta$ is a compactly supported function with*

$$supp(S^{\infty}\delta) = \left[\frac{a}{2}, \frac{b}{2}\right].$$

Exercise 19 *Show that for any binary subdivision scheme the support of the basic limit function is the convex hull of the support of the mask.*

Exercise 20 *Find the support of 4-point DD scheme, 4-point ternary scheme.*

Exercise 21 *Here we present a ternary 4-point stationary subdivision scheme.*

$$\begin{aligned}
f_{3i}^{k+1} &= -\frac{55}{1296}f_{i-1}^{k} + \frac{385}{432}f_{i}^{k} + \frac{77}{432}f_{i+1}^{k} - \frac{35}{1296}f_{i+2}^{k}, \\
f_{3i+1}^{k+1} &= -\frac{1}{16}f_{i-1}^{k} + \frac{9}{16}f_{i}^{k} + \frac{9}{16}f_{i+1}^{k} - \frac{1}{16}f_{i+2}^{k}, \\
f_{3i+2}^{k+1} &= -\frac{35}{1296}f_{i-1}^{k} + \frac{77}{432}f_{i}^{k} + \frac{385}{432}f_{i+1}^{k} - \frac{55}{1296}f_{i+2}^{k},
\end{aligned}$$

Find the masks of this scheme by evaluation at $1/6, 3/6$ and $5/6$ on local cubic interpolation. And investigate the convergence and smoothness of this scheme.

4.8 Ternary 4-point Approximating Scheme

The support of scheme influence the locality and polynomial reproducing property determines the approximation order of the subdivision scheme. We found that a higher regularity does not guarantee a higher approximation order. For mathematical theory, approximation order is a more important property than the support size. But for CAD, support is more important concept. Our objective is to find a improved scheme which has smaller support and a higher smoothness. We choose a ternary scheme because the best way to get a smaller support is to raise arity and use polynomial reproducing to get higher approximation order.

Scheme	Approximation order	Support(size)	C^n
DD(2,4)	4	6	1
DD(2,6)	6	10	2
ternary 3-point	2	4	1
ternary 4-point	3	5	2
our scheme	4	5.5	2

Table 4.6: Comparison of DD and ternary schemes

4.8.1 Construction of Scheme

Our primary concern in the work is to construct a uniform subdivision scheme with higher approximation order and smaller support. To this end, first, we choose a ternary scheme instead of a binary scheme in order to construct a scheme with smaller support size. Secondly, the approximation order can be guaranteed by the polynomial reproducing property (see Theorem 4.2 below). To obtain a uniform subdivision scheme, we derive the mask of this scheme by evaluation at $1/6, 3/6$ and $5/6$ on local cubic interpolation.

In our argument, the Lagrange polynomials play a crucial role. Let $\{L_i(x)\}_{i=-1}^{2}$ be the fundamental Lagrange polynomials to the node points $\{-1, 0, 1, 2\}$ given by

$$L_{-1}(x) = -\frac{x(x-1)(x-2)}{6}, \quad L_0(x) = \frac{(x+1)(x-1)(x-2)}{2},$$

and

$$L_1(x) = -\frac{x(x+1)(x-2)}{2}, \quad L_2(x) = \frac{x(x+1)(x-1)}{6}.$$

The Lagrange polynomials reproduce any cubic polynomial p in the way that

$$p(x) = \sum_{\alpha=-1}^{2} p(\alpha) L_\alpha(x). \tag{4.5}$$

Now we construct the desired 4-point ternary subdivision scheme. We sample the data $(j, f_j), j = i-1, i, i+1, i+2$ from an arbitrarily given cubic polynomial p_3;

$$p_3(j) = f_j, \quad j = i-1, i, i+1, i+2,$$

and request

$$f^1_{3i} = p_3\left(i + \frac{1}{6}\right), \quad f^1_{1+3i} = p_3\left(i + \frac{1}{2}\right), \quad f^1_{2+3i} = p_3\left(i + \frac{5}{6}\right).$$

Since our scheme is stationary, uniform and the space of polynomials up to a fixed degree are shift invariant, it is sufficient to consider the case $k = 0$ and $i = 0$, that is, the cubic polynomial such that $p_3(j) = f_j$ for $j = -1, 0, 1, 2$. Using the Lagrange interpolation property, we have

$$\begin{aligned}
p_3(1/6) &= L_{-1}(1/6)f_{-1} + L_0(1/6)f_0 + L_1(1/6)f_1 + L_2(1/6)f_2, \\
p_3(1/2) &= L_{-1}(1/2)f_{-1} + L_0(1/2)f_0 + L_1(1/2)f_1 + L_2(1/2)f_2, \\
p_3(5/6) &= L_{-1}(5/6)f_{-1} + L_0(5/6)f_0 + L_1(5/6)f_1 + L_2(5/6)f_2.
\end{aligned}$$

For the initial values $x_i^0 = i, i \in \mathbb{Z}$, we can see that the values x_{3i+j}^{k+1} given by

$$x_{3i+j}^{k+1} := \frac{1}{4}\left(1 - \frac{1}{3^{k+1}}\right) + \frac{3i+j}{3^{k+1}}, \quad j = 0, 1, 2, \tag{4.6}$$

are obtained recursively from the subdivision rule,

$$\sum_{\alpha=-1}^{2} L_\alpha\left(\frac{1+2j}{6}\right)\left[\frac{1}{4}(1 - \frac{1}{3^k}) + \frac{\alpha+i}{3^k}\right] = \frac{1}{4}\left(1 - \frac{1}{3^{k+1}}\right) + \frac{3i+j}{3^{k+1}} = x_{3i+j}^{k+1}.$$

Now, as an affine combination of 4 points $f_{i-1}^k, f_i^k, f_{i+1}^k, f_{i+2}^k$, we suppose the $(k+1)$st level points f_{3i+j}^{k+1} to be attached to the values x_{3i+j}^{k+1} instead of being attached to the points $\frac{3i+j}{3^{k+1}}$.

Using the Lagrange polynomials, we propose a 4-point approximating ternary subdivision scheme as

$$f_{3i+j}^{k+1} = \sum_{\alpha=-1}^{2} L_\alpha\left(\frac{1+2j}{6}\right) f_{i+\alpha}^k, \quad j = 0, 1, 2. \tag{4.7}$$

Here, we present the desired ternary 4-point approximating subdivision scheme:

$$\begin{aligned}
f_{3i}^{k+1} &= -\frac{55}{1296} f_{i-1}^k + \frac{385}{432} f_i^k + \frac{77}{432} f_{i+1}^k - \frac{35}{1296} f_{i+2}^k, \\
f_{3i+1}^{k+1} &= -\frac{1}{16} f_{i-1}^k + \frac{9}{16} f_i^k + \frac{9}{16} f_{i+1}^k - \frac{1}{16} f_{i+2}^k, \\
f_{3i+2}^{k+1} &= -\frac{35}{1296} f_{i-1}^k + \frac{77}{432} f_i^k + \frac{385}{432} f_{i+1}^k - \frac{55}{1296} f_{i+2}^k.
\end{aligned}$$

i	$\cdots$	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	$\cdots$
a_i	0	$-\frac{35}{1296}$	$-\frac{1}{16}$	$-\frac{55}{1296}$	$\frac{77}{432}$	$\frac{9}{16}$	$\frac{385}{432}$	$\frac{385}{432}$	$\frac{9}{16}$	$\frac{77}{432}$	$-\frac{55}{1296}$	$-\frac{1}{16}$	$-\frac{35}{1296}$	0

Table 1. Mask of the proposed ternary 4-point approximating scheme.

To obtain the scheme, we borrowed the idea of the derive of the corner-cutting subdivision scheme. And as preparing this work, we became aware that using the similar idea, Dyn, Floaster, and Hormann [12] obtained the binary four point scheme reproducing all the cubic polynomials.

Now, we need to check if the proposed scheme reproduces all the cubic polynomials, indeed.

Lemma 1 *The subdivision scheme reproduces all the cubic polynomials.*

Proof Let p be a polynomials of degree ≤ 3. Assume that the data f_ℓ^k are sampled from $p(x_\ell^k)$ for the given values x_ℓ^k as in (4.6). Using the Lagrange interpolation property (4.5), we obtain

$$
\begin{aligned}
f_{3i+j}^{k+1} &= \sum_{\alpha=-1}^{2} L_\alpha\left(\frac{1+2j}{6}\right) f_{i+\alpha}^{k} \\
&= \sum_{\alpha=-1}^{2} L_\alpha\left(\frac{1+2j}{6}\right) p\left(\frac{1}{4}(1-\frac{1}{3^k})+\frac{i+\alpha}{3^k}\right) \\
&= p\left(\frac{1}{4}(1-\frac{1}{3^k})+\frac{i}{3^k}+\frac{1}{3^k}\frac{1+2j}{6}\right) \\
&= p(x_{3i+j}^{k+1}),
\end{aligned}
$$

which shows the lemma. □

With the same way, we can obtain the mask of a ternary $(2n+2)$ point approximation schemes by local interpolating polynomial p_{2n+1} using Lagrange interpolation polynomials $\{L_k(x)\}_{k=-n}^{n+1}$ defined by

$$L_k(x) = \prod_{j\neq k, j=-n}^{n+1} \frac{x-j}{k-j}, \quad k=-n,\cdots,n+1, \tag{4.8}$$

for which

$$L_k(j) = \delta_{k,j}, \quad k,j=-n,\cdots,n+1, \tag{4.9}$$

and

$$\sum_{k=-n}^{n+1} p(k)L_k(x) = p(x), \quad p \in P_{2n+1}. \tag{4.10}$$

Here, P_{2n+1} denotes the space of all polynomials of degree $\leq 2n+1$ for a nonnegative integer n.

We can generalize the problem of finding a mask $\mathbf{a} = \{a_i\}$ reproducing polynomials of degree $\leq 2n+1$, that is, we can find a $(2n+2)$-point ternary scheme reproducing all polynomials p of degree $\leq 2n+1$ by solving the linear equations,

$$
\begin{aligned}
\sum_k a_{3k}p(k) &= p(\frac{1}{6}), \quad k \in \mathbb{Z}, \\
\sum_k a_{1+3k}p(k) &= p(\frac{1}{2}), \quad k \in \mathbb{Z}, \\
\sum_k a_{2+3k}p(k) &= p(\frac{5}{6}), \quad k \in \mathbb{Z}.
\end{aligned}
$$

4.8.2 Analysis of the subdivision Scheme

In this section, we analyze the smoothness of the proposed 4-point scheme S with the mask $\mathbf{a}$ given in Table 1. As mentioned in the introduction, our refinement rule is defined for an initial data $f^0 = \{f_i^0\}_{i\in\mathbb{Z}}$ by

$$\begin{pmatrix} x_i^k \\ f_i^k \end{pmatrix} = \sum_{j\in\mathbb{Z}} a_{i-3j} \begin{pmatrix} x_j^{k-1} \\ f_j^{k-1} \end{pmatrix}, \quad i \in \mathbb{Z} \quad (x_i^0 = i) \tag{4.11}$$

and the control points f_i^k are attached to the parameter values x_i^k ((a) of Figure1), not to the values $\frac{i}{3^k}$, as an usual rule ((a) of Figure 1).

In general schemes, unlike our scheme, the control points $f_i^k = (S^k f^0)_i$ are attached to the parameter values $\frac{i}{3^k}$. And the analysis of convergence and smoothness for a subdivision scheme has been developed in this setting. However, the following theorem shows that the convergence of any of the rules induces that of the other.

Theorem 23 *Let S be the proposed 4-point ternary subdivision scheme with the mask $\mathbf{a} = \{a_i\}_{i\in\mathbb{Z}}$ given in Table 1. For each $k \geq 0$, let $\{(S^k\delta)_i\}_{i\in\mathbb{Z}}$ be the k-th level points given by*

$$(S^k\delta)_i = \sum_{j\in\mathbb{Z}} a_{i-3j}(S^{k-1}\delta)_j$$

for the initial control points $\delta = \{\delta_{i,0}\}_{i\in\mathbb{Z}}$ and let $\{x_i^k\}_{i\in\mathbb{Z}}$ be the parameter values given by

$$x_i^k = \sum_{j\in\mathbb{Z}} a_{i-3j}x_j^{k-1} = \frac{1}{4}(1-\frac{1}{3^k}) + \frac{i}{3^k}, \quad i \in \mathbb{Z} \quad (x_i^0 = i).$$

Then the two statements are equivalent:

(i) There is a continuous function ϕ on $\mathbb{R}$ such that

$$\lim_{k\to\infty} \sup_{i\in\mathbb{Z}} |(S^k\delta)_i - \phi(x_i^k)| = 0; \tag{4.12}$$

(ii) There is a continuous function ψ on $\mathbb{R}$ such that

$$\lim_{k\to\infty} \sup_{i\in\mathbb{Z}} |(S^k\delta)_i - \psi(\frac{i}{3^k})| = 0.$$

In this case, $\psi = \phi(\cdot + \frac{1}{4})$ and ϕ has a compact support set.

Proof The equivalence is straightforward and we show only the implement of (i) to (ii). We assume that the subdivision scheme S converges uniformly. Then there is a continuous function ϕ satisfying (4.12). From Theorem 29 in section 4, ϕ has a compact support. Let $\psi = \phi(\cdot + \frac{1}{4})$. Then for an arbitrarily given $\epsilon > 0$, the assumption implies the existence of an integer $N_\epsilon > 0$ such that for any $k \geq N_\epsilon$,

$$\sup_{i\in\mathbb{Z}} |(S^k\delta)_i - \psi(\frac{i}{3^k} - \frac{1}{4\cdot 3^k})| \leq \epsilon.$$

On the other hand, we can see that ψ is uniformly continuous on $\mathbb{R}$ for ψ has a compact support. Thus, there is an integer $N_\psi > 0$ such that for any $k \geq N_\psi$,

$$\sup_{x\in\mathbb{R}} |\psi(x) - \psi(x - \frac{1}{4\cdot 3^k})| \leq \epsilon.$$

Combining these two estimates, we have that for any $k \geq N := \max(N_\epsilon, N_\psi)$,

$$\sup_{i\in\mathbb{Z}} |(S^k\delta)_i - \psi(\frac{i}{3^k})| \leq 2\epsilon,$$

which shows the statement (ii). The rest argument follows directly from the uniqueness of the limit of a convergent sequence. □

Due to Theorem 23, we may use well-known sufficient conditions to analyze the convergence and smoothness of our scheme. For each scheme S with a mask $\mathbf{a}$, we define the Laurent polynomial as the symbol of a mask $\mathbf{a}$

$$a(z) := \sum_{i\in\mathbb{Z}} a_i z^i.$$

From the refinement rule of S (3 refinement rules),

$$f_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-3j} f_j^k, \quad i \in \mathbb{Z},$$

we may regard S as a operator of $\ell^\infty(\mathbb{Z})$ into itself and we have an estimate

$$\|f_i^{k+1}\|_\infty \leq \left(\sum_j |a_{i-3j}|\right) \max_j \|f_j^k\|.$$

Then we can calculate the norm of S:

$$\|S\|_\infty = \max\left\{\sum_j |a_{3j}|, \sum_j |a_{1+3j}|, \sum_j |a_{2+3j}|\right\}.$$

We define the generating functions of control points f^k as

$$F^k(z) = \sum_i f_i^k z^i.$$

Since the coefficient of z^i in $F^{k+1}(z)$ is f_i^{k+1} and the coefficient of z^i in $a(z)F^k(z^3)$ is $\sum_j a_{i-3j} f_j^k$, we have

$$F^{k+1}(z) = a(z)F^k(z^3).$$

Let $a^{[L]}(z) = \prod_{j=0}^{L-1} a(z^{3^j}) = \sum_i a_i^{[L]} z^i$. From the relation $F^{k+L}(z) = a^{[L]}(z)F^k(z^{3^L})$, we have the 3^L refinement rules and the norm of S^L:

$$F_i^{k+L} = \sum_j a_{i-3^L j}^{[L]} F_j^k,$$

and the norm of S^L is given by

$$\|S^L\|_\infty = \max\left\{\sum_j |a_{i-3^L j}^{[L]}|, i = 0, 1, \ldots, 3^L - 1\right\}.$$

From Theorem 23, we have the following theorems which play essential roles to analyze the convergence and smoothness of a subdivision scheme.

Theorem 24 *Let S be a convergent ternary subdivision scheme, with a mask* $\mathbf{a}$. *Then*

$$\sum_j a_{3j} = \sum_j a_{3j+1} = \sum_j a_{3j+2} = 1. \tag{4.13}$$

Proof Combining Theorem 1 in [27] and Theorem 23, we have the theorem. □

Applying the polynomial reproduction property (4.5) to the subdivision rule (4.7), the mask of the proposed scheme satisfies the condition (4.13). The symbol of a convergent ternary subdivision scheme satisfies,

$$a(e^{2i\pi/3}) = a(e^{4i\pi/3}) = 0 \text{ and } a(1) = 3,$$

and there exists the Laurent polynomial $a_1(z)$ such that

$$a_1(z) = \frac{3z^2}{(1+z+z^2)} a(z).$$

Then the subdivision S_1 with symbol $a_1(z)$ is related to S with symbol $a(z)$ by the following theorem.

Theorem 25 *([16]) Let S denote a ternary subdivision scheme with symbol $a(z)$ satisfying (4.13). Then there exists a subdivision scheme S_1 with the property*

$$df^k = S_1 df^{k-1},$$

where $f^k = S^k f^0 = \{f_i^k : i \in \mathbb{Z}\}$ and $df^k = \{(df^k)_i = 3^k(f_{i+1}^k - f_i^k) : i \in \mathbb{Z}\}$.

Using the subdivision scheme S_1, we can check the convergence of S as follows:

Theorem 26 *S is a uniformly convergent ternary subdivision scheme if and only if $\frac{1}{3}S_1$ converges uniformly to the zero function for all initial data f^0.*

$$\lim_{k\to\infty} \left(\frac{1}{3}S_1\right)^k f^0 = 0. \tag{4.14}$$

Proof It follows from Theorem 4.2 in [16] and Theorem 23. □

A scheme S_1 satisfying (4.14) for all initial data f^0 is termed contractive. Theorem 26 indicates that checking of the convergence of S is equivalent to checking whether S_1 is contractive, which is equivalent to checking whether $||(\frac{1}{3}S_1)^L||_\infty < 1$, for some integer $L > 0$. After the convergence of S is determined, we need to check the smoothness of the limit functions generated by S. An condition of C^m continuity is expressed in the following theorem.

Theorem 27 *Let us consider a scheme S with Laurent polynomial $a(z)$. If there exists a polynomial $b(z)$ such that*

$$a(z) = \left(\frac{1+z+z^2}{3z^2}\right)^m b(z),$$

and such that the associated scheme $\frac{1}{3}S_b$ is contractive, then the limit function is C^m for any initial data.

Proof It follows from Theorem 4.4 in [16] and Theorem 23. □

Now, we are ready to analyze the smoothness of the proposed scheme. For the given ternary mask:

$$\mathbf{a} = \frac{1}{1296}[-35, -81, -55, 231, 729, 1155, 1155, 729, 231, -55, -81, -35],$$

we have the mask of scheme S_1:

$$\mathbf{a_1} = \frac{3}{1296}[-35, -46, 26, 251, 452, 452, 251, 26, -46, -35],$$

where $a_1(z) = \frac{3z^2 a(z)}{1+z+z^2}$. It is easy to verify that $a(z)$ and $a_1(z)$ satisfy the necessary condition (4.13) for the convergence S and S_1. Since

$$\left\|\frac{1}{3}S_1\right\|_\infty = \max\left\{\frac{524}{1296}, \frac{572}{1296}\right\} < 1,$$

this scheme converges uniformly. We have the mask of S_2

$$\mathbf{a_2} = \frac{9}{1296}[-35, -11, 72, 190, 190, 72, -11, -35],$$

and

$$\left\|\frac{1}{3}S_2\right\|_\infty = \max\left\{\frac{708}{1296}, \frac{432}{1296}\right\} < 1.$$

Hence this scheme has $C^1(\mathbb{R})$. We can verify that $a_2(z)$ satisfies the necessary condition for $C^2(\mathbb{R})$. And the mask of S_3 is

$$\mathbf{a_3} = \frac{27}{1296}[-35, 24, 83, 83, 24, -35],$$

and we get

$$\left\|\frac{1}{3}S_3\right\|_\infty = \max\left\{\frac{1062}{1296}, \frac{432}{1296}\right\} < 1.$$

Hence this scheme is $C^2(\mathbb{R})$. The mask of S_4 is

$$\mathbf{a_4} = \frac{81}{1296}[-35, 59, 59, -35],$$

and we have

$$\left\|\frac{1}{3}S_4\right\|_\infty = \max\left\{\frac{1890}{1296}, \frac{1593}{1296}\right\} > 1.$$

Actually, there exists no integer $L > 0$ such that

$$\left\|(\frac{1}{3}S_4)^L\right\|_\infty < 1,$$

therefore this scheme can not generate $C^3(\mathbb{R})$ functions.

4.8.3 Approximation Order and Support

While the regularity of the limit function for the subdivision scheme is important, another an important issue of subdivision scheme is how to attain the original function as close as possible if a given initial data f^0 is sampled from an underlying function.

Definition 4 *Let us consider the initial grid $X_0 = h\mathbb{Z}$ and initial data $f_i^0 = g(ih)$ sampled a enough smooth function g. Let us denote by f^∞ the limit function obtained through subdivision. The subdivision scheme has approximation order p if*

$$|(g - f^\infty)(x)| \leq Ch^p, \quad x \in \mathbb{R}$$

where C is a real constant and independent of h.

As seen in Theorem 4.2 below, the approximation order of a subdivision scheme can be obtained from its precision set.

Theorem 28 ([12]) *An convergent subdivision scheme that reproduces polynomial P_n has an approximation order of $n + 1$.*

From Lemma 1 and Theorem 28, the proposed scheme has approximation order 4.
Next, we consider the support of the proposed scheme. This is the support of the basic limit function $\phi = S^\infty\delta$ generated by the given control point $f_i^0 = \delta_{i,0}$ as shown in Figure 2.

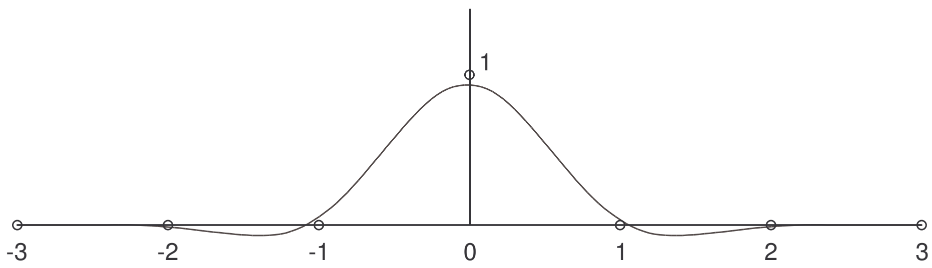

Figure 2. The basic limit function of the proposed scheme.

Theorem 29 *Let S be the proposed 4-point ternary subdivision scheme with a mask* **a** *given in Table 1. Then we have*

$$supp(\phi) = supp(S^\infty\delta) = \left[-\frac{11}{4}, \frac{11}{4}\right].$$

Proof Choose $f^0 = \{f_\alpha^0 : f_\alpha^0 = \delta_{\alpha,0},\ \alpha \in \mathbb{Z}\}$, and let $S^\infty\delta = \phi$. From the subdivision rule

$$(S^k\delta)_i = \sum_{j\in\mathbb{Z}} a_{i-3j}(S^{k-1}\delta)_j,$$

we have that $\mathrm{supp}(S\delta) = \mathrm{supp}(\mathbf{a}) = [-6, 5]$ and for each $k = 2, 3, \ldots,$

$$\begin{aligned} \mathrm{supp}(S^k\delta) &= \{i \in \mathbb{Z} : i - 3j \in \mathrm{supp}(\mathbf{a}),\ j \in \mathrm{supp}(S^k\delta)\} \\ &= \{i \in \mathbb{Z} : i \in \mathrm{supp}(\mathbf{a}) + 3\mathrm{supp}(S^k\delta)\}. \end{aligned}$$

Thus, $\mathrm{supp}(S^k\delta) = \frac{3^k-1}{2}\mathrm{supp}(\mathbf{a})$. The values $S^k\delta$ are attached to the parameter values

$$\frac{1}{4}(1 - \frac{1}{3^k}) + \frac{1}{3^k}\mathrm{supp}(S^k\delta) = \frac{1}{4}(1 - \frac{1}{3^k}) + \frac{1 - 3^{-k}}{2}\mathrm{supp}(S^k\delta).$$

Hence, the support of the limit function ϕ is

$$\mathrm{supp}(\phi) = \mathrm{supp}(S^\infty\delta) = \left[-\frac{11}{4}, \frac{11}{4}\right],$$

which completes the proof. $\square$

The support of the proposed subdivision scheme is smaller than the support $[-3, 3]$ of the Dyn 4-point binary subdivision scheme.

Scheme	Approximation order	Support(size)	C^n
binary 4-point	4	6	1
binary 6-point	6	10	2
ternary 3-point	2	4	1
ternary 4-point	3	5	2
our scheme	4	5.5	2

Table 2. Comparison of the proposed scheme to binary 4-point and 6-point, and ternary 3-point and 4-point schemes.

In this table above, the masks of binary 4-point and 6-point schemes are given by

$$\frac{1}{16}[-1, 0, 9, 16, 9, 0, -1],$$

and

$$\frac{1}{256}[3, 0, -25, 0, 150, 256, 150, 0, -25, 0, 3],$$

respectively. And the ternary 3-point and 4-point schemes are related to tension parameters a and μ, respectively. Here we choose the tension parameters to generate the highest smoothness ($a = -\frac{1}{15}$ and $\mu = \frac{1}{11}$). In this case, the masks of ternary 3-point and 4-point schemes are given by

$$\frac{1}{15}[-1, 0, 4, 12, 15, 12, 4, 0, -1],$$

and

$$\frac{1}{99}[-4, -7, 0, 34, 76, 99, 76, 34, 0, -7, -4],$$

respectively.

We illustrate the proposed scheme by applying to the control points forming the cross polygon in Figure 3. In the figure, the curve interpolating the control points is generated by the 4-point DD scheme and the other is created by the proposed subdivision scheme.

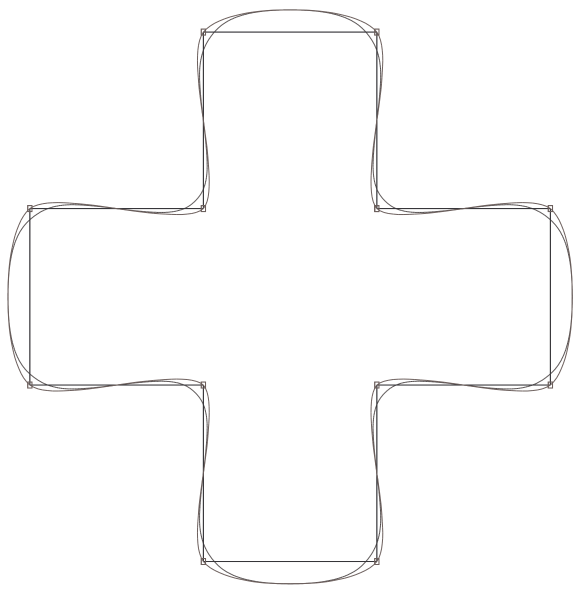

Figure 3. Comparison of the binary 4-point interpolating DD scheme and the proposed 4-point ternary approximating scheme.

4.8.4 Limit Value and Curvature of Limit Function

The limit behavior of a subdivision scheme can be analyzed by examining the eigen structure of the of subdivision matrix S. If there is $n+1$ linearly independent eigenvectors v_i corresponding eigenvalues λ_i, then we can diagonalize S by transforming S by the eigenvectors and their inverse.

$$S = V\Lambda V^{-1}.$$

If the subdivision curve is C^0, then we have limit value

$$f^\infty = \lim_{k\to\infty} f^k = S^\infty f^0.$$

Since the subdivision scheme is affine invariant, we have one eigenvector consisting of all ones $v_0 = [1 \quad 1 \quad \cdots \quad 1]^t$ and corresponding eigenvalue equal to one $\lambda_0 = 1$. If $1 = \lambda_0 > \lambda_i$, using the eigen decomposition of S and since $1 > \lambda_i, i = 1, 2, \cdots, n$, we have

$$\begin{aligned}
S^\infty &= V\Lambda^\infty V^{-1} \\
&= \begin{bmatrix} 1 & v_1 & \cdots & v_n \\ 1 & \downarrow & & \downarrow \\ \cdots & & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} v_0^{-1} & \rightarrow \\ v_1^{-1} & \rightarrow \\ \cdots & \rightarrow \\ v_n^{-1} & \rightarrow \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} v_0^{-1} & \rightarrow \\ v_1^{-1} & \rightarrow \\ \cdots & \rightarrow \\ v_n^{-1} & \rightarrow \end{bmatrix} \\
&= \begin{bmatrix} v_0^{-1} & \rightarrow \\ v_0^{-1} & \rightarrow \\ \cdots & \rightarrow \\ v_0^{-1} & \rightarrow \end{bmatrix}.
\end{aligned}$$

From the property above, we have limit value of local neighborhood of given control point f^0.

$$f^\infty = S^\infty f^0 = \begin{bmatrix} v_0^{-1} & \rightarrow \\ v_0^{-1} & \rightarrow \\ \cdots & \rightarrow \\ v_0^{-1} & \rightarrow \end{bmatrix} f^0.$$

We can write down the subdivision matrix S for ternary four-point approximating scheme

$$S := \begin{bmatrix} -\frac{35}{1296} & \frac{77}{432} & \frac{385}{432} & -\frac{55}{1296} & 0 & 0 \\ 0 & -\frac{55}{1296} & \frac{385}{432} & \frac{77}{432} & -\frac{35}{1296} & 0 \\ 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 \\ 0 & -\frac{35}{1296} & \frac{77}{432} & \frac{385}{432} & -\frac{55}{1296} & 0 \\ 0 & 0 & -\frac{55}{1296} & \frac{385}{432} & \frac{77}{432} & -\frac{35}{1296} \\ 0 & 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} \end{bmatrix}.$$

The subdivision matrix can be written by

$$S = V\Lambda V^{-1},$$

where

$$V := \begin{bmatrix} 1 & -11 & 121 & -1331 & 1 & \frac{13663805089}{2902814} \\ 1 & -7 & 49 & -343 & 0 & \frac{40609429}{30881} \\ 1 & -3 & 9 & -27 & 0 & \frac{3709719}{30881} \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 5 & 25 & 125 & 0 & \frac{-9646170}{30881} \\ 1 & 9 & 81 & 729 & 0 & \frac{-365501646}{216167} \end{bmatrix},$$

$$V^{-1} := \begin{bmatrix} 0 & \frac{-1485}{39584} & \frac{42315}{158336} & \frac{131335}{158336} & \frac{-9639}{158336} & \frac{245}{158336} \\ 0 & \frac{93}{2984} & \frac{-39863}{143232} & \frac{10047}{47744} & \frac{1881}{47744} & \frac{-385}{143232} \\ 0 & \frac{-1}{544} & \frac{101}{2176} & \frac{-211}{2176} & \frac{135}{2176} & \frac{-21}{2176} \\ 0 & \frac{1}{176} & \frac{-107}{4224} & \frac{59}{1408} & \frac{-43}{1408} & \frac{35}{4224} \\ 1 & \frac{-341}{94} & \frac{212}{47} & \frac{-83}{47} & \frac{-23}{47} & \frac{35}{94} \\ 0 & \frac{216167}{86281987} & \frac{-864668}{86281987} & \frac{1297002}{86281987} & \frac{-864668}{86281987} & \frac{216167}{86281987} \end{bmatrix},$$

$$\Lambda := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{27} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-35}{1296} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{59}{1296} \end{bmatrix}.$$

Consider the limit basis function of the scheme when applied to the unit delta sequence $f^0 = \delta$

$$f^0 = \begin{bmatrix} -3 & 0 \\ -2 & 0 \\ -1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix}.$$

$$\begin{aligned} f^n &= S^n f^0 = (V\Lambda^n V^{-1}) f^0 \\ &= \begin{bmatrix} -\frac{1}{4} - \frac{11(\frac{1}{3})^n}{4} & \frac{131355}{158336} - \frac{110517(\frac{1}{3})^n}{47744} - \frac{25531(\frac{1}{9})^n}{2176} - \frac{7139(\frac{1}{27})^n}{128} - \frac{83(\frac{-35}{1296})^n}{47} + \frac{26085446079(\frac{59}{1296})^n}{368659399} \\ -\frac{1}{4} - \frac{7(\frac{1}{3})^n}{4} & \frac{131355}{158336} - \frac{70329(\frac{1}{3})^n}{47744} - \frac{10339(\frac{1}{9})^n}{2176} - \frac{20237(\frac{1}{27})^n}{1408} + \frac{1705596018(\frac{59}{1296})^n}{86281987} \\ -\frac{1}{4} - \frac{3(\frac{1}{3})^n}{4} & \frac{131355}{158336} - \frac{30141(\frac{1}{3})^n}{47744} - \frac{1899(\frac{1}{9})^n}{2176} - \frac{1593(\frac{1}{27})^n}{1408} + \frac{155808198(\frac{59}{1296})^n}{86281987} \\ -\frac{1}{4} + \frac{(\frac{1}{3})^n}{4} & \frac{131355}{158336} + \frac{10047(\frac{1}{3})^n}{47744} - \frac{211(\frac{1}{9})^n}{2176} + \frac{59(\frac{1}{27})^n}{1408} + \frac{1297002(\frac{59}{1296})^n}{86281987} \\ -\frac{1}{4} + \frac{5(\frac{1}{3})^n}{4} & \frac{131355}{158336} + \frac{50235(\frac{1}{3})^n}{47744} - \frac{5275(\frac{1}{9})^n}{2176} - \frac{7375(\frac{1}{27})^n}{1408} - \frac{405139140(\frac{59}{1296})^n}{86281987} \\ -\frac{1}{4} + \frac{9(\frac{1}{3})^n}{4} & \frac{131355}{158336} + \frac{90423(\frac{1}{3})^n}{47744} - \frac{17091(\frac{1}{9})^n}{2176} + \frac{43011(\frac{1}{27})^n}{1408} - \frac{2193009876(\frac{59}{1296})^n}{86281987} \end{bmatrix}. \end{aligned}$$

We have the limit value of the scheme for the given control point f^0.

$$S^\infty f^0 = \begin{bmatrix} -\frac{1}{4} & \frac{131355}{158336} \end{bmatrix}.$$

We can define an approximation to the discrete curvature at a vertex by calculating the circumcircle of the triangle formed by the vertex and its immediate neighbors. Calculating the curvature as the inverse of the radius of the circumcircle through the middle 3 points, we get

$$\frac{1}{r} =$$

and the curvature of the limit function at the vertex is given by

$$\lim_{n\to\infty} \frac{1}{r} =$$

4.9 Ternary 3-point Interpolating Scheme

Here we present ternary 3-point interpolating scheme.

$$\begin{aligned}
p_{3i}^{k+1} &= p_i^k, \\
p_{3i+1} &= ap_{i-1} + (1-a-b)p_i^k + bp_{i+1}^k, \\
p_{3i-1} &= bp_{i-1} + (1-a-b)p_i^k + ap_{i+1}^k,
\end{aligned}$$

where the weights are chosen such that the scheme is symmetric and the weights sum to unity for affine invariance. For this scheme we have mask:

$$\mathbf{a} = [\ldots, 0, 0, a, 0, b, 1-a-b, 1, 1-a-b, b, 0, a, 0, 0, \ldots], \tag{4.15}$$

and

$$\mathbf{a_1} = 3[\ldots, 0, 0, a, -a, b, 1-2b, b, -a, a, 0, 0, \ldots],$$

where $a_1(z) = \frac{3z^2 a(z)}{1+z+z^2}$. It is easy to verify that $a(z)$ satisfies the necessary condition (4.3) for the convergence of S. If

$$\left\|\frac{1}{3}S_1\right\|_\infty = \max\{|1-2b| + 2|a|, |a| + |b|, |a| + |b|\} < 1, \tag{4.16}$$

then this scheme converges to continuous limit function. For C^1 continuity, $a_1(z)$ should satisfy (4.3). This implies

$$b = a + \frac{1}{3}.$$

From this fact, we have the mask of S_1 and S_2

$$\mathbf{a_1} = 3[\ldots, 0, 0, b - \frac{1}{3}, \frac{1}{3} - b, b, 1-2b, b, \frac{1}{3} - b, b - \frac{1}{3}, 0, 0, \ldots],$$

and

$$\mathbf{a_2} = 9[\ldots, 0, 0, b - \frac{1}{3}, \frac{2}{3} - 2b, 2b - \frac{1}{3}, \frac{2}{3} - 2b, b - \frac{1}{3}, 0, 0, \ldots],$$

If

$$\left\|\frac{1}{3}S_2\right\|_\infty = \max\{9|b - \frac{1}{3}|, 9|b - \frac{1}{3}|, 3|2b - \frac{1}{3}|\} < 1, \tag{4.17}$$

then we have $C^1(\mathbb{R})$.
It is easy to check that

$$\frac{2}{9} < b < \frac{3}{9}, \quad a = b - \frac{1}{3},$$

satisfies (4.16) and (4.17).
From the necessary condition for $C^2(\mathbb{R})$-continuity, we have $b = \frac{2}{9}$. For $b = \frac{2}{9}$, we get

$$\mathbf{a_2} = [\ldots, 0, 0, -1, 2, 1, 2, -1, 0, 0, \ldots].$$

Exercise 22 • *Using the fact* $a_2^{[2]}(z) = a_2(z)a_2(z^3)$, *find the mask of* $a_2^{[2]}(z)$.

- *Calculate the norms*

$$\left\|\frac{1}{3}S_2\right\|_\infty, \quad \left\|\left(\frac{1}{3}S_2\right)^2\right\|_\infty.$$

Furthermore it can be shown that

$$\left\|\left(\frac{1}{3}S_2\right)^n\right\|_\infty \geq 1, \quad \forall n \in \mathbb{Z}_+.$$

Hence a C^1 ternary 3-point interpolating subdivision scheme can be defined with the mask (4.15), where $\frac{2}{9} < b < \frac{1}{3}$ and $a = b - \frac{1}{3}$.

Exercise 23 *Find the support of ternary 3-point interpolating scheme.*

Consider the configuration around mark point (vertex). We have from given ternary scheme

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} a & \frac{2}{3}-2a & a+\frac{1}{3} & 0 & 0 \\ 0 & a+\frac{1}{3} & \frac{2}{3}-2a & a & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & a & \frac{2}{3}-2a & a+\frac{1}{3} & 0 \\ 0 & 0 & a+\frac{1}{3} & \frac{2}{3}-2a & a \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix}.$$

The eigenvalues for this matrix are $1, \frac{1}{3}, 2a+\frac{1}{3}, a, a$.
For the mid-point, we have

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a & \frac{2}{3}-2a & a+\frac{1}{3} & 0 \\ 0 & a+\frac{1}{3} & \frac{2}{3}-2a & a \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}.$$

The eigenvalues are $1, \frac{1}{3}, -3a, -a$.

Exercise 24 *Find bounds on a which are necessary for C^1 using the eigen analysis.*

Suppose $\{p_i\}$ is a sequence of points lying at equally spaced parameter values on a straight line. Without loss of generality we can express straight line

$$P(t) = (1-t)p_0 + tp_1,$$

so that $P(i) = p_i$. After on subdivision step, we have

$$\begin{aligned} p^1_{3i} &= p_i, \\ p^1_{3i+1} &= ap_{i-1} + \left(\frac{2}{3} - 2a\right)p_i + \left(a + \frac{1}{3}\right)p_{i+1}, \\ p^1_{3i-1} &= \left(a + \frac{1}{3}\right)p_i + \left(\frac{2}{3} - 2a\right)p_{i+1} + ap_{i+2}. \end{aligned}$$

We can see that

$$\begin{aligned} p_{3i}^1 &= P(i), \\ p_{3i+1}^1 &= P\left(i+\frac{1}{3}\right), \\ p_{3i-1}^1 &= P\left(i-\frac{1}{3}\right). \end{aligned}$$

Exercise 25 *Verify the last equation.*

Hence the subdivided points lie on the original line. This is not for quadratic so this scheme has linear precision and has an approximation order 2.

4.9.1 Hölder Continuity

First we selected 20 values for a equally distributed within the range for which the scheme is $C^1(-\frac{1}{9} < a < 0)$. For each of these values we calculated ν_k such that

$$3^{-k\nu_k} = \left\| \left(\frac{1}{3}S_2\right)^k \right\|_\infty ,$$

for $k = 1, 2, \ldots, 20$. We found that

$$\left\| \left(\frac{1}{3}S_2\right)^k \right\|_\infty = \left(\|(\frac{1}{3}S_2)\|_\infty \right)^k ,$$

for all a and k. This yields

$$3^{-\nu_k} = \left\| \left(\frac{1}{3}S_2\right) \right\|_\infty .$$

Recalling (4.17), we can get the Hölder continuity (C^r) for this scheme

$$r = \begin{cases} 1 - \log_3(-9a), & -\frac{1}{9} < a \leq -\frac{1}{15} \\ 1 - \log_3(1+6a), & -\frac{1}{15} \leq a < 0. \end{cases}$$

For $a = -\frac{1}{15}$ the scheme is $C^{1.46}$.

4.10 Ternary 3-point Approximating Scheme

For this scheme we have mask:

$$\mathbf{a} = \frac{1}{27}[\ldots, 0, 0, 1, 4, 10, 16, 19, 16, 10, 4, 1, 0, 0, \ldots], \tag{4.18}$$

and

$$\mathbf{a_1} = \frac{1}{9}[\ldots, 0, 0, 1, 3, 6, 7, 6, 3, 1, 0, 0, \ldots].$$

It is easy to verify that $a(z)$ satisfies the necessary condition (4.3) for the convergence of S.

Exercise 26 *Find the mask of (4.18) by obtaining trisection for cubic spline by convolution.*

Since

$$\left\|\frac{1}{3}S_1\right\|_\infty = \max\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\} < 1,$$

this scheme converges to continuous limit function. It is easy to check that $a_1(z)$ satisfies (4.3). We have the mask of S_2

$$\mathbf{a_2} = \frac{1}{3}[\ldots, 0, 0, 1, 2, 3, 2, 1, 0, 0, \ldots],$$

and

$$\left\|\frac{1}{3}S_2\right\|_\infty = \max\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\} < 1.$$

Hence this scheme has $C^1(\mathbb{R})$. We can verify that $a_2(z)$ satisfies the necessary condition for $C^2(\mathbb{R})$. And the mask of S_3 is

$$\mathbf{a_3} = [\ldots, 0, 0, 1, 1, 1, 0, 0, \ldots],$$

and

$$\left\|\frac{1}{3}S_3\right\|_\infty = \max\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\} < 1.$$

Hence this scheme is $C^2(\mathbb{R})$.

Exercise 27 *Find the mask of S_4 and calculate the norm $\|\frac{1}{3}S_4\|_\infty$ and $\|(\frac{1}{3}S_4)^2\|_\infty$.*

4.11 Binary 3-point Approximating Scheme

For this scheme we have mask:

$$\mathbf{a} = [a, b, 1-a-b, 1-a-b, b, a],$$

and

$$\mathbf{a_1} = 2[\ldots, 0, 0, a, b-a, 1-2b, b-a, a, 0, 0, \ldots],$$

It is easy to verify that $a(z)$ satisfies the necessary condition (2.1) for the convergence of S. If

$$\left\|\frac{1}{2}S_1\right\|_\infty = \max\{|1-2b| + 2|a|, 2|b-a|\} < 1,$$

then this scheme converges to continuous limit function. For C^1 continuity, $a_1(z)$ should satisfy (2.1). This implies

$$b = a + \frac{1}{4}.$$

From this fact, we have the mask of S_2

$$\mathbf{a_2} = 4[\ldots, 0, 0, a, \frac{1}{4} - a, \frac{1}{4} - a, a, 0, 0, \ldots].$$

For C^2 continuity $a_2(z)$ should satisfy (2.1),which is true. And

$$\left\|\frac{1}{2}S_2\right\|_\infty = \max\{2|a| + 2|\frac{1}{4} - a|\} < 1.$$

We have

$$\mathbf{a_3} = 8[\ldots, 0, 0, a, \frac{1}{4} - 2a, a, 0, 0, \ldots],$$

and

$$\left\|\frac{1}{2}S_3\right\|_\infty = \max\{|8a|, |1 - 8a|\} < 1$$

which implies that $0 < a < \frac{1}{8}$.

For C^3 continuity, $a_3(z)$ should satisfy (2.1). This implies that $a = \frac{1}{16}$ and we have

$$\mathbf{a_4} = [\ldots, 0, 0, 1, 1, 0, 0, \ldots],$$

and

$$\left\|\frac{1}{2}S_4\right\|_\infty = \max\{1/2, 1/2\} < 1.$$

Thus binary 3-point approximating scheme with the mask $\mathbf{a} = \frac{1}{16}[1, 5, 10, 10, 5, 1]$ has C^3 continuity.

Exercise 28 *Verify that this scheme does not generate C^4 curve.*

4.12 Comparison

In this section we compare ternary schemes to the well-known binary schemes.

Table 4.7: Mask of binary and ternary schemes

Scheme	Mask
DD(2,4)	$\frac{1}{16}[\ldots, 16, 9, 0, -1]$
DD(3,4)	$\frac{1}{81}[\ldots, 81, 60, 30, 0, -5, -4]$
DD(2,6)	$\frac{1}{256}[\ldots, 256, 150, 0, -25, 0, 3]$
DD(3,6)	$\frac{1}{729}[\ldots, 729, 560, 280, 0, -70, -56, 0, 8, 7]$
ternary 3-point	$\frac{1}{15}[\ldots, 15, 12, 4, 0, -1]$
ternary 4-point	$\frac{1}{99}[\ldots, 99, 76, 34, 0, -7, -4]$

The masks of DD schemes, together with the masks of ternary schemes are shown in Table. The tension parameters of the ternary schemes have been chosen to give the highest smoothness, ($a = -\frac{1}{15}, w = \frac{1}{11}$ respectively). In Table, we compare of main properties (approximation order, support and regularity) of DD schemes and ternary schemes.

We found that a higher smoothness does not automatically equal to a higher approximation order/precision set. While DD schemes have better approximation order than ternary schemes, the main advantage of ternary schemes is their smaller support for a given continuity class.

Table 4.8: Comparison of DD and ternary schemes

Scheme	Approximation order	Support(size)	C^n
DD(2,4)	3	6	1
DD(3,4)	3	5	1
DD(2,6)	5	10	2
DD(3,6)	5	8	2
ternary 3-point	1	4	1
ternary 4-point	2	5	2

Exercise 29 *The DD b-ary $2N$-point scheme, denoted by DD(b,2N), derive their coefficients by fitting the highest degree polynomial through the points at nodal values $\{-N+1, -N+2, \ldots, N-1, N\}$ and calculating the coefficients at the nodal values $\{r/b\}$ where $r \in \{1, 2, \ldots, b-1\}$. Clearly this construction gives the highest possible approximation order/precision set. Calculate the mask of DD(2,4), DD(3,4),DD(2,6) and DD(3,6).*

Exercise 30 *For DD(3,4) verify that*

$$\begin{aligned}
\mathbf{a} &= \frac{1}{81}[0, -4, -5, 0, 30, 60, 81, 60, 30, 0, -5, -4, 0],\\
\mathbf{a_1} &= \frac{1}{27}[0, -4, -1, 5, 26, 29, 26, 5, -1, -4, 0],\\
\mathbf{a_2} &= \frac{1}{9}[0, -4, 3, 6, 17, 6, 3, -4, 0],\\
\mathbf{a_3} &= \frac{1}{3}[0, -4, 7, 3, 7, -4, 0].
\end{aligned}$$

Show that

$$\begin{aligned}
\left\|\frac{1}{3}S_1\right\|_\infty &= \frac{35}{81} < 1,\\
\left\|\frac{1}{3}S_2\right\|_\infty &= \frac{25}{27} < 1,\\
\left\|\frac{1}{3}S_3\right\|_\infty &= \frac{11}{9} > 1.
\end{aligned}$$

In fact, we have $\left\|(\frac{1}{3}S_3)^k\right\|_\infty > 1, \quad \forall k \in \mathbb{Z}_+$. Hence this scheme is C^1, not for C^2.

Exercise 31 *For DD(3,6) verify that*

$$\begin{aligned}
\mathbf{a} &= \frac{1}{729}[0,7,8,0,-56,-70,0,280,560,729\\
&\quad 560,280,0,-70,-56,0,8,7,0],\\
\mathbf{a_1} &= \frac{1}{243}[0,7,1,-8,-49,-13,62,231,267,\\
&\quad 231,62,-13,-49,-8,1,7,0],\\
\mathbf{a_2} &= \frac{1}{81}[0,7,-6,-9,-34,30,66,135,66,30,-34,-9,-6,7,0]\\
\mathbf{a_3} &= \frac{1}{27}[0,7,-13,-3,-18,51,33,51,-18,-3,-13,7,0].
\end{aligned}$$

Using

$$a^{k+1}(z) = a(z)a^k(z^3),$$

find the mask of $a_3^{[2]}$. *Show that*

$$\begin{aligned}
\left\|\frac{1}{3}S_1\right\|_\infty &= \frac{357}{729} < 1,\\
\left\|\frac{1}{3}S_2\right\|_\infty &= \frac{217}{243} < 1,\\
\left\|\frac{1}{3}S_3\right\|_\infty &= \frac{89}{81} > 1,\\
\left\|\left(\frac{1}{3}S_3\right)^2\right\|_\infty &= \frac{6450}{6561} < 1.
\end{aligned}$$

Hence this scheme to be C^2, *not for* C^3.

Exercise 32 *Consider the Laurent polynomial*

$$a(z) = \frac{1}{64}(1+z)^3(1-6z+z^2)^2.$$

Determine the mask of the corresponding subdivision scheme and analyze the convergence by the formalism of Laurent polynomials.

Chapter 5

Subdivision for Surfaces

Subdivision surfaces are polygon mesh surfaces generated from a given mesh through a refinement process makes the mesh smooths while increasing its density. Complex smooth surfaces can be derived in a reasonably predictable way from relatively simple meshes.

Classification :

- stationary or non-stationary
- binary or ternary
- type of mesh (triangle or quadrilateral)
- approximating or interpolating
- linear or non-linear

Subdivision Zoo:

- Vertex insertion (primal): Insert a vertex on the interior of each edge and one on the interior of each face.– Loop, Kobbelt, Catmull-Clark, Modified Butterfly.
- Corner cutting (dual): Insert a face in the middle of each old face and connect faces in adjacent old faces.– Doo-Sabin.
- Interpolating- Control the limit surface in a more intuitive manner. Simplify algorithms.
- Approximating- Higher quality surfaces. Faster convergence.

5.1 Tensor Product B-spline Surfaces

A piecewise polynomial tensor product B-spline surface $S^{r,s}(\mathbf{u})$ is defined by

$$S^{r,s}(\mathbf{u}) = \sum_{\mathbf{i}\in\mathbb{Z}^2} d_{\mathbf{i}}^{r,s} B^{r,s}(\mathbf{u}-\mathbf{i}), \tag{5.1}$$

where $\mathbf{u} \in \mathbb{R}^2$. A tensor product B-spline is the product of two independently parameterized univariate B-spline

$$B^{r,s}(\mathbf{u}-\mathbf{i}) = B^r(u-i)B^s(v-j).$$

Subdivision of tensor product B-spline surfaces is analogous to subdivision of B-spline curves. The knot set $\mathbf{i}$ is refined to $\mathbf{i}/2 = \mathbf{j}$, so $\mathbf{j} \in \{(i/2, j/2)|i, j \in \mathbb{Z}\}$. The corresponding equation can is written over the refined grid to become

$$S^{r,s}(\mathbf{u}) = \sum_{\mathbf{i}} \hat{d}_{\mathbf{i}}^{r,s} B^{r,s}(2(\mathbf{u}-\mathbf{i})).$$

A single translated tensor product B-spline is written over the refined grid as

$$B^{r,s}(\mathbf{u}-\mathbf{i}) = \sum_{\mathbf{j}\in\mathbb{Z}^2/2} c_{\mathbf{j}-\mathbf{i}}^{r,s} B^{r,s}(2(\mathbf{u}-\mathbf{j})).$$

This is substituted into (5.1) to give

$$S^{r,s}(\mathbf{u}) = \sum_{\mathbf{i}\in\mathbb{Z}^2} d_{\mathbf{i}}^{r,s} \sum_{\mathbf{j}\in\mathbb{Z}^2/2} c_{\mathbf{j}-\mathbf{i}}^{r,s} B^{r,s}(2(\mathbf{u}-\mathbf{j})).$$

It follows that

$$\hat{d}_{\mathbf{j}}^{r,s} = \sum_{\mathbf{i}\in\mathbb{Z}^2} c_{\mathbf{j}-\mathbf{i}}^{r,s} d_{\mathbf{i}}^{r,s}, \quad \mathbf{j} \in \mathbb{Z}^2/2,$$

where

$$c_{\mathbf{j}}^{r,s} = c_i^r c_j^s = 2^{-(r+s)} \binom{r+1}{2i} \binom{s+1}{2j}.$$

For example, if $r = s = 2$ then

$$\begin{aligned} \hat{d}_{0,0}^{2,2} &= \cdots + \left(\frac{9}{16}\right) d_{-1,-1}^{2,2} + \left(\frac{3}{16}\right) d_{0,-1}^{2,2} + \cdots \\ &\quad \cdots + \left(\frac{3}{16}\right) d_{-1,0}^{2,2} + \left(\frac{1}{16}\right) d_{0,0}^{2,2} + \cdots \end{aligned}$$

From the above example, the mask set:
We can also get the mask set by convolution of single quadratic B-spline mask $\frac{1}{4}[1 \quad 3 \quad 3 \quad 1]$.

$$\begin{array}{ll} \frac{1}{4}[3 \quad 1] * \frac{1}{4}[3 \quad 1], & \frac{1}{4}[3 \quad 1] * \frac{1}{4}[1 \quad 3], \\ \frac{1}{4}[1 \quad 3] * \frac{1}{4}[3 \quad 1], & \frac{1}{4}[1 \quad 3] * \frac{1}{4}[1 \quad 3]. \end{array}$$

Exercise 33 • *Find* $\hat{d}_{1/2,0}^{2,2}, \hat{d}_{0,1/2}^{2,2}, \hat{d}_{1/2,1/2}^{2,2}$.

- *Find the masks for cubic* $(r = s = 3)$.

The tensor product subdivision scheme inherits the convergence and smoothness property of the univariate schemes. The mask generating the bi-quadratic and bi-cubic B-spline is defined by the symbols

$$\begin{aligned} a(z_1, z_2) &= 2^{-4}(1+z_1)^3(1+z_2)^3, \\ a(z_1, z_2) &= 2^{-6}(1+z_1)^4(1+z_2)^4. \end{aligned}$$

But, the tensor product schemes are not ideal with respect to the size of the support of the mask per given smoothness.

5.2 Subdivision of Triangular Spline

Another class of splines sharing of univariate and tensor product B-splines are triangular splines. A triangular spline surfaces $S^{r,s,t}(\mathbf{u})$ can be written by

$$S^{r,s,t}(\mathbf{u}) = \sum_{\mathbf{i}} d_{\mathbf{i}}^{r,s,t} B^{r,s,t}(\mathbf{u} - \mathbf{i}), \tag{5.2}$$

where $\mathbf{u} \in \mathbb{R}^2$ and $\mathbf{i} \in \mathbb{Z}^2$. $B^{r,s,t}(\mathbf{u})$ is a normalized triangular spline of degree $(r+s+t-2)$ over the grid $\mathbf{i}$.

The procedure for subdivision triangular splines exactly parallels the subdivision schemes so far. The grid $\mathbf{i}$ is refined to a grid $\mathbf{j} = \mathbf{i}/2$. The surface corresponding to equation (5.2) is written over the refined grid to become

$$S^{r,s,t}(\mathbf{u}) = \sum_{\mathbf{i}} \hat{d}_{\mathbf{i}}^{r,s,t} B^{r,s,t}(2(\mathbf{u} - \mathbf{i})). \tag{5.3}$$

A single triangular spline is decomposed into splines of identical degree over the refined grid

$$B^{r,s,t}(\mathbf{u} - \mathbf{i}) = \sum_{\mathbf{j}} c_{\mathbf{j}-\mathbf{i}}^{r,s,t} B^{r,s,t}(2(\mathbf{u} - \mathbf{j})).$$

By substituting into (5.2) and rearranging the order of summation it is found that

$$\hat{d}_{\mathbf{j}}^{r,s,t} = \sum_{\mathbf{i}} c_{\mathbf{j}-\mathbf{i}}^{r,s,t} d_{\mathbf{i}}^{r,s,t}.$$

For the special case of binary subdivision it is possible to show that

$$c_{\mathbf{j}}^{r,s,t} = 2^{-(r+s+t)} \sum_{k=0}^{t} \binom{r}{2i-k} \binom{s}{2j-k} \binom{t}{k}.$$

Of particular interest are the binary subdivision masks for triangular spline $B^{2,2,2}(\mathbf{u})$.

Exercise 34 *Find the mask set for triangular spline* $B^{2,2,2}(\mathbf{u})$.

5.3 Doo-Sabin Scheme

- Approximating Scheme
- Tensor Product Quadratic B-Spline
- Locally C^1 Continuity
- Dual Quadrilateral

Doo-Sabin scheme comes from the regular tensor product of quadratic B-spline. **Problem:** Subdivision for tensor product quadratic B-spline surface has rigid restrictions on the topology. Each vertex must have order 4. This restriction makes the design of many surfaces difficult. Doo/Sabin presented an algorithm that eliminated this restriction by generalizing the bi-quadratic B-spline subdivision rules to include arbitrary topology. And the behavior of the limit surface defined by a recursive division construction can be analyzed in terms of the eigenvalues of a set of matrices.

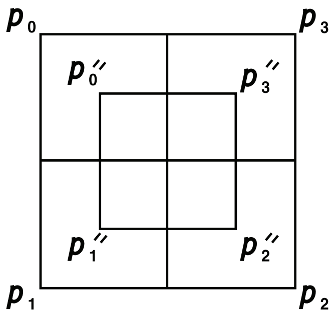

Figure 5.1: The Doo-Sabin scheme:

The subdivision masks for bi-quadratic B-spline is as follow:

$$[P_0", P_1", P_2", P_3]^t = S_4^{DS}[P_0, P_1, P_2, P_3]^t,$$

where

$$S_4^{DS} = \frac{1}{16}\begin{bmatrix} 9 & 3 & 1 & 3 \\ 3 & 9 & 3 & 1 \\ 1 & 3 & 9 & 3 \\ 3 & 1 & 3 & 9 \end{bmatrix}.$$

Recall that Laurent polynomial for quadratic B-spline is $(1+z)^3/4$. The Laurent polynomial for a tensor product of quadratic B-spline is

$$a(z_1, z_2) = \frac{(1 + 3z_1 + 3z_1^2 + z_1^3)(1 + 3z_2 + 3z_2^2 + z_2^3)}{16}.$$

The coefficients of the $a(z_1, z_2)$ is the mask of the scheme. Actually for tensor product B-spline, we have 4 rules for determining the positions on the next subdivision level. They are the Even-Even, Even-Odd, Odd-Even and Odd-Odd rules.

Geometric view of bi-quadratic B-spline subdivision: the new points are centroid of the sub-face formed by the face centroid, a corner vertex and the two mid-edge points next to the corner. The new points then are connected. There will be two vertices along each side of each edge in the old mesh, by construction. These pairs are connected, forming quadrilaterals across the old edges. The new mesh, therefore, will create quadrilaterals for each edge in the old mesh, will create a smaller n-sided polygon for each n-sided polygon in the old mesh, and will create an n-sided polygon for each n-valence vertex (valence being the number of edges that share the vertex). After one application of the scheme all vertices will have a valence of four, so subsequent applications will create quadrilaterals for the vertices. (The original n-sided polygons are retained, however, and shrink to extraordinary points where the mesh is not as smooth, as the scheme is repeatedly applied.)

5.3.1 Eigen-analysis for Doo-Sabin Scheme

Consider the dual scheme at an extraordinary face with N sides. The subdivision matrix S is circulant with elements $S_{lk} = a_{k-l}$. That is, the matrix is

$$S = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{N-1} \\ a_{N-1} & a_0 & a_1 & \cdots & a_{N-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}.$$

Discrete Fourier transform is defined via multiplication with matrix F, where $F_{pq} := z^{pq}, z = e^{2\pi i/N}$. The inverse transform is performed with the matrix $F^{-1} = (1/N)F^*$. Here F^* is conjugate matrix with elements $F^*_{pq} = z^{-pq}$.
The Fourier transform of the subdivision matrix produces diagonal subdivision matrix $\hat{S} = FSF^{-1}$. $\hat{S}$ is diagonal with values on diagonal being the eigenvalues and given by

$$\hat{a}_t = \sum_q a_q z^{qt}.$$

So the transformed matrix is

$$\hat{S} = \begin{bmatrix} \hat{a}_0 & 0 & 0 & \cdots & 0 \\ 0 & \hat{a}_1 & 0 & \cdots & 0 \\ 0 & 0 & \hat{a}_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \hat{a}_{N-1} \end{bmatrix}$$

The inverse transform give us the a_s:

$$a_s = \frac{1}{N} \sum \hat{a}_t z^{-ts}.$$

The regular case has the spectrum:

$$\hat{a}_0 = 1, \hat{a}_1 = \hat{a}_3 = \frac{1}{2}, \hat{a}_2 = \frac{1}{4}.$$

In the irregular case we will enforce $1, 1/2, 1/2, 1/4, \ldots, 1/4$. This produces the mask of Doo-Sabin scheme. We set $\hat{a}_0 = 1, \hat{a}_1 = \hat{a}_{N-1} = 1/2$ and all other $\hat{a}_t = 1/4$. For an N-sided face, we get the mask of Doo/Sabin subdivision matrix $S_N^{DS} = (\alpha_{ij})_{N\times N}$.

$$\begin{aligned}\alpha_{ij} &= \frac{5+N}{4N}, \quad i=j\\ \alpha_{ij} &= \frac{3+2\cos(\frac{2\pi(i-j)}{N})}{4N}, \quad i\neq j.\end{aligned}$$

Since bi-quadratic B-splines are C^1, the surfaces generated by the Doo/Sabin algorithm are locally C^1.

Exercise 35 *Read the paper [7]*

The following figures show how the Doo-Sabin scheme can proceed.

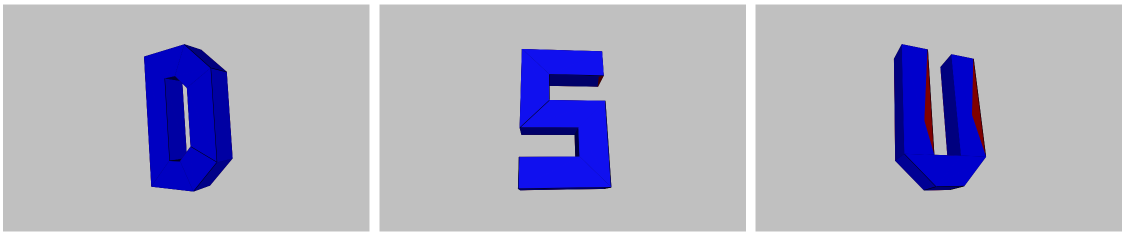

Figure 5.2: Doo-Sabin scheme-step 0.

The base mesh consists of a mere 62 polygons (24 vertices, 24 faces and 48 edges in the D, 24 vertices, 22 faces and 44 edges in the S, and 20 vertices, 16 faces and 34 edges in U).

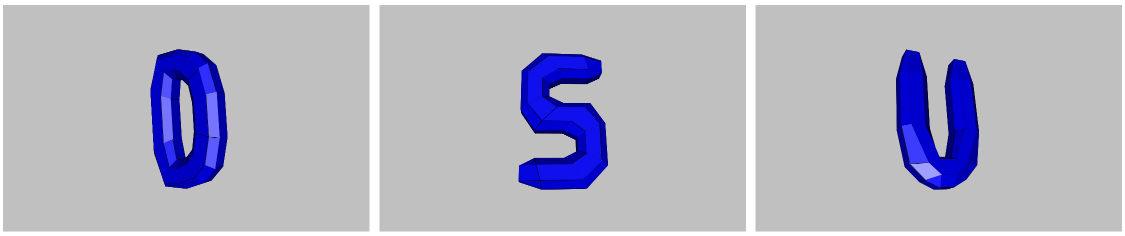

Figure 5.3: Doo-Sabin scheme-after one iteration.

After one application of the Doo-Sabin scheme, the mesh has become 256 polygons (96 vertices, 96 faces and 192 edges in the D, 88 vertices, 90 faces and 176 edges in the S, and 68 vertices, 70 faces and 136 edges in U). The sharpest points have been nicely rounded off. The lengths of the letters have become smoother, the sharpest points have been nicely rounded off.

After one more iteration of subdivision, the Doo-Sabin surface now consists of 1,012 polygons (384 vertices, 384 faces and 768 edges in the D, 352 vertices, 354 faces and 704 edges in the S, and 272 vertices, 274 faces and 544 edges in U). The surface is already quite smooth.

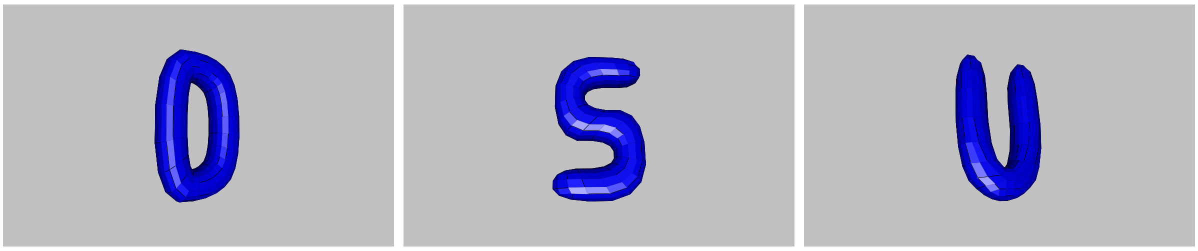

Figure 5.4: Doo-Sabin scheme-after two iterations.

5.4 Catmull-Clark Scheme

Catmull-Clark devised recursively subdivide a surface patch into four sub-patches until the resulting patch is roughly the size of a picture element of the raster display on which it is to be rendered. This method is presented as a generalization of a recursive bi-cubic B-spline subdivision algorithm.

- Approximating Scheme
- Tensor Product Bicubic B-spline
- Continuity
  - C^2 regular regions
  - C^1 extraordinary vertices
- Primal Quadrilateral

Rectangular B-spline Patch Splitting

The bi-cubic B-spline patch can be expressed in matrix form by

$$S(u,v) = UMGM^tV^t,$$

where

$$M = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix},$$

is the cubic B-spline basis matrix, and

$$G = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix},$$

is the set of control points, and

$$U = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}, \quad V = \begin{bmatrix} v^3 & v^2 & v & 1 \end{bmatrix},$$

is the primitive basis vectors.

We will consider the sub-patch of this patch corresponding to $0 < u, v < 1/2$. The other sub-patch need not to be considered due to the symmetry of the B-spline.

$$S(u_1, v_1) = USMGM^tS^tV^t,$$

where

$$S = \begin{bmatrix} \frac{1}{8} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This patch must still be B-spline with its own control point mesh G_1, satisfying

$$S(u_1, v_1) = UMG_1M^tV^t.$$

We get

$$MG_1M^t = SMGM^tVS^t.$$

Assuming that M is invertible,

$$G_1 = [M^{-1}SM]G[M^tS^tM^{-t}] = H_1GH_1^t,$$

here

$$H_1 = M^{-1}SM,$$

is called the splitting matrix. Carrying out the matrix multiplication, it is found

$$H_1 = \begin{bmatrix} 4 & 4 & 0 & 0 \\ 1 & 6 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 6 & 1 \end{bmatrix}.$$

Hence the control point mesh corresponding to the sub-patch is related to old control point mesh by the expression

$$G_1 = H_1GH_1^t.$$

The new face point gives

$$q_{11} = \frac{p_{11} + p_{12} + p_{21} + p_{22}}{4}.$$

Likewise, the new edge point is given by

$$q_{12} = \frac{\frac{C+D}{2} + \frac{p_{12}+p_{22}}{2}}{2},$$

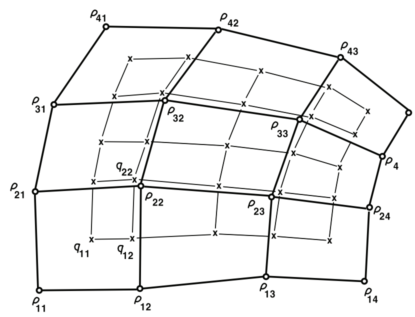

Figure 5.5: Catmull-Clark scheme :

where

$$q_{11} = C, \quad q_{13} = D = \frac{p_{12} + p_{13} + p_{22} + p_{23}}{4}.$$

The new vertex point q_{22} is given by

$$q_{22} = \frac{Q}{4} + \frac{R}{2} + \frac{p_{22}}{4},$$

where

$$Q = \frac{q_{11} + q_{13} + q_{31} + q_{33}}{4},$$

and

$$R = \frac{1}{4}\left[\frac{p_{22} + p_{12}}{2} + \frac{p_{22} + p_{21}}{2} + \frac{p_{22} + p_{32}}{2} + \frac{p_{22} + p_{23}}{2}\right].$$

Since these expression were deduced from the standard B-spline basis, we can generate a bi-cubic B-spline surface.

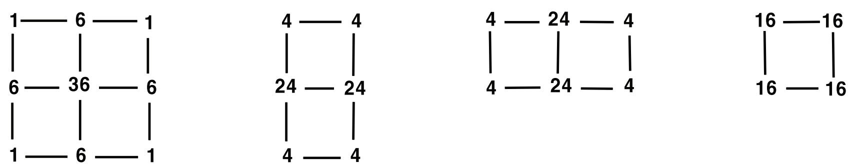

Figure 5.6: The Masks set for bi-cubic.

Exercise 36 *Find the mask set by convolution of single cubic B-spline mask.*

Arbitrary Topology

For generalizing the expressions to arbitrary topology, a set of rules which are dependent on the valence. Of course, the rules yields the expressions when valence is four. The rules are:

- New face point-the average of all the old points defining the face.
- New edge point-the average of the midpoints of the old edge with the average of the two new face points of the faces sharing the edge.
- New vertex point-the average
$$\frac{Q}{n} + \frac{2R}{n} + \frac{S(n-3)}{n},$$
where
  - Q-average of new face points of all faces adjacent to old vertex point.
  - R-average of the midpoint of all edges incident on the old vertex point.
  - S-old vertex point.

The initial convex combination Catmull-Clark proposed was
$$S = \frac{1}{4}Q + \frac{1}{2}R + \frac{1}{4}S.$$
It should be noted that after one iteration all faces are four-sided, hence all new vertices created subsequently will have four incident edges. Therefore after one iteration the number of extraordinary points on the surface remains constant. It was observed that in some arbitrary meshes, tangent plane continuity was not maintained at extraordinary points. The modified rule is
$$\hat{S} = \frac{1}{N}Q + \frac{2}{N}R + \frac{N-3}{N}S,$$
where N is the order of the vertex. Catmull-Clark surfaces have the convex-hull property, local control, and are locally C^2 everywhere except at the extraordinary points. A proof have a continuous tangent plane at the extraordinary points was given by Doo and Sabin.

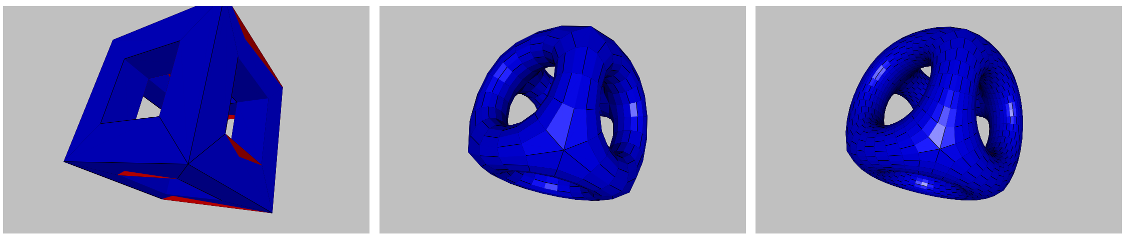

Figure 5.7: Catmull-Clark scheme-step 0, 2 iterations and 3 iterations.

The base mesh (box cube) consists of 48 vertices, 54 faces and 108 edges. After two iterations of the Catmull-Clark scheme, the mesh has become 858 vertices, 864 faces and 1728 edges. After one more iteration of subdivision, the surface now consists of 3450 vertices, 3456 faces and 6912 edges.

5.5 Butterfly Scheme

- Interpolating Scheme
- General Scheme
- Continuity C^1 regular regions
- Primal Triangular Quadrisection

The Butterfly scheme is an extension of the 4-point interpolating subdivision scheme to the case of surfaces defined by control points with topology of general triangulations.

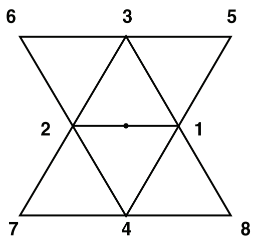

Figure 5.8: The butterfly scheme:

Given a set of control points $\{p_i^k\}$ which comprise the vertices of a triangulation T^k, the scheme associate with each edge $e \in T^k$ a new point q_e^k defined by

$$q_e^k = \frac{1}{2}(p_{e,0}^k + p_{e,1}^k) + 2w(p_{e,2}^k + p_{e,3}^k) - w\sum_{j=4}^{7} p_{e,j}^k.$$

The butterfly scheme defines the control points at stage $k+1$ as

$$\{p_i^{k+1}\} = \{p_i^k\} \cup \{q_e^k : e \in T^k\},$$

and the triangulation T^{k+1} as the collection of edge

$$\{(q_e^k, p_{e,j}^k), j = 0, 1, \quad (q_e^k, q_{e_{ij}}^k), i = 0, 1, j = 2, 3 : e \in T^k\},$$

where $e_{ij} = (p_{e,i}^k, p_{e,j}^k)$. With this construction of T^{k+1}, the number of edges having p_i^k as vertex in T^{k+1} is the same as in T^k, while each new vertex is regular, namely a vertex of six edges in T^{k+1}. Therefore, with the exclusion of the irregular points in T^0, all vertices of T^k are regular. A triangulation with regular vertices is topologically equivalent to a three direction grid.
The mask of Butterfly scheme is

$$\begin{aligned}
a_{0,0} &= 1, \\
a_{1,0} &= a_{-1,0} = a_{-1,-1} = a_{1,1} = 1/2, \\
a_{1,-1} &= a_{-1,-2} = a_{1,2} = 2w, \\
a_{1,-2} &= a_{-3,-2} = a_{-1,2} = a_{3,2} = a_{-1,-3} = a_{1,3} = -w, \\
a_{i,j} &= a_{j,i},
\end{aligned}$$

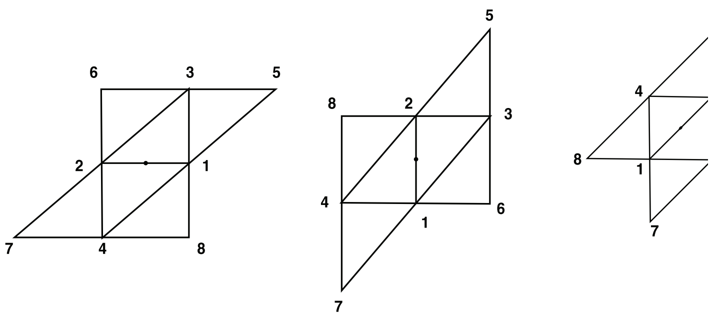

Figure 5.9: The mask of butterfly scheme:

and zero otherwise. Refinement rules are

$$\begin{aligned}
P^{k+1}_{2i,2j} &= P^k_{i,j}, \\
P^{k+1}_{2i+1,2j} &= \frac{1}{2}(P^k_{i,j} + P^k_{i+1,j}) + 2w(P^k_{i,j-1} + P^k_{i+1,j+1}) \\
&\quad -w(P^k_{i-1,j-1} + P^k_{i+1,j-1} + P^k_{i,j+1} + P^k_{i+2,j+1}), \\
P^{k+1}_{2i,2j+1} &= \frac{1}{2}(P^k_{i,j} + P^k_{i,j+1}) + 2w(P^k_{i-1,j} + P^k_{i+1,j+1}) \\
&\quad -w(P^k_{i-1,j-1} + P^k_{i-1,j+1} + P^k_{i+1,j} + P^k_{i+1,j+2}), \\
P^{k+1}_{2i+1,2j+1} &= \frac{1}{2}(P^k_{i,j} + P^k_{i+1,j+1}) + 2w(P^k_{i+1,j} + P^k_{i,j+1}) \\
&\quad -w(P^k_{i,j-1} + P^k_{i-1,j} + P^k_{i+2,j+1} + P^k_{i+1,j+2}).
\end{aligned}$$

The bivariate Laurent polynomial has the form

$$a(z_1, z_2) = 2^{-1}(1 + z_1^{-1})(1 + z_2^{-1})(1 + z_1^{-1}z_2^{-1})z_1 z_2(1 + wq(z_1, z_2)),$$

where

$$\begin{aligned}
q(z_1, z_2) &= 2z_1^{-2}z_2^{-1} + 2z_1^{-1}z_2^{-2} - 4z_1^{-1}z_2^{-1} - 4z_1^{-1} - 4z_2^{-1} + 2z_1^{-1}z_2 \\
&\quad +2z_1 z_2^{-1} + 12 - 4z_1 - 4z_2 - 4z_1 z_2 + 2z_1^2 z_2 + 2z_1 z_2^2.
\end{aligned}$$

It is easy to check that the butterfly scheme maps f^k which is constant in one of the three grid directions to f^{k+1} with the same property. The scheme for such f^k reduces to the 4-point interpolating subdivision scheme applied along lines of the other two directions.

Exercise 37 *Verify the last statement.*

Exercise 38 *Show that the butterfly scheme with* $w = 1/16$ *on a regular grid, which reproduce cubic polynomial.*

The averaged butterfly scheme: The butterfly scheme is not symmetric relative to the regular grid. A symmetric scheme can be obtained by averaging two butterfly schemes, corresponding to the choices, (1,1) and (-1,1) of the third direction. The Laurent polynomial A corresponding to this new scheme is

$$A(z_1, z_2) = \frac{1}{2}[a(z_1, z_2) + a(z_1^{-1}, z_2)]$$

Exercise 39 *Show that the new scheme also reproduce cubic polynomial for $w = 1/16$ and find the mask of the new scheme.*

A truncated tensor-product scheme: Consider the tensor product of two 4-point Dyn schemes has the mask $t_w(z_1, z_2) = a_w(z_1)a_w(z_2)$ with support size of $4 \times 4 = 16$ points, where

$$a_w(z) = \frac{1}{2z}(1+z)^2(1+wb(z)), \quad b(z) = -2z^{-2}(z-1)^2(z^2+1).$$

To reduce the support size, interpolating and with the same polynomial precision and smoothness, by removing all the w^2-terms in t_w. The resulting Laurent polynomial is

$$A_w^t(z_1, z_2) = \frac{1}{4}(1+z_1)^2(1+z_2)^2 z_1^{-1} z_2^{-1}(1 - w(b(z_1) + b(z_2))).$$

Some basic characteristics of the new scheme:

1. The rules for new points corresponding to centers of edge coincide with the univariate 4-point Dyn scheme.
2. It reproduces cubic polynomial for $w = 1/16$.
3. It can be reduced to the 4-point Dyn scheme in one direction, when the data is constant along the other directions.

Exercise 40 *Find the mask of this new scheme.*

The butterfly scheme generates smooth interpolating surface only for regular meshes. Smoothness, however, is not guaranteed at extraordinary points. It exhibits undesirable artifacts in the case of irregular meshes. A modification of this scheme with special coefficients for computing new vertices adjacent to the extraordinary vertices generates that are smooth for almost everywhere. The coefficients of this scheme [31] are given by

$$s_j = \frac{1}{n}\left(\frac{1}{4} + \cos\frac{2\pi j}{n} + \frac{1}{2}\cos\frac{4\pi j}{n}\right), \quad j = 0, \ldots, n-1$$

if the extraordinary vertex has valence ≥ 5. The coefficients are $5/12, -1/12, -1/12$ for $n = 3$ and $3/8, 0, -1/8, 0$ for $n = 4$.

The base mesh consists of 891 vertices, 1704 faces and 2592 edges. After one iteration of the Modified butterfly scheme, the mesh has become 3848 vertices, 6816 faces and 10296 edges.

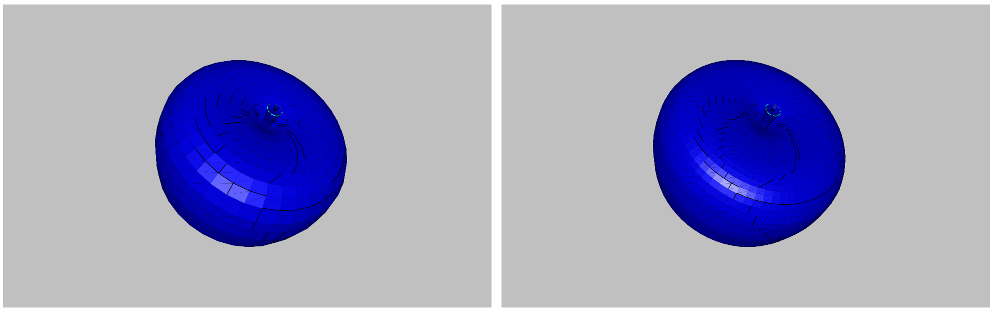

Figure 5.10: Modified butterfly scheme

5.6 Loop Scheme

- Approximating Scheme
- Three Directional Box-Spline
- Continuity -C^2 for regular regions C^1 for extraordinary vertices
- Primal Triangular Quadrisection

Special attention was given to the triangular spline $B^{2,2,2}(u)$. This triangular spline was chosen for study because it is the lowest order (in this case degree 4) triangular spline that trilateral symmetry and C^2 smoothness. The derivation of the generalized subdivision rules for this new algorithm begins with an abstraction of the geometric properties of the subdivision masks for $B^{2,2,2}(u)$.

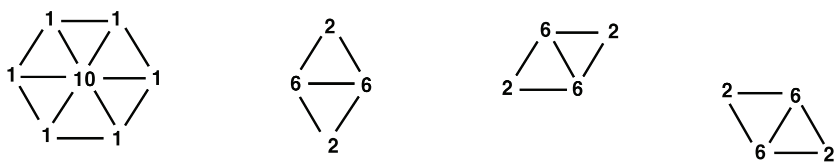

Figure 5.11: The subdivision masks for $B^{2,2,2}(u)$

Mask A generates new control points for each vertex, and masks B generate new control points for each edge of the original regular triangular mesh.

The generalization of a vertex point rule is more difficult. To derive the new vertex point rule, consider mask A. The new vertex, can be computed as a convex combination of the old vertex and all vertices that share an edge with it. These rule are: V, the old vertex point, and Q, the average of old points that share an edge with V. The new vertex point is computed as

$$\hat{V} = \frac{5}{8}V + \frac{3}{8}Q.$$

The Loop scheme is defined for triangle meshes only, not general polygonal meshes. At each step of the scheme each triangle is split into four smaller triangles. Edge points are constructed on each edge. These points are three eighths of the sum of the two end points of the edge plus one eighth of the sum of the two other points that form the two triangles that share the edge in question. Vertex points are constructed for each old vertex.

Each old triangle will have three edge points, one for each edge, and three vertex points, one for each vertex. To form the new triangles these points are then connected, vertex-edge-edge, creating four triangles. One new triangle touches each old vertex, and the last new triangle sits in the center, connecting the three edge points. The associated Laurent polynomial for Loop's scheme is

$$a(z_1, z_2) = \frac{1}{16} z_1^{-2} z_2^{-2} (1+z_1)^2 (1+z_2)^2 (1+z_1 z_2)^2.$$

The refinement rules derived from this mask are

$$\begin{aligned}
P^{k+1}_{2i,2j} &= \frac{5}{8} P^k_{i,j} + \frac{1}{16}(P^k_{i-1,j-1} + P^k_{i-1,j} + P^k_{i,j-1} + P^k_{i+1,j+1} + P^k_{i+1,j} + P^k_{i,j+1}),\\
P^{k+1}_{2i+1,2j} &= \frac{3}{8}(P^k_{i,j} + P^k_{i+1,j}) + \frac{1}{8}(P^k_{i,j-1} + P^k_{i+1,j+1}),\\
P^{k+1}_{2i,2j+1} &= \frac{3}{8}(P^k_{i,j} + P^k_{i,j+1}) + \frac{1}{8}(P^k_{i-1,j} + P^k_{i+1,j+1}),\\
P^{k+1}_{2i+1,2j+1} &= \frac{3}{8}(P^k_{i,j} + P^k_{i+1,j+1}) + \frac{1}{8}(P^k_{i+1,j} + P^k_{i,j+1}).
\end{aligned}$$

$P^{k+1}_{2i,2j}$ is the v-vertex, the others are midpoint for the edges of the triangulation, e-vertex.

This same idea may be applied to an arbitrary triangular mesh. Because of the properties inherited from $B^{2,2,2}(u)$ it will be shown that the underlying surface of this algorithm is locally C^2 everywhere, except at the extraordinary points. The tangent plane continuity is lost at one of extraordinary points. This may be remedied by considering the order of the vertex when taking the convex combination of V and Q. This result in a new vertex point rule of the form

$$\hat{V} = \alpha_N V + (1 - \alpha_N) Q,$$

where α_N is a function of the vertex order N

$$\alpha_N = \left(\frac{3}{8} + \frac{1}{4}\cos\left(\frac{2\pi}{N}\right)\right) + \frac{3}{8}.$$

Exercise 41 *Read the thesis [22].*

The Butterfly scheme is interpolating. This means that all the vertices on a mesh after any number of iterations of this scheme belong to the limit surface. On the other hand, the Loop scheme is non-interpolating, since each on each iterations the old vertices of the mesh are replaced by new vertices. The vertices of the mesh do not belong to the limit surface, but they converge to points on the limit surface after a small number of iterations. In the Loop scheme the mesh is always contained in the convex hull of the original mesh, and it becomes smaller after each iteration. On the other hand, in the Butterfly scheme the mesh is not contained in the convex hull of the original mesh, and it becomes bigger than the original mesh. This is caused by the negative coefficients used in the formula of the new vertices. Both schemes produce smooth surfaces for ordinary vertices, but the Loop scheme gives smooth surfaces for extraordinary vertices, unlike the Butterfly scheme.

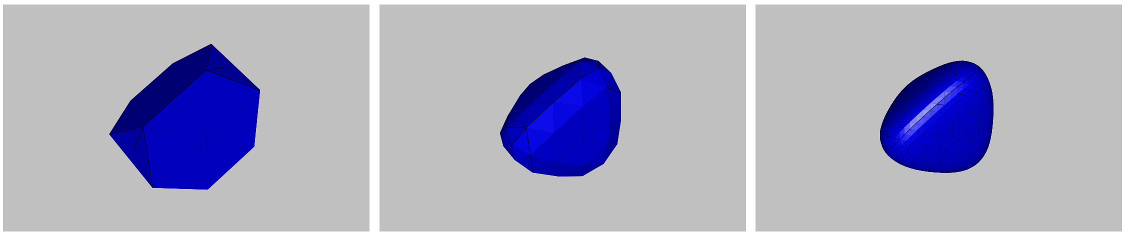

Figure 5.12: Loop scheme-step 0, 1 iteration and 3 iterations.

The base mesh (truncated tetrahedron) consists of 20 vertices, 36 faces and 54 edges. After one iteration of the Loop scheme, the mesh has become 74 vertices, 144 faces and 216 edges. After three iterations of subdivision, the surface now consists of 1154 vertices, 2304 faces and 3456 edges.

5.7 $\sqrt{3}$ Subdivision

- Approximating Scheme
- Continuity: C^2 regular regions C^1 extraordinary vertices
- Triangular Trisection
- Adaptive Refinement

A subdivision operator for polygonal meshes can be considered as being composed by a (topological) split operation followed by a (geometric) smoothing operation. The split operation performs the actual refinement by introducing new vertices and the smoothing operation changes the vertex position by computing averages of neighboring vertices. The most wide-spread way to uniformly refine a given triangle mesh is the dyadic split which bi-sects all the edges by inserting a new vertex between every adjacent pair of old ones. Each triangular face is then split into four smaller triangles by mutually connecting the new vertices sitting on a face. That is, subdivision schemes on triangle meshes are usually based on the 1-to-4 split operation which inserts a new vertex for every edge of the given mesh and then connects the new vertices.

This scheme is based on a split operation which first inserts a new vertex for every face of the given mesh. Flipping the original edges. Applying this scheme twice leads to a 1-to-9 refinement

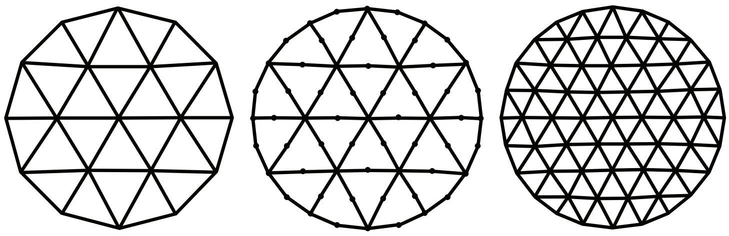

Figure 5.13: Subdivision schemes on triangle meshes are usually based on the 1-to-4 split operation which inserts a new vertex for every edge of the given mesh and then connects the new vertices.

of the original mesh. Analyzing the action of the $\sqrt{3}$-subdivision operator on arbitrary triangle meshes, we find that all newly inserted vertices have exactly valence six. The valences of the old vertices are not changed such that after a sufficient number of refined steps.

5.7.1 Stationary Smoothing Rules

To complete the definition of this scheme, we have to find the two smoothing rules, one for the placement of the newly inserted vertices and one for the relaxation of the old ones. There are well-known necessary and sufficient criteria which tell whether a subdivision scheme S is convergent or not and what smoothness properties the limit surface has. Such criteria check if the eigenvalues of the subdivision matrix have a certain distribution and if a local regular parametrization exists. The subdivision matrix is a square matrix S which maps a sub-mesh V to a topologically equivalent sub-mesh $S(V)$ of the refined mesh. Every row of this matrix is a rule to compute the position of a new vertex. Every column of this matrix tells how one old vertex contributes to the vertex positions in the refined mesh.

To derive the weight coefficients for the new subdivision scheme, we use **Reverse Engineering Process**: instead of analyzing a given scheme, we derive one which by construction satisfies the known necessary criteria.

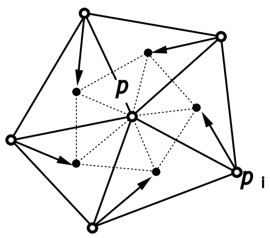

Figure 5.14: The application of the subdivision matrix S causes a rotational around p since the neighborhood vertices are replaced by the center of the adjacent triangles.

The reasonable choice for smoothing rule is

$$q := \frac{1}{3}(p_i + p_j + p_k),$$

i.e., the new vertex q is simply inserted at the center of the triangle $\Delta(p_i, p_j, p_k)$.
The smallest non-trivial stencil for the relaxation of the old vertices is the 1-ring neighborhood containing the vertex itself and its direct neighbor. Let p be a vertex with valence n and $p_0, \ldots, p_{n-1}$ its directly adjacent neighbors in the unrefined mesh, we define

$$S(p) := (1 - \alpha_n)p + \alpha_n \frac{1}{n} \sum_{i=0}^{n-1} p_i.$$

Arranging all vertices in a vector $[p, p_0, \ldots, p_{n-1}]$ we derive the subdivision matrix

$$S = \frac{1}{3} \begin{bmatrix} u & v & v & v & \cdots & v \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 1 & 0 & \ddots & \ddots & \ddots & 1 \\ 1 & 1 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

with $u = 3(1 - \alpha_n)$ and $v = 3\alpha_n/n$. When analyzing the eigen-structure of this matrix, we find that it is not suitable for the construction of a convergent subdivision scheme. The reason for this defect is the rotation around p which is caused by the application of S and which makes all eigenvalues are complex. We know that applying the $\sqrt{3}$-subdivision operator two times corresponds to a triadic split. So instead of analyzing one single subdivision step, we can combine two successive steps since after the second application of S, the neighbor hood of $S^2(p)$ is again aligned to the original configuration around p. Hence, the back-rotation can be written as a simple permutation matrix

$$R = \frac{1}{3} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \vdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

The resulting matrix $\tilde{S} = RS^2$ has eigenvalues:

$$\frac{1}{9}[9, (2 - 3\alpha_n)^2, 2 + 2\cos(2\pi/n), \ldots, 2 + 2\cos(2\pi(n-1)/n)].$$

The necessary condition for C^1:

$$\lambda_1 = 1 > \lambda_2 = \lambda_3 > \lambda_i, \quad i = 4, \ldots, n + 1.$$

Additionally, a natural choice for the eigenvalue λ_4 is

$$\lambda_4 = \lambda_2^2.$$

Consequently, we define the value for α_n by solving

$$\left(\frac{2}{3} - \alpha_n\right)^2 = \left(\frac{2 + 2\cos(2\pi/n)}{9}\right)^2,$$

which leads to

$$\alpha_n = \frac{4 - 2\cos(2\pi/n)}{9}.$$

5.7.2 Boundaries

When topologically refining a given open control mesh by the $\sqrt{3}$-operator we split all triangular faces 1-to-3 but flip only the interior edges. Edge flipping at the boundaries is not possible since the opposite triangle-mate is missing.
The application of a second $\sqrt{3}$-operator has overall effect of a triadic split where each original triangle is replaced by 9 new ones. Consequently, we have to apply a univariate trisection rule to the boundary polygon and connect the new vertices to the corresponding interior ones such that a uniform 1-to-9 split is established for each boundary triangle.
We choose a univariate boundary subdivision scheme which reproduces cubic splines. From the trivial trisection for linear splines, we can obtain trisection mask for cubic splines by convolution

$$\frac{1}{3}[1,2,3,2,1] * \left(\frac{1}{3}[1,1,1]\right)^2 = \frac{1}{27}[1,4,10,16,19,16,10,4,1].$$

Hence the resulting rules are

$$\begin{aligned}
p_{3i-1}^{k+1} &= \frac{1}{27}(10p_{i-1}^k + 16p_i^k + p_{i+1}^k),\\
p_{3i}^{k+1} &= \frac{1}{27}(4p_{i-1}^k + 19p_i^k + 4p_{i+1}^k),\\
p_{3i+1}^{k+1} &= \frac{1}{27}(p_{i-1}^k + 16p_i^k + 10p_{i+1}^k).
\end{aligned}$$

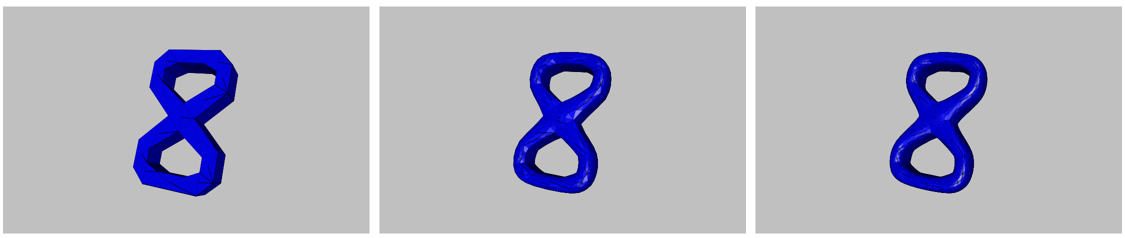

Figure 5.15: $\sqrt{3}$-scheme-step 0, 2 iterations and 3 iterations.

The base mesh consists of 114 vertices, 232 faces and 348 edges. After two iterations of the this scheme, the mesh has become 1042 vertices, 2088 faces and 3132 edges. After three iterations of subdivision, the surface now consists of 3130 vertices, 6264 faces and 9396 edges.

Chapter 6

Generalized Schemes

We present explicitly a new general formula for the mask of $(2n+4)$-point interpolating symmetric subdivision schemes (ISSS) with two parameters. It can be used to compute the mask of $(2n+4)$-point ISSS with parameter simply and rapidly. This new formula will be useful to analyze the convergence and smoothness of $(2n+4)$-point ISSS for further study. In this chapter, we presented a new class of subdivision scheme. This scheme reproduce polynomials up to certain degree π_{2N+1}. And this scheme unifies not only the ISSS but DD-scheme. We generalize the mask of symmetric subdivision scheme with two parameters.

6.1 Preliminaries

Throughout the work, we denote by $\mathbb{Z}$ the set of all integers and by π_{2N+1} the space of all polynomials of degree $\leq 2N+1$ for a nonnegative integer $\mathbb{N}$. In our argument, the Lagrange fundamental polynomials $\{L_k(x)\}_{k=-N}^{N+1}$ corresponding to the nodes $\{k\}_{k=-N}^{N+1}$ play quite an important rule. We define the Lagrange fundamental polynomials $\{L_k(x)\}_{k=-N}^{N+1}$ by

$$L_k(x) = \prod_{j\neq k, j=-N}^{N+1} \frac{x-j}{k-j}, \quad k = -N, \cdots, N+1, \tag{6.1}$$

for which

$$L_k(j) = \delta_{k,j}, \quad k, j = -N, \cdots, N+1, \tag{6.2}$$

and

$$\sum_{k=-N}^{N+1} p(k) L_k(x) = p(x), \quad p \in \pi_{2N+1}. \tag{6.3}$$

We have

$$\begin{aligned}
\prod_{j\neq k, j=-N}^{N+1} (k-j) &= (-1)^{N+1+k}(N+k)!(N+1-k)!, \\
\prod_{j\neq k, j=-N}^{N+1} \left(\frac{1}{2} - j\right) &= \frac{1}{2^{2N+1}} \frac{1}{1-2k} \prod_{j=-N-1}^{N} (2j+1) = \frac{(-1)^N}{2^{4N+1}} \frac{1}{2k-1} \left[\frac{(2N+1)!}{N!}\right]^2.
\end{aligned}$$

Then it is easy to see that for each $j = -N-1, \cdots, N$,

$$L_{-j}\left(\frac{1}{2}\right) = (-1)^j \frac{(N+1)}{2^{4N+1}(2j+1)} \binom{2N+1}{N} \binom{2N+1}{N+j+1}, \tag{6.4}$$

$$L_{-j}(N+2) = (-1)^{j+N+1} \frac{(2N+2)!}{(N-j)!(N+j+1)!(N+j+2)}, \tag{6.5}$$

$$L_{-j}(-N-1) = (-1)^{j+N} \frac{(2N+2)!}{(N-j)!(N+j+1)!(N-j+1)}. \tag{6.6}$$

and

$$L_{-j}(N+2) + L_{-j}(-N-1) = (-1)^{j+N} \binom{2N+1}{N+j+1} \frac{(2N+2)(2j+1)}{(N+j+2)(N-j+1)}. \tag{6.7}$$

These quantities are crucial to find the explicit form of masks considered in the following sections.

Exercise 42 *Verify (6.4),(6.5) and (6.6).*

We obtain a general rule about the mask of $(2n+4)$-point interpolating symmetric subdivision schemes with a parameter, which reproduces all polynomials of degree $\leq 2n+1$. Also, we generalize the masks of interpolating symmetric subdivision scheme such as Dyn 4-point and Weissman 6-point schemes and the Deslauriers and Dubuc scheme.

Precisely, for an integer $n \geq 0$, we provide a symmetric mask $\{a_i\}_{i=-2n-3}^{2n+3}$ of which even-indexed elements of masks are given as $a_{2n+2} = a_{-2n-2} = v$ and

$$a_{2j} = \delta_{j,0} + (-1)^{j+n+1} \binom{2n+2}{n+j+1} v, \quad \text{for} \quad i = -n, -n+1, \ldots, n,$$

and of which odd-indexed elements of masks are given as $a_{2n+3} = a_{-2n-3} = w$ and

$$\begin{aligned} a_{2j+1} = {} & \frac{n+1}{2^{4n+1}} \binom{2n+1}{n} \frac{(-1)^j}{2j+1} \binom{2n+1}{n+j+1} \\ & + (-1)^{j+n+1} w \binom{2n+1}{n+j+1} \frac{(2n+2)(2j+1)}{(n+j+2)(n-j+1)}, \quad \text{for} \quad j = -n-1, -n, \ldots, n. \end{aligned}$$

The masks v and w play roles as tension parameters. This mask has the properties:

- The mask is symmetric, that is, $a_j = a_{-j}$;
- the scheme corresponding to the mask reproduces all polynomials of degree $\leq 2n+1$,

$$\sum_k a_{j-2k} p(k) = p\left(\frac{j}{2}\right), \quad j \in \mathbb{Z}, \quad p \in \pi_{2n+1}$$

- the mask $\{a_{2j+1}\}_{-n-2}^{n+1}$ becomes the mask of the $(2n+4)$-Deslauriers and Dubuc scheme when we choose $v = 0$ and

$$w = (-1)^{n+1} a_{2n+3} = \frac{(n+2)}{2^{4n+5}(2n+3)} \binom{2n+3}{n+1}.$$

Furthermore, the mask generalizes some masks:

- when $w = 0$, the mask becomes the interpolating symmetric scheme;
- when $v = 0$, the mask generalizes the mask given by Choi et al. Actually, Choi et al. found an explicit mask up to $n = 3$. Since they found the masks step by step, it is difficult to apply their method in order to obtain general masks for $n \geq 4$.
- When $v = w = 0$, the mask becomes the DD mask.

6.2 Masks of Generalized Scheme

We consider the problem of finding a mask $\mathbf{a} = \{a_j\}_{j=-2N-3}^{2N+3}$ reproducing polynomials of degree $\leq 2N+1$, that is

$$\sum_k a_{j-2k} p(k) = p\left(\frac{j}{2}\right), \quad j \in \mathbb{Z}, \quad p \in \pi_{2N+1}. \tag{6.8}$$

Throughout this section, we let $v = a_{2N+2}$ and $w = a_{2N+3}$, for convenience's sake. Setting $j = 0$ in (6.23) and using (6.2), (6.3), the equation (6.3) implies

$$\sum_{k=-N-1}^{N+1} a_{-2k} L_{-j}(k) = \delta_{j,0}, \quad j = -N-1, \cdots, N. \tag{6.9}$$

We split the summation on the left-hand side of the equation (6.24) as

$$\begin{aligned} \sum_{k=-N-1}^{N+1} a_{-2k} L_{-j}(k) &= \sum_{k=-N}^{N+1} a_{-2k} L_{-j}(k) + a_{2N+2} L_{-j}(-N-1) \\ &= a_{2j} + a_{2N+2} L_{-j}(-N-1). \end{aligned}$$

Thus substituting (6.6) gives the explicit form of a_{2j} for $j = -N-1, \cdots, N$,

$$\begin{aligned} a_{2j} &= \delta_{j,0} - v L_{-j}(-N-1) \\ &= \delta_{j,0} + (-1)^{j+N+1} v \frac{(2N+2)!}{(N-j)!(N+j+1)!(N-j+1)} \\ &= \delta_{j,0} + (-1)^{j+N+1} \binom{2N+2}{N+j+1} v. \end{aligned} \tag{6.10}$$

Also setting $j = 1$ in (6.23), we get

$$\sum_{k=-N-1}^{N+2} a_{1-2k} L_{-j}(k) = L_{-j}\left(\frac{1}{2}\right), \quad j = -N-1, \cdots, N. \tag{6.11}$$

We split the summation on the left-hand side of the equation (6.25) as

$$\sum_{k=-N-1}^{N+2} a_{1-2k} L_{-j}(k) = \sum_{k=-N}^{N+1} a_{1-2k} L_{-j}(k) + a_{2N+3}[(L_{-j}(N+2) + L_{-j}(-N-1)].$$

By applying the relation (6.2), we get

$$\sum_{k=-N-1}^{N+2} a_{1-2k} L_{-j}(k) = a_{1+2j} + w[L_{-j}(N+2) + L_{-j}(-N-1)].$$

Using the identities (6.4)-(6.7), we have the explicit form for a_{2j+1}

$$\begin{aligned}
a_{2j+1} &= L_{-j}\left(\frac{1}{2}\right) - w[L_{-j}(N+2) + L_{-j}(-N-1)] \\
&= \frac{N+1}{2^{4N+1}}\binom{2N+1}{N}\frac{(-1)^j}{2j+1}\binom{2N+1}{N+j+1} \\
&+ (-1)^{j+N+1} w \binom{2N+1}{N+j+1}\frac{(2N+2)(2j+1)}{(N+j+2)(N-j+1)}, \quad j = -N-1, \cdots, N.
\end{aligned} \tag{6.12}$$

Now it remains to show that the mask $\{a_j\}_{j=-2N-3}^{2N+3}$ with a_{2j} as given in (6.10) and a_{2j+1} as given in (6.26) satisfies the conditions of symmetry and polynomial reproduction of degree $\leq 2N+1$, that is, for any polynomial $p(x)$ of degree $\leq 2N+1$,

$$\sum_k a_{j-2k} p(k) = p\left(\frac{j}{2}\right), \quad j \in \mathbb{Z}.$$

Actually, this proof is unnecessary, because this property is the starting point of the construction of the mask, as formulated in (6.23). From the expression of the mask, it is easy to see that $\{a_j\}_{j=-2N-3}^{2N+3}$ is symmetric, that is

$$a_{-j} = a_j, \quad j = 1, 2, \cdots, 2N+3.$$

Let $p(x)$ be any polynomial of degree $\leq 2N+1$. For an even integer $j = 2m$, we have

$$\begin{aligned}
\sum_k a_{j-2k} p(k) &= \sum_k a_{2m-2k} p(k) = \sum_k a_{2k} p(m-k) \\
&= p(m) - v \sum_{k=-N-1}^{N} L_{-k}(-N-1) p(m-k) + v p(m-N-1) \\
&= p\left(\frac{j}{2}\right),
\end{aligned}$$

where we used the relation (6.3),

$$\sum_{k=-N-1}^{N} L_{-k}(-N-1) p(m-k) = \sum_{k=-N}^{N+1} L_k(-N-1) p(m+k) = p(m-N-1).$$

And for an odd integer $j = 2m+1$, we have

$$\begin{aligned}
\sum_k a_{j-2k} p(k) &= \sum_k a_{2m+1-2k} p(k) = \sum_{k=-N-2}^{N+1} a_{2k+1} p(m-k) \\
&= w[p(m-N-1) + p(m+N+2)] \\
&+ \sum_{k=-N-1}^{N} \left\{ L_{-k}\left(\frac{1}{2}\right) - w[L_{-k}(N+2) + L_{-k}(-N-1)] \right\} p(m-k).
\end{aligned}$$

On the other hand, by the interpolation property of the Lagrange fundamental polynomials $\{L_j(x)\}_{j=-N}^{N+1}$, we have that

$$\begin{aligned}
\sum_{k=-N-1}^{N} L_{-k}\left(\frac{1}{2}\right) p(m-k) &= \sum_{k=-N}^{N+1} L_k\left(\frac{1}{2}\right) p(m+k) = p\left(m + \frac{1}{2}\right), \\
\sum_{k=-N-1}^{N} L_{-k}(N+2) p(m-k) &= \sum_{k=-N}^{N+1} L_k(N+2) p(m+k) = p(m+N+2), \\
\sum_{k=-N-1}^{N} L_{-k}(-N-1) p(m-k) &= \sum_{k=-N}^{N+1} L_k(-N-1) p(m+k) = p(m-N-1).
\end{aligned}$$

Hence we obtain that

$$\begin{aligned}
\sum_k a_{j-2k} p(k) &= w[p(m-N-1) + p(m+N+2)] \\
&+ \sum_{k=-N-1}^{N} \left\{ L_{-k}\left(\frac{1}{2}\right) - w[L_{-k}(N+2) + L_{-k}(-N-1)] \right\} p(m-k) \\
&= p\left(m + \frac{1}{2}\right) = p\left(\frac{j}{2}\right).
\end{aligned}$$

Thus, the mask $\{a_j\}_{j=-2N-3}^{2N+3}$ satisfies the polynomial reproduction of degree $\leq 2N+1$, which completes the proof.
Note that by applying $p(x) = 1$ to the relation (6.23), we have the identity

$$\sum_{j \in \mathbb{Z}} a_{2j} = \sum_{j \in \mathbb{Z}} a_{2j+1} = 1.$$

and when $v = 0$ the scheme becomes $(2N+4)$- point symmetric and interpolating scheme. However, it is not an interpolating scheme if $v \neq 0$ since, in this case , we have

$$a_{2j} \neq \delta_{j,0},$$

in general.

Remark

- We obtain the symmetric subdivision scheme which reproduces all polynomials of degree $\leq 2N+1$ and which is not interpolating.
- In case when $v = 0$, it becomes the $(2N+4)$-point Dyn symmetric and interpolating scheme.
- In case when $v = w = 0$, it becomes the $(2N+2)$-point DD scheme.
- S.W.Choi et al. [5] presented a new class of subdivision schemes. These schemes unified not only DD-scheme but the quadratic and cubic B-spline schemes. They proved the convergence, smoothness and approximation order. But they did not get the explicit masks of those schemes. They proposed the forms of the mask of S_L for $L = 1, \ldots, 10$. And they indicated the smoothness of S_L and the corresponding range of one parameter, which are obtained by computing $\|(\frac{1}{2}S_L)^{13}\|_\infty < 1$ with Maple 8.
- We obtain the mask of subdivision schemes which S.W.Choi et al. [5] has proposed(L is even) in case $w = 0$.

Example 2 *(1) For $N = 0$, we have the mask of non-interpolation scheme:*

$$\left[w, v, \frac{1}{2} - w, 1 - 2v, \frac{1}{2} - w, v, w\right].$$

Exercise 43 *Verify that the scheme with this mask generate C^4 curve.*

In case when $v = 0$, it becomes the 4-point N. Dyn scheme:

$$\left[w, 0, \frac{1}{2} - w, 1, \frac{1}{2} - w, 0, w\right].$$

In case $w = 0$ we get

$$\left[v, \frac{1}{2}, 1 - 2v, \frac{1}{2}, v\right].$$

(2) For $N = 1$, we have:

$$\left[w, v, -\frac{1}{16} - 3w, -4v, \frac{9}{16} + 2w, 1 + 6v, \frac{9}{16} + 2w, -4v, -\frac{1}{16} - 3w, v, w\right].$$

Exercise 44 *Find the smoothness of this scheme.*

In case when $v = 0$, it becomes the 6-point Weissman scheme:

$$\left[w, 0, -\frac{1}{16} - 3w, 0, \frac{9}{16} + 2w, 1, \frac{9}{16} + 2w, 0, -\frac{1}{16} - 3w, 0, w\right].$$

In the case of $v = w = 0$, it becomes the 4-point DD scheme:

$$\left[-\frac{1}{16}, 0, \frac{9}{16}, 1, \frac{9}{16}, 0, -\frac{1}{16}\right].$$

(3) For $N = 2$, we have mask:

$$[w, v, \frac{3}{256} - 5w, -6v, -\frac{25}{256} + 9w, 15v, \frac{75}{128} - 5w,$$
$$1 - 20v,$$
$$\frac{75}{128} - 5w, 15v, -\frac{25}{256} + 9w, -6v, \frac{3}{256} - 5w, v, w].$$

In case when $v = 0$, it becomes the 8-point ISSS:

$$[w, 0, \frac{3}{256} - 5w, 0, -\frac{25}{256} + 9w, 0, \frac{75}{128} - 5w,$$
$$1,$$
$$\frac{75}{128} - 5w, 0, -\frac{25}{256} + 9w, 0, \frac{3}{256} - 5w, 0, w].$$

In case of $v = w = 0$, it becomes the 6-point DD scheme:

$$\left[\frac{3}{256}, 0, -\frac{25}{256}, 0, \frac{75}{128}, 1, \frac{75}{128}, 0, -\frac{25}{256}, 0, \frac{3}{256}\right].$$

Exercise 45 *Find a symmetric mask $\mathbf{a} = \{a_j\}_{j=-2N-3}^{2N+3}$ reproducing polynomials of degree $\leq 2N+1$, that is*

$$\sum_k a_{j-2k} p(k) = p\left(\frac{2j+1}{4}\right), \quad j \in \mathbb{Z}, \quad p \in \pi_{2N+1} \tag{6.13}$$

where we set $a_{2N+2} = v, a_{2N+3} = w$.

We shall illustrate the performance of the subdivision scheme with a mask. We shall also present some numerical examples by setting the tension parameter v and w to various values, which shows how these parameters affect the limit function.

To the end, we introduce a symbol called the Laurent polynomial

$$a(z) := \sum_{n \in \mathbb{Z}} a_n z^n$$

of a mask $\{a_n\}_{n \in \mathbb{Z}}$ with finite support. With the symbol, we can simplify the presentation of the subdivision schemes and their analysis. As mentioned in Introduction, the Laurent polynomial $a(z)$ corresponding to a uniformly convergent subdivision scheme is divided by $z + 1$, that is to say, $a(-1) = 0$.

By the linearity, the smoothness of the limit function $S^\infty f^0$ for a given sequence f^0 of control points is equivalent to that of $\varphi = S^\infty \delta, \delta = \{\delta_{n,0}\}_{n \in \mathbb{Z}}$. The function φ is called the basic limit function of a subdivision scheme.

We investigate the smoothness range of two tension parameters v and w for the 4-point and 6-point subdivision schemes.

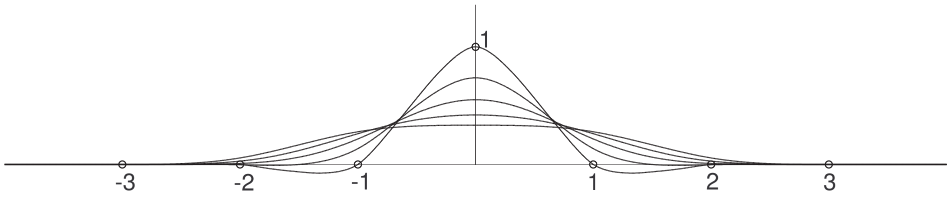

Figure 6.1: We illustrate a few basic limit functions. The effect of the tension parameters v and w on the shape of the basic limit functions of the 4-point subdivision scheme. Here, $w = (v - 1/8)/2, v = 0, 3/32, 3/16, 9/32, 3/8$ from the top at the origin.

The 4-point subdivision scheme with a mask: $[w,\ v,\ \frac{1}{2} - w,\ 1 - 2v,\ \frac{1}{2} - w,\ v,\ w]$.
From the given mask

$$\mathbf{a} = [w,\quad v,\quad \frac{1}{2} - w,\quad 1 - 2v,\quad \frac{1}{2} - w,\quad v,\quad w],$$

we have the mask of subdivision scheme S_1

$$\mathbf{a}_1 = 2[w, v - w, \frac{1}{2} - v, \frac{1}{2} - v, v - w, w],$$

where $a_1(z) = \frac{2z}{1+z}a(z)$. It is easy to verify that $a(z)$ and $a_1(z)$ satisfy the necessary condition (3.1) for the convergence of S and S_1. If

$$\|\frac{1}{2}S_1\|_\infty = \max\{|w| + |\frac{1}{2} - v| + |v - w|\} < 1,$$

then this scheme converges to continuous limit function. We have the mask of scheme S_2 by using relation $a_2(z) = \frac{2z}{1+z}a_1(z)$.

$$\mathbf{a}_2 = 4[w, v - 2w, \frac{1}{2} - 2v + 2w, v - 2w, w].$$

If

$$\|\frac{1}{2}S_2\|_\infty = \max\{4|w| + 2|\frac{1}{2} - 2v + 2w|, 4|v - 2w|\} < 1,$$

then this scheme is $C^1(\mathbb{R})$.

For C^2 continuity, $a_2(z)$ should satisfy necessary condition (3.1). This implies

$$w = \frac{v}{2} - \frac{1}{16}.$$

From the relation $a_3(z) = \frac{2z}{1+z}a_2(z)$, we have the mask of scheme S_3

$$\mathbf{a}_3 = 8[\frac{v}{2} - \frac{1}{16}, -\frac{v}{2} + \frac{3}{16}, -\frac{v}{2} + \frac{3}{16}, \frac{v}{2} - \frac{1}{16}].$$

and
$$\|\frac{1}{2}S_3\|_\infty = \max\{4|\frac{v}{2}-\frac{1}{16}|+4|-\frac{v}{2}+\frac{3}{16}|\} < 1,$$
which implies that $0 < v < \frac{1}{2}$. Hence for the case $w = \frac{v}{2} - \frac{1}{16}$ and $0 < v < \frac{1}{2}$, this scheme is $C^2(\mathbb{R})$.

For C^3 continuity, $a_3(z)$ should satisfy necessary condition (3.1), which is always true. The mask of S_4 is
$$\mathbf{a}_4 = 16[\frac{v}{2}-\frac{1}{16}, -v+\frac{1}{4}, \frac{v}{2}-\frac{1}{16}],$$
and
$$\|\frac{1}{2}S_4\|_\infty = \max\{16|\frac{v}{2}-\frac{1}{16}|, 8|-v+\frac{1}{4}|\} < 1,$$
which implies that $\frac{1}{8} < v < \frac{3}{8}$. This scheme is $C^3(\mathbb{R})$ in case $w = \frac{v}{2} - \frac{1}{16}$ and $\frac{1}{8} < v < \frac{3}{8}$. From the fact that $a_4(z)$ should satisfy necessary condition (3.1) for C^4 continuity, we get
$$v = \frac{3}{16}, \quad w = \frac{1}{32},$$
and we have the mask of scheme S_5
$$\mathbf{a}_5 = [1, 1],$$
and
$$\|\frac{1}{2}S_5\|_\infty = \frac{1}{2}.$$
Hence this scheme is $C^4(\mathbb{R})$.

	4-point Dyn scheme	proposed scehme
support of limit function	$[-3, 3]$	$[-3, 3]$
maximal smoothness	C^1	C^4

Table 6.1: Comparison of 4-point Dyn scheme and proposed scheme

In Table 6.1, we compare some properties of 4-point Dyn scheme with those of the proposed scheme. We can see that for a given same support of limit function, the proposed scheme provides good smoothness in Table 6.1.

We say that a subdivision scheme is said to be C^m if for any initial data the basic limit function has continuous derivatives up to order m. The segment $w = 1/2(v - 1/8)$ represents the ranges of C^2 and C^3 smoothness for $0 < v < 1/2$ and $1/8 < v < 3/8$, respectively. When $v = 3/16$ and $w = 1/32$, the scheme becomes the 6-th order B-spline scheme which induces C^4 smoothness, as known well.

The 6-point subdivision scheme with a mask: $[a_0, a_1, a_2, a_3, a_4, a_5] = [1+6v, \frac{9}{16}+2w, -4v, -\frac{1}{16} - 3w, v, w]$.

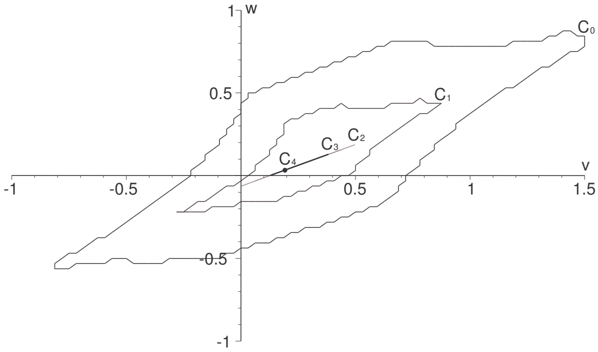

Figure 6.2: Ranges of v and w for the 4-point subdivision scheme S with MAPLE 8, Digits:=30.

Smoothness	Range of v	Range of w
C^0	given in Figure 2	given in Figure 2
C^1	given in Figure 2	given in Figure 2
C^2	$0 < v < 1/2$	$w = 1/2(v - 1/8)$
C^3	$1/8 < v < 3/8$	$w = 1/2(v - 1/8)$
C^4	$3/16$	$1/32$

Table 6.2: By computing $||(\frac{1}{2}S_m)^L||_\infty < 1, m = 1, 2, 3, 4, 5$ for 4-point subdivision scheme S, we obtain the ranges of v and$w$ with MAPLE 8, Digits:=30.

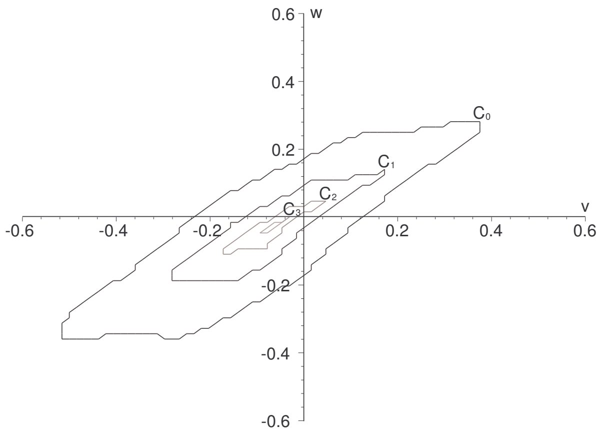

Figure 6.3: Ranges of v and w for the 6-point subdivision scheme S with MAPLE 8, Digits:=30.

6.3 Generalization of $(2n + 4)$-point approximating subdivision scheme

We present explicitly a general formula for the mask of $(2n+4)$-point approximating subdivision schemes with two parameters which reproduces all polynomials of degree $\leq 2n+1$. The proposed

Smoothness	Range of v	Range of w
C^0	given in Figure 3	given in Figure 3
C^1	given in Figure 3	given in Figure 3
C^2	given in Figure 3	given in Figure 3
C^3	given in Figure 3	given in Figure 3
C^4	$-0.0654296875000000 < v < -0.0290527343750000$	$w = v/2 + 3/256$
C^5	$-0.0468750000000000 < v < -0.0382050771680549$	$w = v/2 + 3/256$

Table 6.3: By computing $||(\frac{1}{2}S_m)^L||_\infty < 1, m = 1, 2, \cdots, 6$ for the 6-point subdivision scheme S, we obtain the ranges of v and w with MAPLE 8, Digits:=30.

scheme generalizes several subdivision schemes such as the Chainkin's algorithm, the 4-point approximating scheme and the $(2n+2)$-point approximating schemes.

6.3.1 Construction of scheme

We denote by P_{2n+1} the space of all polynomials of degree $\leq 2n+1$ for a nonnegative integer n. We define the Lagrange fundamental polynomials $\{L_k(x)\}_{k=-n}^{n+1}$ corresponding to the nodes $\{k\}_{k=-n}^{n+1}$ by

$$L_k(x) = \prod_{j\neq k, j=-n}^{n+1} \frac{x-j}{k-j}, \quad k = -n, \cdots, n+1, \tag{6.14}$$

for which

$$L_k(j) = \delta_{k,j}, \quad k, j = -n, \cdots, n+1, \tag{6.15}$$

and

$$\sum_{k=-n}^{n+1} p(k)L_k(x) = p(x), \quad p \in P_{2n+1}. \tag{6.16}$$

Then it is easy to see that for each $j = -n-1, \cdots, n$,

$$\begin{aligned} L_{-j}(\frac{1}{2}) &= (-1)^j \frac{(n+1)}{2^{4n+1}(2j+1)} \binom{2n+1}{n} \binom{2n+1}{n+j+1}, & (6.17)\\ L_{-j}(n+2) &= (-1)^{j+n+1} \frac{(2n+2)!}{(n-j)!(n+j+1)!(n+j+2)}, & (6.18)\\ L_{-j}(-n-1) &= (-1)^{j+n} \frac{(2n+2)!}{(n-j)!(n+j+1)!(n-j+1)}, & (6.19)\end{aligned}$$

$$L_{-j}(n+2) + L_{-j}(-n-1) = (-1)^{j+n} \binom{2n+1}{n+j+1} \frac{(2n+2)(2j+1)}{(n+j+2)(n-j+1)}, \tag{6.20}$$

and

$$L_{-j}(\frac{1}{4}) = \frac{\prod_{j=-n-1}^{n}(4j+1)}{(-1)^{n+1-j}4^{2n+1}(4j+1)(n-j)!(n+j+1)!}, \tag{6.21}$$

$$L_{-j}(\frac{3}{4}) = \frac{\prod_{j=-n-1}^{n}(4j+3)}{(-1)^{n+1-j}4^{2n+1}(4j+3)(n-j)!(n+j+1)!}. \tag{6.22}$$

These quantities are crucial to find the explicit form of masks considered in the following process.

We consider the problem of finding masks $\mathbf{a} = \{a_j\}_{j=-2n-4}^{2n+3}$ reproducing polynomials of degree $\leq 2n+1$, that is

$$\sum_k a_{j-2k} p(k) = p(\frac{2j+1}{4}), \quad j \in \mathbb{Z}, \quad p \in P_{2n+1}. \tag{6.23}$$

Throughout this section, we let $v = a_{2n+2} = a_{-2n-3}$ and $w = a_{2n+3} = a_{-2n-4}$, for convenience's sake. Setting $j = 0$ in (6.23) and using (6.15) and (6.16), the equation (6.23) implies

$$\sum_{k=-n-1}^{n+2} a_{-2k} L_{-j}(k) = L_{-j}(\frac{1}{4}), \quad j = -n-1, \cdots, n. \tag{6.24}$$

We split the summation on the left-hand side of the equation (6.24) as

$$\begin{aligned} \sum_{k=-n-1}^{n+2} a_{-2k} L_{-j}(k) &= \sum_{k=-n}^{n+1} a_{-2k} L_{-j}(k) + a_{2n+2} L_{-j}(-n-1) + a_{-2n-4} L_{-j}(n+2) \\ &= a_{2j} + a_{2n+2} L_{-j}(-n-1) + a_{-2n-4} L_{-j}(n+2). \end{aligned}$$

Thus we get the explicit form of a_{2j} for $j = -n-1, \cdots, n$,

$$a_{2j} = L_{-j}(\frac{1}{4}) - v L_{-j}(-n-1) - w L_{-j}(n+2). \tag{6.25}$$

Also setting $j = 1$ in (6.23), we get

$$\sum_{k=-n-1}^{n+2} a_{1-2k} L_{-j}(k) = L_{-j}(\frac{3}{4}), \quad j = -n-1, \cdots, n. \tag{6.26}$$

By splitting the summation on the left-hand side of the equation (6.26) and applying the relation (6.15), we get

$$\begin{aligned} \sum_{k=-n-1}^{n+2} a_{1-2k} L_{-j}(k) &= \sum_{k=-n}^{n+1} a_{1-2k} L_{-j}(k) + a_{-2n-3} L_{-j}(n+2) + a_{2n+3} L_{-j}(-n-1) \\ &= a_{1+2j} + a_{-2n-3} L_{-j}(n+2) + a_{2n+3} L_{-j}(-n-1). \end{aligned}$$

Hence we have the explicit form for a_{2j+1}

$$a_{2j+1} = L_{-j}(\frac{3}{4}) - v L_{-j}(n+2) - w L_{-j}(-n-1), \tag{6.27}$$

for $j = -n-1, \cdots, n$.

We can see that the proposed scheme with mask a_{2j} as given in (6.25) and a_{2j+1} as given in (6.27) satisfies the polynomial reproducing property up to degree $2n+1$, because this property is the starting point of the construction of the mask as formulated (6.23).

Example 3 *For $n = 0$, we have the mask*

$$[w, \quad v, \quad \frac{1}{4} + v - 2w, \quad \frac{3}{4} - 2v + w, \quad \frac{3}{4} - 2v + w, \quad \frac{1}{4} + v - 2w, \quad v, \quad w].$$

In case when $v = w = 0$, it becomes the Chaikin's scheme:

$$[\frac{1}{4}, \quad \frac{3}{4}, \quad \frac{3}{4}, \quad \frac{1}{4}].$$

If we set $w = -\frac{5}{128}, v = -\frac{7}{128}$, we have the mask of Dyn et al.[11] 4-point approximating scheme:

$$[\frac{-5}{128}, \frac{-7}{128}, \frac{35}{128}, \frac{105}{128}, \frac{105}{128}, \frac{35}{128}, \frac{-7}{128}, \frac{-5}{128}].$$

When we set $v = -\frac{3}{32} - w$, we get the same mask as Choi et al.[5] proposed for $L = 3$ case:

$$[w, \frac{-3}{32} - w, \frac{5}{32} - 3w, \frac{15}{16} + 3w, \frac{15}{16} + 3w, \frac{5}{32} - 3w, \frac{-3}{32} - w, w].$$

Also, when we set $v = \frac{7}{64}, w = \frac{1}{64}$, this subdivision scheme becomes the B-spline of degree 6 subdivision scheme:

$$[\frac{1}{64}, \quad \frac{7}{64}, \quad \frac{21}{64}, \quad \frac{35}{64}, \quad \frac{35}{64}, \quad \frac{21}{64}, \quad \frac{7}{64}, \quad \frac{1}{64}].$$

6.3.2 Analysis of scheme

From the given mask

$$\mathbf{a} = [w, \quad v, \quad \frac{1}{4} + v - 2w, \quad \frac{3}{4} - 2v + w, \quad \frac{3}{4} - 2v + w, \quad \frac{1}{4} + v - 2w, \quad v, \quad w],$$

we have

$$\mathbf{a_1} = 2[w, v - w, \frac{1}{4} - w, \frac{1}{2} - 2v + 2w, \frac{1}{4} - w, v - w, w],$$

where $a_1(z) = \frac{2z}{1+z}a(z)$. It is easy to verify that $a(z)$ and $a_1(z)$ satisfy the necessary condition (3.1) for the convergence of S and S_1. If

$$\|\frac{1}{2}S_1\|_\infty = \max\{2|w| + 2|\frac{1}{4} - w|, |\frac{1}{2} - 2v + 2w| + 2|v - w|\} < 1,$$

then this scheme converges to continuous limit function. We have the mask of S_2 using equation $a_2(z) = \frac{2z}{1+z}a_1(z)$.

$$\mathbf{a_2} = 4[w, v - 2w, \frac{1}{4} - v + w, \frac{1}{4} - v + w, v - 2w, w].$$

If

$$\|\frac{1}{2}S_2\|_\infty = \max\{2|w| + 2|\frac{1}{4} - v + w| + 2|v - 2w|\} < 1,$$

then this scheme is $C^1(R)$.

For C^2 continuity, $a_2(z)$ should satisfy (3.1), which is true. From the relation $a_3(z) = \frac{2z}{1+z}a_2(z)$, we have the mask of S_3

$$\mathbf{a_3} = 8[w, v-3w, \frac{1}{4} - 2v + 4w, v-3w, w].$$

For C^3 continuity, $a_3(z)$ should satisfy (3.1). This implies $v = 3w + \frac{1}{16}$. From this fact, we have

$$\mathbf{a_3} = 8[w, \frac{1}{16}, \frac{1}{8} - 2w, \frac{1}{16}, w],$$

and if

$$\|\frac{1}{2}S_3\|_\infty = \max\{8|w| + 4|\frac{1}{8} - 2w|, \frac{1}{2}\} < 1,$$

then this scheme is $C^2(R)$.

We get the mask of S_4

$$\mathbf{a_4} = 16[w, \frac{1}{16} - w, \frac{1}{16} - w, w],$$

and

$$\|\frac{1}{2}S_4\|_\infty = \max\{8|w| + 8|\frac{1}{16} - w|\} < 1,$$

which implies that $-\frac{1}{32} < w < \frac{3}{32}$. Hence this scheme is $C^3(R)$ in case $v = 3w + \frac{1}{16}$ and $-\frac{1}{32} < w < \frac{3}{32}$. We can see that $a_4(z)$ satisfy (3.1) for C^4 continuity. From the mask of S_4, we have the mask of scheme S_5

$$\mathbf{a_5} = 32[w, \frac{1}{16} - 2w, w].$$

From the necessary condition for C^5 continuity, we get $w = \frac{1}{64}, v = \frac{7}{64}$ and

$$\mathbf{a_5} = [\frac{1}{2}, 1, \frac{1}{2}].$$

Since

$$\|\frac{1}{2}S_5\|_\infty = \max\{\frac{1}{2}, \frac{1}{2}\} < 1,$$

this scheme is $C^4(R)$. We get the mask of S_6

$$\mathbf{a_5} = [1, 1],$$

and

$$\|\frac{1}{2}S_6\|_\infty = \max\{\frac{1}{2}, \frac{1}{2}\} < 1.$$

Hence this scheme is $C^5(R)$.

In Table 6.4, we compare support and maximal smoothness of 4-point Dyn scheme with those of the proposed scheme. We can see that for a given same support of limit function, the proposed scheme provides good smoothness.

	4-point approximating scheme	proposed scheme
support	[-4,3]	[-4,3]
maximal regularity	C^2	C^5

Table 6.4: Comparison of 4-point Dyn approximating scheme and proposed scheme

Chapter 7

$2n$-point Subdivision Scheme

In this chapter we construct a new general formula for the mask of $2n$-point interpolating symmetric subdivision schemes(ISSS) with parameter. Also, we introduce a new matrix formula for the mask of $2n$-point Deslauriers and Dubuc(DD) scheme. Also, we find some relations between the masks of the $(2n+4)$-point interpolating symmetric subdivision scheme and the $(2n+2)$-point Deslauriers and Dubuc scheme.

7.1 Introduction

Weissman [30] generated 6-point interpolating subdivision scheme of the form

$$\begin{aligned} f_{2i}^{k+1} &= f_i^k, \\ f_{2i+1}^{k+1} &= \left(\frac{9}{16}+2\theta\right)(f_i^k+f_{i+1}^k) - \left(\frac{1}{16}+3\theta\right)(f_{i-1}^k+f_{i+2}^k) + \theta(f_{i-2}^k+f_{i+3}^k). \end{aligned}$$

For $\theta = 0$ this scheme reduces to the 4-point scheme with $w = \frac{1}{16}$. Weissman proved that for $0 < \theta < 0.02$, this scheme guaranteed the continuity of the curvature of the curve.

Dyn [12] stated that we can construct the Dyn 4-point and the Weissman 6-point schemes by taking a convex combination of the two DD schemes. And K.P.Ko et al. [20] generated the mask of interpolating symmetric subdivision schemes–4-point and 6-point interpolating subdivision schemes, ternary 4-point interpolating scheme, butterfly scheme and modified butterfly scheme–by using symmetry and necessary condition for smoothness.

The mask of a subdivision scheme consists of a set of coefficients, which measure the influence of a value at a location on the values at neighboring locations after subdivision. Since the schemes we consider are interpolating, there is always the coefficient 1, due to the interpolating rule $f_{2i}^{k+1} = f_i^k$. Here are few examples of ISSS:

- The mask of 4-point DD scheme:

$$\left[-\frac{1}{16}, 0, \frac{9}{16}, 1, \frac{9}{16}, 0, -\frac{1}{16}\right].$$

- The mask of 4-point Dyn scheme:

$$\left[-w, 0, \frac{1}{2}+w, 1, \frac{1}{2}+w, 0, -w\right].$$

- The mask of 6-point Weissman scheme:

$$\left[w, 0, -3w-\frac{1}{16}, 0, 2w+\frac{9}{16}, 1, 2w+\frac{9}{16}, 0, -3w-\frac{1}{16}, 0, w\right].$$

We use here the convention that the coefficients which are not specified in the mask are all zero. There is no unique or best way of obtaining a mask. Deslauries and Dubuc obtained the mask by using polynomial reproducing property. K.P.Ko et al.[20] obtained the mask of interpolating symmetric subdivision schemes-binary 2n-point schemes, ternary 4-point scheme, butterfly scheme, and modified butterfly scheme-by using symmetry and necessary condition for smoothness.

7.2 Relation between ISSS and DD Scheme

We can easily obtain the mask of 4-point, 6-point, 8-point and 10-point ISSS by using the same method.

- 4-point scheme: $[a_1, a_3] = [w_2 + \frac{1}{2}, -w_2]$.
- 6-point scheme: $[a_1, a_3, a_5] = [2w_3 + \frac{9}{16}, -3w_3 - \frac{1}{16}, w_3]$.
- 8-point scheme: $[a_1, \ldots, a_7] = [5w_4 + \frac{75}{128}, -9w_4 + \frac{25}{256}, 5w_4 - \frac{3}{256}, -w_4]$.
- 10-point scheme: $[a_1, \ldots, a_9] = [14w_5 + \frac{1225}{2048}, -28w_5 - \frac{245}{2048}, 20w_5 + \frac{49}{2048}, -7w_5 - \frac{5}{2048}, w_5]$.

From these masks, our objective is to find out a general form for the mask of $2n$-point ISSS with one parameter w_i:

$$[a^{2n}_{-2n+1}, a^{2n}_{-2n+3}, \cdots, a^{2n}_{-1}, a^{2n}_{1}, \cdots, a^{2n}_{2n-3}, a^{2n}_{2n-1}].$$

Corresponding to result above, by symmetric property, the coefficients of w_i can be arranged as follows:

Table 7.1: Coefficients of w_i.

2 (n=1)	1	0	0	0	0	0
4 (n=2)	1	-1	0	0	0	0
6 (n=3)	2	-3	1	0	0	0
8 (n=4)	5	-9	5	-1	0	0
10 (n=5)	14	-28	20	-7	1	0
12 (n=6)	42	-90	75	-35	9	-1

From the TABLE above, the coefficient of w_i can be expressed in a matrix form:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 2 & -3 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 5 & -9 & 5 & -1 & 0 & 0 & \cdots & 0 \\ 14 & -28 & 20 & -7 & 1 & 0 & \cdots & 0 \\ 42 & -90 & 75 & -35 & 9 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & (-1)^{n-1} \end{bmatrix} := \begin{bmatrix} A_1^1 & 0 & \cdots & 0 \\ A_1^2 & A_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1^n & A_2^n & \cdots & A_n^n \end{bmatrix}.$$

We found out that matrix A satisfied the particular rule. And we derived the following formula for the elements of matrix A.

$$A_i^n = \begin{cases} A_i^{n-1} - A_{i+1}^{n-1} & i = 1; n = 2, 3 \ldots \\ (-1)^{n-1} & i = n \\ 2A_i^{n-1} - A_{i-1}^{n-1} - A_{i+1}^{n-1} & i = 2, 3 \ldots, n; n = 3, 4, \ldots \\ 0 & i > n, \end{cases} \tag{7.1}$$

where we define $A_1^1 := 1$ for convenience's sake. Villiers [28] has already given a general formula for the mask of $2n$-point DD scheme explicitly. The Dubuc-Deslauriers mask sequence $\mathbf{a}$ has explicit formulation.

$$a_{2j+1} = \frac{n}{2^{4n-3}} \binom{2n-1}{n-1} \frac{(-1)^j}{2j+1} \binom{2n-1}{n+j}, \quad j = -n, -n+1 \ldots, n-1.$$

Throughout this section, we denote by $\{a_{2i+1}^{DD,2n+2}\}$ the mask of the $(2n+2)$-point DD scheme given by

$$a_{2i+1}^{DD,2n+2} = \frac{n+1}{2^{4n+1}} \binom{2n+1}{n} \frac{(-1)^i}{2i+1} \binom{2n+1}{n+i+1}, \quad i = -n-1, \cdots, n \tag{7.2}$$

and by $\{a_{2i+1}^{ISSS,2n+4}\}$ the mask of $(2n+4)$-point interpolating symmetric scheme given by

$$a_{2i+1}^{ISSS,2n+4} = a_{2i+1}^{DD,2n+2} + w_{n+2} A_{i+1}^{n+2}, \quad i = -n-1, \cdots, n, \tag{7.3}$$

Table 7.2: Masks of D-D scheme

2 (n=1)	$a_1^{DD,2} = \frac{1}{2}$	0	0	0	0
4 (n=2)	$a_1^{DD,4} = \frac{9}{16}$	$a_3^{DD,4} = -\frac{1}{16}$	0	0	0
6 (n=3)	$a_1^{DD,6} = \frac{150}{256}$	$a_3^{DD,6} = -\frac{25}{256}$	$a_5^{DD,6} = \frac{3}{256}$	0	0
8 (n=4)	$a_1^{DD,8} = \frac{1225}{2048}$	$a_3^{DD,8} = -\frac{245}{2048}$	$a_5^{DD,8} = \frac{49}{2048}$	$a_7^{DD,8} = -\frac{5}{2048}$	0

where w_{n+2} is a free(tension) parameter given by $w_{n+2} = (-1)^{n+1} a_{2n+3}^{ISSS,2n+4}$ and A_{i+1}^{n+2} are the quantities given by

$$A_{i+1}^{n+2} = (-1)^i \binom{2n+1}{n+i+1} \frac{(2n+2)(2i+1)}{(n+i+2)(n-i+1)}, \quad i = -n-1, \cdots, n. \tag{7.4}$$

We omit the subscript of w_{n+2} when there occurs no confusion.

In the section, we find some relations between the masks $\{a_{2i+1}^{DD,2n+2}\}$ and $\{a_{2i+1}^{ISSS,2n+4}\}$ by observation on the masks only for $n = 0, 1, 2, 3$. Comparing between the masks of $(2n+4)$-point ISSS with parameter and $(2n+2)$-point DD scheme, we can find out the relation between $2n$-point ISSS and $(2n-2)$-point DD scheme. For example, for 4-point scheme($n = 2$), the mask of 4-point scheme has the following property:

$$a_1^{ISSS,4} = w + \frac{1}{2} = A_1^2 w + a_1^{DD,2}, \quad a_3^{ISSS,4} = -w = A_2^2 w + a_3^{DD,2}.$$

And if we set $w = 1/16$, then we can get the mask of 4-point of DD scheme from 4-point ISSS scheme, that is, for $w_2 = 1/16$

$$a_1^{ISSS,4}|_{w=\frac{1}{16}} = \frac{9}{16} = a_1^{DD,4}, \quad a_3^{ISSS,4}|_{w=\frac{1}{16}} = -\frac{1}{16} = a_3^{DD,4}.$$

For 6-point scheme($n = 3$), we can find the rule.

$$a_1^{ISSS,6} = A_1^3 w + a_1^{DD,4}, \quad a_3^{ISSS,6} = A_2^3 w + a_3^{DD,4}, \quad a_5^{ISSS,6} = A_3^3 w + a_5^{DD,4}.$$

Also, for $w_3 = 3/256$, we can get the mask of 6-point DD scheme.

$$a_1^{ISSS,6}|_{\frac{3}{256}} = a_1^{DD,6} = \frac{150}{256}, \quad a_3^{ISSS,6}|_{\frac{3}{256}} = a_3^{DD,6} = -\frac{25}{256}, \quad a_5^{ISSS,6}|_{\frac{3}{256}} = a_5^{DD,4} = \frac{3}{256}.$$

In case of 8-point scheme, we can easily check the property.

$$\begin{aligned} a_1^{ISSS,8} &= A_1^4 w + a_1^{DD,6}, a_3^{ISSS,8} = A_2^4 w + a_3^{DD,6}, \\ a_5^{ISSS,8} &= A_3^4 w + a_5^{DD,6}, a_7^{ISSS,8} = A_4^4 w + a_7^{DD,6}. \end{aligned}$$

And for $w_4 = 5/2048$, we can obtain the mask of 8-point DD scheme.

Obviously, for $n = 0, 1, 2, 3$, when we set the parameter w_{n+2} by

$$w_{n+2} = (-1)^{n+1} a_{2n+3} = \frac{(n+2)}{2^{4n+5}(2n+3)} \binom{2n+3}{n+1}, \tag{7.5}$$

we can see that the masks of $(2n+4)$-point DD scheme and $(2n+2)$-point DD scheme satisfy the relation

$$a_{2i-1}^{DD,2n+4} = a_{2i-1}^{DD,2n+2} + A_i^{n+2} w_{n+2}, \quad i = 1, 2, \ldots, n, \quad n = 0, 1, 2, 3. \tag{7.6}$$

In fact, we predict the value w_{n+2} given in (7.5) from the equation

$$w_{n+2} = (a_{2i-1}^{DD,2n+4} - a_{2i-1}^{DD,2n+2})/A_i^{n+2}$$

where $a_{2i-1}^{DD,2n+4}, a_{2i-1}^{DD,2n+2}$, and A_i^{n+2} are already known factors as (7.2) and (7.4).
However, this relation holds in general by reference [28], we have the general from of w_n

$$w_n = \frac{n}{2^{4n-3}} \binom{2n-1}{n-1} \frac{1}{2n-1}. \tag{7.7}$$

Theorem 30 *For an integer $n \geq 0$, let $\{a_{2i+3}^{ISSS,2n+4}\}$ and $\{a_{2i+1}^{DD,2n+2}\}$ be the masks of the $(2n+4)$-point interpolating symmetric scheme in (7.3) and of the $(2n+2)$-point DD scheme in (7.2), respectively. When we take*

$$v = a_{2n+2} = 0 \quad \textit{and} \quad w_{n+2} = (-1)^{n+1} a_{2n+3} = \frac{(n+2)}{2^{4n+5}(2n+3)} \binom{2n+3}{n+1},$$

then the $(2n+4)$-point interpolating symmetric scheme becomes the $(2n+4)$-point DD scheme.

Proof. It is straightforward that from the equations (7.2), (7.3), and (7.4), we have

$$\begin{aligned}
a_{2i+1}^{ISSS,2n+4} &= a_{2i+1}^{DD,2n+2} + w_{n+2} A_{i+1}^{n+2} \\
&= \frac{n+1}{2^{4n+1}} \binom{2n+1}{n} \frac{(-1)^i}{2i+1} \binom{2n+1}{n+i+1} \\
&\quad + \frac{n+2}{2^{4n+5}} \binom{2n+3}{n+1} \frac{1}{2n+3} (-1)^i \binom{2n+1}{n+i+1} \frac{(2n+2)(2i+1)}{(n+i+2)(n-i+1)} \\
&= \frac{n+2}{2^{4n+5}} \binom{2n+3}{n+1} \frac{(-1)^i}{2i+1} \binom{2n+3}{n+i+2} \\
&= a_{2i+1}^{DD,2n+4},
\end{aligned}$$

which shows the theorem. ♣
Finally, we can get a new general formula for the mask of $2n$-point ISSS.

Theorem 31 *We have a relation between the masks of $(2n+4)$-point interpolating symmetric subdivision scheme and of $(2n+2)$-point DD scheme:*

$$\begin{aligned}
a_{2i+1}^{ISSS,2n+4} &= a_{-2i-1}^{ISSS,2n+4}, \\
a_{2i+1}^{ISSS,2n+4} &= a_{2i+1}^{DD,2n+2} + A_{i+1}^{n+2} w_{n+2}, \quad i = 0, 1, 2, 3 \ldots, n+1, n \in \mathbb{N}_0,
\end{aligned} \tag{7.8}$$

where

$$A_{i+1}^{n+2} = \begin{cases} A_{i+1}^{n+1} - A_{i+2}^{n+1} & i = 0; \quad n = 0, 1, \dots \\ (-1)^{n+1} & i = n+1 \\ 2A_{i+1}^{n+1} - A_{i}^{n+1} - A_{i+2}^{n+1} & i = 1, 2, \dots, n; \quad n = 1, 2, \dots \\ 0 & i > n+1. \end{cases} \tag{7.9}$$

and $a_1^{DD,2} = \frac{1}{2}, \quad a_3^{DD,2} = \cdots = a_{2n+3}^{DD,2n+2} = 0.$

Proof. From (6.26), we see that the masks $a_{2i+1}^{ISSS,2n+4}$ and $a_{2i+1}^{DD,2n+2}$ have the relation

$$a_{2i+1}^{ISSS,2n+4} = a_{2i+1}^{DD,2n+2} + A_{i+1}^{n+2} w, \quad i = -n-1, \cdots, n$$

where we set $w_{n+2} = (-1)^{n+1} a_{2n+3}^{ISSS,2n+4}$ for the sake of our argument and $a_{2i+1}^{DD,2n+2}$ are the $(2n+2)$-point DD masks given in (7.2)

$$a_{2i+1}^{DD,2n+2} = \frac{n+1}{2^{4n+1}} \binom{2n+1}{n} \frac{(-1)^i}{2i+1} \binom{2n+1}{n+i+1}, \quad i = -n-1, \cdots, n,$$

and A_i^{n+2} are quantities given in (7.4)

$$A_{i+1}^{n+2} = (-1)^i \binom{2n+1}{n+i+1} \frac{(2n+2)(2i+1)}{(n+i+2)(n-i+1)}, \quad i = -n-1, \cdots, n.$$

It follows from the expression (7.4) that the relations in (7.9) hold for $n \geq 0$, which completes the proof. ♣

Example 4 *For* $n = 1$, *we have*

$$\begin{aligned} a_1^{ISSS,6} &= a_1^{DD,4} + A_1^3 w = \frac{9}{16} + 2w, \\ a_3^{ISSS,6} &= a_3^{DD,4} + A_2^3 w = -\frac{1}{16} - 3w, \\ a_5^{ISSS,6} &= a_5^{DD,4} + A_3^3 w = w. \end{aligned}$$

Therefore we generate the mask of 6-point interpolating symmetric subdivision scheme.

$$\left[w, -\frac{1}{16} - 3w, \frac{9}{16} + 2w, \frac{9}{16} + 2w, -\frac{1}{16} - 3w, w \right].$$

As a corollary, we obtain a new general matrix formula for the mask of $2n$-point DD scheme.

Corollary 2 *Let* $\mathbf{a} = [a_1^{DD,2n}, a_3^{DD,2n,} \dots, a_{2n-1}^{DD,2n}]^T$ *be the vector of the mask of the* $2n$*-point DD scheme and* $\mathbf{w} = [w_1, w_2, \cdots w_n,]^T$, *the vector of* w_i *given by*

$$w_1 = \frac{1}{2} \quad and \quad w_i = \frac{(i)}{2^{4i-3}(2i-1)} \binom{2i-1}{i-1}, \quad i = 2, 3, \dots, n.$$

Then the matrix formula for the mask of $2n$-point DD scheme is:

$$\mathbf{a} = A^T * \mathbf{w}.$$

That is, this formula can be written in the form

$$\mathbf{a} = \begin{bmatrix} a_1^{DD,2n} \\ a_3^{DD,2n} \\ \vdots \\ a_{2n-1}^{DD,2n} \end{bmatrix} = \begin{bmatrix} A_1^1 & A_1^2 & \cdots & A_1^n \\ 0 & A_2^2 & \cdots & A_2^n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & A_n^n \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = A^T * \mathbf{w},$$

where we define $A_1^1 = 1$ and $w_1 = 1/2$ so that $a_1^{DD,2} = A_1^1 w_1$.

Proof. It is straightforward from Theorem and (7.8). ♣

Example 5 *For $n = 1$, we have*

$$A = \begin{bmatrix} A_1^1 & 0 & 0 \\ A_1^2 & A_2^2 & 0 \\ A_1^3 & A_2^3 & A_3^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -3 & 1 \end{bmatrix},$$

and $w_n = [w_1, w_2, w_3] = [\frac{1}{2}, \frac{1}{16}, \frac{3}{256}]$. We obtain the mask of 6-point DD scheme:

$$\left[\frac{3}{256}, -\frac{25}{256}, \frac{150}{256}, \frac{150}{256}, -\frac{25}{256}, \frac{3}{256}\right].$$

The results of smoothness are shown in Table. We have expressed the range of w with Maple.

$2n+2$	Range of w	Smoothness
4	$0 < w < .183$	C^1
6	$0 < w < .042$	C^2
8	$.0016 < w < .0084$	C^3
10	$.0005 < w < .0016$	C^4

Table 7.3: By computing $||(\frac{1}{2}S_m)^{10}||_\infty < 1, m = 2, 3, 4, 5$ for $2n+2$-point ISSS scheme S, we obtain the range of w with MAPLE 8, Digits:=30.

Chapter 8

Non-Stationary Subdivision Scheme

8.1 Non-Stationary Subdivision Scheme Reproducing Circle

We present a non-stationary subdivision scheme for interpolating a set of given data. This scheme is a generalization of the 4-point DD scheme to the non-stationary case. This scheme reproduces elements of the linear space spanned by $1, \cos(\alpha x), \sin(\alpha x)$.

Preliminaries

Given a set of control points $P^0 = \{p_i \in \mathbb{R}^d | i \in \mathbb{Z}\}$ at level 0, a subdivision scheme $\{S_{a_k}\}$ generates a new set of control points $P^k = \{p_i^k | i \in \mathbb{Z}\}$ at the kth level by a subdivision rule:

$$p_i^k = (S_{a_k} P^{k-1})_i = (S_{a_k} S_{a_{k-1}} \cdots S_{a_1} P^0)_i = \sum_{j \in \mathbb{Z}} a_{i-2j}^{(k)} p_j^{k-1}, i \in \mathbb{Z}$$

where the set $\mathbf{a}^{(k)} = \{a_i^{(k)} | i \in \mathbb{Z}, a_i^{(k)} \neq 0\}$ is finite for every $k \in \mathbb{Z}_+$. If the mask is independent of k, then the scheme is called stationary, otherwise it is called non-stationary. To each subdivision scheme $\{S_{a_k}\}$ defined by the mask $\{a_i^k\}$, we assign the polynomial

$$a_k(z) = \sum_{i \in \mathbb{Z}} a_i^{(k)} z^i, \quad k \geq 1$$

called the kth level Laurent polynomial.

Definition 5 *Two subdivisions $\{S_{a_k}\}$ and $\{S_{b_k}\}$ are asymptotically equivalent if*

$$\sum_{k=1}^{\infty} \|S_{a_k} - S_{b_k}\|_\infty < \infty,$$

where $\|S_{a_k}\|_\infty = \max\{\sum_\alpha |a_{2\alpha}^{(k)}|, \sum_\alpha |a_{1+2\alpha}^{(k)}|\}$.

The following result relates the convergence of a non-stationary scheme to its asymptotically equivalent stationary scheme.

Theorem 32 *Let $\{S_{a_k}\}$ and $\{S_a\}$ be two asymptotically equivalent subdivision schemes having finite masks of the same support. Suppose $\{S_{a_k}\}$ is non-stationary and $\{S_a\}$ is stationary. If $\{S_a\}$ is C^m and*

$$\sum_{k=0}^{\infty} 2^{mk} \|S_{a_k} - S_a\|_\infty < \infty,$$

then the non-stationary scheme $\{S_{a_k}\}$ is C^m.

8.1.1 The Subdivision Scheme

To define non-stationary scheme, for $k \geq 0$, we denote

$$w_k = \frac{\sin^2(\frac{\alpha}{2^{k+2}})}{2\sin(\frac{\alpha}{2^k})\sin(\frac{\alpha}{2^{k+1}})} = \frac{1}{16\cos^2(\frac{\alpha}{2^{k+2}})\cos(\frac{\alpha}{2^{k+1}})}.$$

Exercise 46 *Consider the function*

$$L(x) = \sum_{j=0}^{3} f(x_j) L_j(x),$$

where

$$L_j(x) = c\left(\frac{x - x_j}{2}\right) \prod_{k=0, k\neq j}^{3} \frac{s(\frac{x-x_k}{2})}{s(\frac{x_j - x_k}{2})},$$

where $c(x) = \cos(\alpha x)$ and $s(x) = \sin(\alpha x)$.
Define $x_j = 0, 1, 2, 3$ and let $w_0 = \sin^2(\alpha/4)/[2\sin(\alpha/2)\sin\alpha]$,

- *Verify that*

$$\begin{aligned} L_0(3/2) = L_3(3/2) &= -\frac{\sin^2(\alpha/4)}{2\sin(\alpha/2)\sin(\alpha)}, \\ L_1(3/2) = L_2(3/2) &= \frac{\sin^2(3\alpha/4)}{2\sin(\alpha/2)\sin(\alpha)}. \end{aligned}$$

- *Prove that*

$$L(3/2) = -w_0[f(x_0) + f(x_3)] + \left(\frac{1}{2} + w_0\right)[f(x_1) + f(x_2)].$$

Some estimates of w_k which are useful in scheme are given in the following lemma.

Lemma 2 *For $k \geq 0$ and $0 \leq \alpha \leq \pi/2$:*

- $\frac{1}{8} \geq w_k \geq \frac{1}{16}$,

- $|w_k - \frac{1}{16}| \le C/2^{2k}$

We now present the basic algorithm which is a non-stationary subdivision scheme.
Algorithm: Given control points $\{p_i^0 \in \mathbb{R} | i = -2, -1, \ldots, n+2\}$, the control points $\{p_i^{k+1} | i = -2, -1, \ldots, 2^{k+1}n + 1\}$ at level $k+1$ are given by the following rule:

$$\begin{aligned} p_{2i}^{k+1} &= p_i^k, \\ p_{1+2i}^{k+1} &= \left(\frac{1}{2} + w_k\right)[p_i^k + p_{i+1}^k] - w_k[p_{i-1}^k + p_{i+2}^k]. \end{aligned}$$

If we take $w_k = \frac{1}{16}$ for all k, then this scheme coincides with 4-point DD scheme.
Let us denote non-stationary scheme by $\{S_{a^k}\}$. The mask of $\{S_{a^k}\}$ at the kth level is:

$$a_{-3}^{(k)} = a_3^{(k)} = -w_k, \quad a_0^{(k)} = 1, \quad a_{-1}^{(k)} = a_1^{(k)} = \frac{1}{2} + w_k.$$

Note that 4-point DD scheme S has the mask:

$$a_{-3} = a_3 = -\frac{1}{16}, \quad a_0 = 1, \quad a_{-1} = a_1 = \frac{9}{16}.$$

Theorem 33 *The non-stationary scheme $\{S_{a^k}\}$ is asymptotically equivalent to the stationary scheme $\{S\}$. Moreover, the limit function belongs to $C^1(\mathbb{R})$.*

Proof. We have

$$\sum_{i\in\mathbb{Z}} |a_{2i}^{(k)} - a_{2i}| = 0, \quad \sum_{i\in\mathbb{Z}} |a_{1+2i}^{(k)} - a_{1+2i}| = 4|w_k - 1/16|.$$

By Lemma, we get

$$\sum_{i\in\mathbb{Z}} |a_{1+2i}^{(k)} - a_{1+2i}| \le \frac{4C}{2^{2k}},$$

and

$$\|S_{a^k} - S\|_\infty \le \frac{4C}{2^{2k}}.$$

Hence $\sum_{k=0}^{\infty} \|S_{a^k} - S\|_\infty < \infty$ and the schemes $\{S_{a^k}\}$ and $\{S\}$ are asymptotically equivalent. It is clear that

$$\sum_{k=0}^{\infty} 2^k \|S_{a^k} - S\|_\infty < \infty.$$

Since the 4-point DD scheme S is C^1, the scheme $\{S_{a^k}\}$ is also C^1.♣

8.1.2 Basic Limit Function

The basic limit function of the scheme $\{S_{a^k}\}$ is the limit function of the scheme for the data $p_i^0 = \delta_{i,0}$. By Theorem, the basic limit function denoted by F belongs to C^1. Let

$$D_n := \{j/2^n | j \in \mathbb{Z}\}.$$

It is easy to check that restriction of F to D_n satisfies $F(\frac{i}{2^k}) = p_i^k$ for all k and F is symmetric about the y-axis. Next we show that F is a compactly supported function with support in $[-3, 3]$. i.e., F vanishes outside $[-3, 3]$.

Exercise 47 *Verify that F is symmetric and has support* $[-3, 3]$.

8.1.3 Reproducing Circle

In this section we show that certain functions can be reconstructed by this non-stationary scheme. It is easy to check that $p_i^k = 1$ for all i then $p_i^{k+1} = 1$ for all i. This shows that the function $f(x) = 1$ is reproduced by this scheme.
The functions $\cos(\alpha x)$ and $\sin(\alpha x)$ can be also reconstructed by this scheme which follows from the next lemma.

Lemma 3 *Let $k \geq 0$ and $p_j^k = \cos(j\frac{\alpha}{2^k}), -2 \leq j \leq 2^k n + 2$. Then we have for $-1 \leq i \leq 2^k n$,*

$$p_{2i}^{k+1} = \cos\left(\frac{2i\alpha}{2^{k+1}}\right), \quad p_{2i}^{k+1} = \cos\left((2i+1)\frac{\alpha}{2^{k+1}}\right).$$

Proof. Note that

$$p_{2i}^{k+1} = p_i^k = \cos\left(i\frac{\alpha}{2^k}\right) = \cos\left(2i\frac{\alpha}{2^{k+1}}\right).$$

Since

$$\frac{1}{2} + w_k = \frac{\sin^2(3\frac{\alpha}{2^{k+2}})}{2\sin(\frac{\alpha}{2^{k+2}})\sin(\frac{\alpha}{2^k})},$$

and

$$\begin{aligned} p_i^k + p_{i+1}^k &= 2\cos\left((2i+1)\frac{\alpha}{2^{k+1}}\right)\cos\left(\frac{\alpha}{2^{k+1}}\right), \\ p_{i-1}^k + p_{i+2}^k &= 2\cos\left((2i+1)\frac{\alpha}{2^{k+1}}\right)\cos\left(3\frac{\alpha}{2^{k+1}}\right), \end{aligned}$$

we get

$$p_{2i+1}^{k+1} = \cos\left((2i+1)\frac{\alpha}{2^{k+1}}\right). \clubsuit$$

Exercise 48 *Verify the last equation.*

Corollary 3 *If we choose a set of equidistant points*

$$p_i^0 = \left(\cos\left(k\frac{2\pi}{n}\right), \sin\left(k\frac{2\pi}{n}\right)\right), \quad k = 0, 1, \ldots, n$$

on a circle, and $\alpha = 2\pi/n$, then the limit curve is the original unit circle.

8.2 A Non-Stationary Interpolating Scheme

The goal of this section is to generate a new non-stationary interpolating subdivision scheme with a tension parameter, that is capable of reproducing circles and all other conic sections exactly. We are going to define an interpolating 4-point scheme that unifies three different curves schemes which are capable of representing trigonometric, polynomial and hyperbolic functions respectively. We begin by introducing three schemes separately and derive a common insertion rule which unifies all the schemes.

Consider an insertion rule can be obtained by interpolation with a function from the linear space spanned by $\{1, x, \cos(x), \sin(x)\}$, working in the following way.
Without loss of generality, we define the points $\{p_i^k | i \in \mathbb{Z}\}$ at level k and interpolate the data $p_{i+h}^k, h = -1, 0, 1, 2$ by a function of the form $f(x) = a_0 + a_1 x + a_2 \cos(x) + a_3 \sin(x)$, we get the following system of equations

$$\begin{aligned}
f\left(-\frac{t}{2^k}\right) &= p_{i-1}^k, \\
f(0) &= p_i^k, \\
f\left(\frac{t}{2^k}\right) &= p_{i+1}^k, \\
f\left(\frac{2t}{2^k}\right) &= p_{i+2}^k,
\end{aligned}$$

from which it follows

$$\begin{aligned}
p_{i-1}^k &= a_0 - a_1 \frac{t}{2^k} + a_2 \cos\left(\frac{t}{2^k}\right) - a_3 \cos\left(\frac{t}{2^k}\right), \\
p_i^k &= a_0 + a_2, \\
p_{i+1}^k &= a_0 + a_1 \frac{t}{2^k} + a_2 \cos\left(\frac{t}{2^k}\right) + a_3 \cos\left(\frac{t}{2^k}\right), \\
p_{i+2}^k &= a_0 + 2a_1 \frac{t}{2^k} + a_2 \left(\cos^2\left(\frac{t}{2^k}\right) - \sin^2\left(\frac{t}{2^k}\right)\right) + 2a_3 \sin\left(\frac{t}{2^k}\right) \cos\left(\frac{t}{2^k}\right).
\end{aligned}$$

Exercise 49 *Solve the above system with MatLab, Maple or Mathematica.*

To get the insertion rule, we only need to compute the value of the interpolating function $f(x)$ at the grid point $\frac{t}{2^{k+1}}$, defining the new point $p_{i+\frac{1}{2}}^k$ as a linear combination of the four consecutive points $p_{i-1}^k, p_i^k, p_{i+1}^k, p_{i+2}^k$.

$$\begin{aligned}
f\left(\frac{t}{2^{k+1}}\right) &= \frac{2\cos(t/2^k)\cos(t/2^{k+1}) - \cos(t/2^{k+1}) - 1}{4\cos(t/2^{k+1})(\cos(t/2^k) - 1)}(p_i^k + p_{i+1}^k) + \\
&- \frac{1}{8\cos(t/2^{k+1})(1 + \cos(t/2^{k+1}))}(p_{i-1}^k + p_{i+2}^k) = p_{i+\frac{1}{2}}^k.
\end{aligned}$$

In this way we define the rule:

$$\begin{aligned} p_{2i}^{k+1} &= p_i^k, \\ p_{1+2i}^{k+1} &= p_{i+\frac{1}{2}}^k = \left(\frac{1}{2} + w_k\right)(p_i^k + p_{i+1}^k) - w_k(p_{i-1}^k + p_{i+2}^k), \end{aligned}$$

where

$$w_k = \frac{1}{8\cos(t/2^{k+1})(1+\cos(t/2^{k+1}))}.$$

Secondly, we drive the insertion rule by interpolation with a function from the span of the polynomial $\{1, x, x^2, x^3\}$, proceeding in the same way as above.

Exercise 50 *Consider the function of the form*

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$

Prove that

$$\begin{aligned} a_0 &= p_i^k, \\ a_1 &= \frac{2^k(-2p_{i-1}^k - 3p_i^k + 6p_{i+1}^k - p_{i+2}^k)}{6t}, \\ a_2 &= \frac{2^{2k}(p_{i-1}^k - 2p_i^k + p_{i+1}^k)}{2t^2}, \\ a_3 &= \frac{2^{3k}(-p_{i-1}^k + 3p_i^k - 3p_{i+1}^k + p_{i+2}^k)}{6t^3} \end{aligned}$$

We define the rule:

$$\begin{aligned} p_{2i}^{k+1} &= p_i^k, \\ p_{1+2i}^{k+1} &= \left(\frac{1}{2} + w_k\right)(p_i^k + p_{i+1}^k) - w_k(p_{i-1}^k + p_{i+2}^k), \end{aligned}$$

where

$$w_k = \frac{1}{16}.$$

Note that in this case the scheme coincides with 4-point DD scheme.

Finally we drive the insertion rule by interpolation with a function from the span of the four functions $\{1, x, \cosh(x), \sinh(x)\}$, proceeding in the same way.

To get insertion rule, we define the new point $p_{i+\frac{1}{2}}^k$ as a linear combination of the four consecutive points:

$$\begin{aligned} f\left(\frac{t}{2^{k+1}}\right) &= \frac{2\cosh(t/2^k)\cosh(t/2^{k+1}) - \cosh(t/2^{k+1}) - 1}{4\cosh(t/2^{k+1})(\cosh(t/2^k) - 1)}(p_i^k + p_{i+1}^k) + \\ &- \frac{1}{8\cosh(t/2^{k+1})(1+\cosh(t/2^{k+1}))}(p_{i-1}^k + p_{i+2}^k) = p_{i+\frac{1}{2}}^k. \end{aligned}$$

In this way we define the rule:

$$\begin{aligned} p_{2i}^{k+1} &= p_i^k, \\ p_{1+2i}^{k+1} &= \left(\frac{1}{2}+w_k\right)(p_i^k+p_{i+1}^k) - w_k(p_{i-1}^k+p_{i+2}^k), \end{aligned}$$

where

$$w_k = \frac{1}{8\cosh(t/2^{k+1})(1+\cosh(t/2^{k+1}))}.$$

8.2.1 Unified Scheme Reproducing Conics

All three of these schemes can be expressed by a single insertion rule of the form

$$\begin{aligned} p_{2i}^{k+1} &= p_i^k, \\ p_{1+2i}^{k+1} &= \left(\frac{1}{2}+w_k\right)(p_i^k+p_{i+1}^k) - w_k(p_{i-1}^k+p_{i+2}^k), \end{aligned}$$

where

$$w_k = \frac{1}{8v_{k+1}(1+v_{k+1})},$$

with v_{k+1} equal to $\cos(t/2^{k+1}), 1, \cosh(t/2^{k+1})$ depending on the scheme we choose. However, the following result makes these cases unnecessary once the initial parameter v_0 has been chosen.

Theorem 34 *If $v_0 \in (-1,\infty)$, for all three cases the parameter v_k satisfy the recurrence*

$$v_{k+1} = \sqrt{\frac{1+v_k}{2}}, \quad k = 0,1,2,\ldots. \tag{8.1}$$

In this way, we can formulate a unified non-stationary subdivision scheme which combines the three previous schemes in a very elegant manner.
Given a set of control points P^0 at level 0, we define a unified subdivision scheme that generates a new set of control points P^{k+1}, by the rule:

$$\begin{aligned} p_{2i}^{k+1} &= p_i^k, \qquad (8.2) \\ p_{1+2i}^{k+1} &= \frac{(2v_{k+1}+1)^2}{8v_{k+1}(1+v_{k+1})}(p_i^k+p_{i+1}^k) - \frac{1}{8v_{k+1}(1+v_{k+1})}(p_{i-1}^k+p_{i+2}^k), \end{aligned}$$

where the parameter v_{k+1} is easily updated at each subdivision step through equation (8.1). Thus, given parameter v_k, the subdivision rules are derived by first computing v_{k+1} using equation (8.1) and by then substituting v_{k+1} into equation. As a consequence, depending on the choice of the initial parameter v_0, we can generate the scheme exact for trigonometric, polynomial and hyperbolic functions whenever $-1 < v_0 < 1, v_0 = 1, v_0 > 1$, respectively.

Remark: Whenever $v_0 = 1$, by choosing four initial points on the parabola (t, at^2), the limit curve is exactly the parabola itself. If we take as initial the four point $p_i^0 = (\cos(it), \sin(it)), i = -1,0,1,2, \quad t = \pi/2$, are equidistant points on the circle, and choosing the initial tension parameter $v_0 = \cos(\pi/2) = 0$, the resulting limit curve is the circle itself. If we take as initial points $p_i^0 = (a\cos(i\pi/2), b\sin(i\pi/2))$ with $v_0 = 0$, the resulting limit curve is exactly ellipse itself. In the same way, we can get hyperbola by taking $p_i^0 = (a\cosh(it), b\sinh(it))$ with $v_0 = \cosh(t)$.

8.2.2 Convergence Analysis

The goal of this section is to show that the non-stationary subdivision scheme converges to a $C^1(\mathbb{R})$ limit curve. Ir order to prove this, we use the convergence of a non-stationary scheme to its asymptotically equivalent stationary scheme.

It is easy to verify that

$$\lim_{k\to\infty} v_k = 1.$$

Lemma 4 *Due to the recurrence (8.1), the parameter v_k satisfy the relation*

$$\frac{1-v_k}{1-v_{k-1}} < \frac{1}{2},$$

for any $v_{k-1} \in (-1, \infty)$.

Theorem 35 *The non-stationary scheme is asymptotically equivalent to the four point DD scheme. Moreover, it generates C^1 limit curves.*

Exercise 51 *Read the paper [2] and [24].*

Exercise 52 *Goal: We want to construct non-stationary approximating four point subdivision scheme. The mask of this scheme can be obtained by interpolating from the space by $\{1, x, \cos(x), \sin(x)\}$. Without loss of generality, given the point $\{p_i^k\}$, which are defined on a grid $2^{-k}t\mathbb{Z}$, and interpolating the data $(\frac{t\cdot h}{2^k}, p_{i+h}^k), h = -1, 0, 1, 2$ by a function of the form*

$$f(x) = a + bx + c\cos(x) + d\sin(x)$$

We get the system of equations

$$\begin{aligned}
f\left(-\frac{t}{2^k}\right) &= p_{i-1}^k, \\
f(0) &= p_i^k, \\
f\left(\frac{t}{2^k}\right) &= p_{i+1}^k, \\
f\left(\frac{2t}{2^k}\right) &= p_{i+2}^k.
\end{aligned}$$

Using Maple, we have the coefficients of $f(x)$

$$\begin{aligned}
a &= \frac{-p_{i-1}^k + 2\cos(t/2^k)p_i^k - p_{i+1}^k}{2\left(\cos(t/2^k) - 1\right)}, \\
b &= \frac{p_{i-1}^k - (1 + 2\cos(t/2^k))p_i^k + (1 + 2\cos(t/2^k))p_{i+1}^k - p_{i+2}^k}{\frac{t}{2^{k-1}}\left(\cos(t/2^k) - 1\right)}, \\
c &= \frac{p_{i-1}^k - 2p_i^k + p_{i+1}^k}{2(\cos(t2^k) - 1)}, \\
d &= \frac{-\cos(t/2^k)p_{i-1}^k + (1 + 2\cos(t2^k))p_i^k - (1 + 2\cos(t/2^k))p_{i+1}^k + p_{i+2}^k}{2\sin(t/2^k)\left(\cos(t/2^k) - 1\right)}.
\end{aligned}$$

To obtain subdivision scheme, we only need to compute the values of the interpolating function $f(x)$ *at the grid point* $t/2^{k+2}$ *and* $3t/2^{k+2}$ *respectively, define the new point* $p^k_{i+1/4}$ *and* $p^k_{i+3/4}$ *as a linear combination of the 4-point* $p^k_{i-1}, p^k_i, p^k_{i+1}, p^k_{i+2}$

$$\begin{aligned} f^{k+1}_{2i} &= f(t/2^{k+2}) = p^{k+1}_{i+1/4} = Ap^k_{i-1} + Bp^k_i + Cp^k_{i+1} + Dp^k_{i+2}, \\ f^{k+1}_{2i+1} &= f(3t/2^{k+2}) = p^{k+1}_{i+3/4} = Dp^k_{i-1} + Cp^k_i + Bp^k_{i+1} + Ap^k_{i+2}, \end{aligned}$$

where set $v_{k+1} := \cos(t/2^{k+1})$ *and*

$$\begin{aligned} A &= -\frac{1}{32}\frac{6v^2_{k+2} + 2v_{k+2} - 1}{v_{k+1}(1+v_{k+1})v_{k+2}(1+v_{k+2})}, \\ D &= -\frac{1}{32}\frac{2v^2_{k+2} + 2v_{k+2} + 1}{v_{k+1}(1+v_{k+1})v_{k+2}(1+v_{k+2})}. \end{aligned}$$

Find the masks B *and* C*.*

Chapter 9

Eigen Analysis of Subdivision

Let us assume that the subdivision matrix S has eigenvalues $\{\lambda_0, \lambda_1, \ldots, \lambda_{m-1}\}$ and corresponding left eigenvectors $\{x_0, x_1, \ldots, x_{m-1}\}$, respectively, with eigenvalues organized in decreasing order $|\lambda_i| \geq |\lambda_{i+1}|$. The following summarizes some important properties of subdivision surfaces in relation to the eigen structure of the subdivision matrix.

- Affine invariance: The subdivision scheme is affine invariant if and only if $\lambda_0 = 1$.
- Convergence: A subdivision scheme converges if and only if $1 = \lambda_0 > \lambda_1$.
- C^1 continuity: The limit of the control vertex v_0 is C^1 continuous provided a) the characteristic map of the subdivision is regular and injective and b) the sub-dominant eigenvalues satisfy

  $$1 = \lambda_0 > \lambda_1 \geq \lambda_2 > \lambda_3 \cdots .$$

  or preferably the following for binary subdivision schemes

  $$1 = \lambda_0 > \frac{1}{2} = \lambda_1 = \lambda_2 > \lambda_3 \cdots .$$

- Bounded curvature: The quality of curvature can be evaluated by

  $$\rho = \lambda_3/\lambda_1^2.$$

  In case $\rho < 1$, one obtains flat/zero curvature. In case $\rho > 1$, the curvature would diverge. Only in case $\rho = 1$, one achieves bounded curvature. For well behaved binary subdivision surfaces, the sub-dominant eigenvalues should satisfy

  $$1 = \lambda_0 > \frac{1}{2} = \lambda_1 = \lambda_2 > \frac{1}{4} = \lambda_3 = \lambda_4 = \lambda_5 > \lambda_6 \cdots$$

  for obtaining bounded curvatures at the limit position.

- The corresponding limit position of a control vertex v_0 is defined by

$$v_0^\infty = x_0^T v^0.$$

The tangent vectors at the limit position are defined by

$$c_1 = x_1^T v^0, \quad c_2 = x_2^T v^0.$$

The surface normal is defined by

$$n = c_1 \times c_2.$$

9.1 Subdivision Limit Position

The limit behavior of a stationary subdivision scheme can be analyzed by examining the structure of the eigen decomposition of subdivision matrix S. For a non-defective $n \times n$ matrix S, if there is n linearly independent eigenvectors v_i corresponding eigenvalues λ_i, then it is possible to diagonalize S by transforming S by the eigenvectors and their inverse.

$$S = V \Lambda V^{-1}.$$

If the subdivision curves/surfaces is C^0, then all the vertices in the local neighborhood vector P^∞ will shrink to the same point

$$P^\infty = \lim_{k \to \infty} P^k = S^\infty P^0.$$

If each of the rows of S sum to one, then S will have one eigenvector consisting of all ones $v_0 = [1 \quad 1 \quad \cdots \quad 1]^t$ and at least one eigenvalue equal to one $\lambda_0 = 1$.
If $1 = \lambda_0 > \lambda_i$, then S is C^0 continuous.

$$\begin{aligned}
S &= V \Lambda V^{-1} \\
&= \begin{bmatrix} v_0 & v_1 & \cdots & v_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \lambda_0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} v_0^{-1} & \rightarrow \\ v_1^{-1} & \rightarrow \\ \cdots & \rightarrow \\ v_n^{-1} & \rightarrow \end{bmatrix} \\
&= \begin{bmatrix} 1 & v_1 & \cdots & v_n \\ 1 & \downarrow & & \downarrow \\ \cdots & & & \\ 1 & & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} v_0^{-1} & \rightarrow \\ v_1^{-1} & \rightarrow \\ \cdots & \rightarrow \\ v_n^{-1} & \rightarrow \end{bmatrix}.
\end{aligned}$$

Using the eigen decomposition of S and since $1 > \lambda_i$, we have

$$\begin{aligned}
S^\infty &= V\Lambda^\infty V^{-1} \\
&= \begin{bmatrix} 1 & v_1 & \cdots & v_n \\ 1 & \downarrow & & \downarrow \\ \cdots & & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} v_0^{-1} & \rightarrow \\ v_1^{-1} & \rightarrow \\ \cdots & \rightarrow \\ v_n^{-1} & \rightarrow \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} v_0^{-1} & \rightarrow \\ v_1^{-1} & \rightarrow \\ \cdots & \rightarrow \\ v_n^{-1} & \rightarrow \end{bmatrix} \\
&= \begin{bmatrix} v_0^{-1} & \rightarrow \\ v_0^{-1} & \rightarrow \\ \cdots & \rightarrow \\ v_0^{-1} & \rightarrow \end{bmatrix}.
\end{aligned}$$

Because all of the rows of S^∞ are identical, all of the vertices of the limit local neighborhood P^∞ converges to the same position.

$$P^\infty = S^\infty P^0 = \begin{bmatrix} v_0^{-1} & \rightarrow \\ v_0^{-1} & \rightarrow \\ \cdots & \rightarrow \\ v_0^{-1} & \rightarrow \end{bmatrix} P^0.$$

9.2 Subdivision Derivatives

It is also possible to compute derivatives of subdivision curves/surfaces from the subdivision matrix S. We define the first derivative of the subdivision for a parameter direction u.

$$\frac{\partial S}{\partial u} = \lim_{k\to\infty} \frac{1}{\lambda_1^k} \frac{p_{i+1}^k - p_i^k}{\|p_{i+1}^0 - p_i^0\|}.$$

The limit of the finite difference between different pairs of points along the same parameter direction u will be the same if the subdivision is C^1 continuous. By similar reasoning, a second finite difference between these quantities scaled by $1/\lambda_2^k$ defines the second derivatives with respect to u.

$$\begin{aligned}
\frac{\partial^2 S}{\partial u^2} &= \lim_{k\to\infty} \frac{1}{\lambda_2^k} \frac{(p_{i+1}^k - p_i^k) - (p_i^k - p_{i-1}^k)}{\|p_{i+1}^0 - p_i^0\|^2}, \\
\frac{\partial^3 S}{\partial u^3} &= \lim_{k\to\infty} \frac{1}{\lambda_3^k} \frac{(p_{i+2}^k - 2p_{i+1}^k + p_i^k) - (p_{i+1}^k - 2p_i^k + p_{i-1}^k)}{\|p_{i+1}^0 - p_i^0\|^3}.
\end{aligned}$$

9.3 Example: Uniform Cubic B-spline

We can write down the subdivision matrix S for uniform cubic B-spline

$$S = \frac{1}{8}\begin{bmatrix} 1 & 6 & 1 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 6 & 1 \end{bmatrix}.$$

Since the eigen decomposition of S for cubic B-spline

$$\begin{aligned} S &= V\Lambda V^{-1} \\ &= \begin{bmatrix} 1 & -1 & 1 & 1 & 0 \\ 1 & -\frac{1}{2} & \frac{2}{11} & 0 & 0 \\ 1 & 0 & -\frac{1}{11} & 0 & 0 \\ 1 & \frac{1}{2} & \frac{2}{11} & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & \frac{11}{6} & -\frac{22}{6} & \frac{11}{6} & 0 \\ 1 & -3 & 3 & -3 & 0 \\ 0 & -1 & 3 & -3 & 1 \end{bmatrix}, \end{aligned}$$

we can calculate the limit position

$$\begin{aligned} V_{0,0,0} &= v_0^{-1} P^0 \\ &= \begin{bmatrix} 0 & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & 0 \end{bmatrix} \begin{bmatrix} V_{-3,-2,-1} \\ V_{-2,-1,0} \\ V_{-1,0,1} \\ V_{0,1,2} \\ V_{1,2,3} \end{bmatrix} \\ &= \frac{1}{6}V_{-2,-1,0} + \frac{4}{6}V_{-1,0,1} + \frac{1}{6}V_{0,1,2}. \end{aligned}$$

The limit of finite difference between adjacent vertices in the local neighborhood will be the same if the subdivision is C^1 continuous. Difference p_{i+1}^k and p_i^k is the same as difference the corresponding rows $i+1$ and i if S^k and the apply that new matrix to P^0.

$$\frac{\partial S^k}{\partial u} = D_u S^k,$$

where

$$D_u = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

From this fact, we get

$$\frac{\partial S^\infty}{\partial u} = \lim_{k\to\infty} \frac{1}{\lambda_1^k}\frac{\partial S^k}{\partial u} = \begin{bmatrix} 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

Therefore, we get the first derivatives of S

$$\frac{\partial V_{0,0,0}}{\partial u} = \frac{\partial S^{\infty}}{\partial u} P^0 = -\frac{1}{2} V_{-2,-1,0} + \frac{1}{2} V_{0,1,2}.$$

Exercise 53 *1. Calculate the second derivative of S.*

2. Prove that the third derivative of the cubic spline is discontinuous.

Appendix A

Binary Refinement Rules

Let $P^k = \{P_i^k | i \in \mathbb{Z}\}$ be control points at refinement level k and $\mathbf{a} = \{a_i : i \in \sigma(\mathbf{a}) \subset \mathbb{Z}\}$ be the mask. Let $\sigma(\mathbf{a}) = \{j | a_j \neq 0\}$ be the support of the mask $\mathbf{a}$.

The refinement rules are:

$$\begin{aligned} P_{2i}^{k+1} &= \sum_{2j \in \sigma(\mathbf{a})} a_{2j} P_{i-j}^k, \\ P_{1+2i}^{k+1} &= \sum_{1+2j \in \sigma(\mathbf{a})} a_{1+2j} P_{i-j}^k. \end{aligned}$$

This refinement rules can be written as combined rule:

$$P_i^{k+1} = \sum_{i-2j \in \sigma(\mathbf{a})} a_{i-2j} P_j^k.$$

Let P^k be control polygon at level k, i.e., the polygonal line through the control points $\{P_i^k\}$ and $P^k(t)$ be a parametric representation of P^k:

$$P^k(t) = \frac{t - 2^{-k} i}{2^{-k}} P_{i+1}^k + \frac{2^{-k}(i+1) - t}{2^{-k}} P_i^k, \quad 2^{-k} i \leq t \leq 2^{-k}(i+1).$$

A.1 Basic Limit Function

A convergent subdivision scheme S_a with mask $\mathbf{a}$ defines a basic limit function

$$\phi_a = S_a^\infty \delta,$$

with δ the initial data

$$\delta_0 = 1, i = 0, \quad \delta_i = 0, i \neq 0.$$

By the linearity and uniformity of the refinement rule (each refinement rule operates in the same way at all locations), we have

$$(S_a^\infty P^0)(t) = \sum_i P_i^0 \phi(t - i).$$

Thus $\phi(t) \neq 0$. And $\mathrm{supp}(\phi_a)$ is the convex hull of $\sigma(\mathbf{a})$.

In the above example, the basic limit function is the hat function.

A.2 Refinement Equation

Starting from $P_i^0 = \delta_i$, after one refinement step

$$P_i^1 = \sum_j a_{i-2j} P_j^0 = \sum_j a_{i-2j} \delta_j = a_i.$$

Thus support of P^1 is $\sigma(a)$.

By the stationarity of the scheme

$$S_a^\infty P^1(t) = \sum_i P_i^1 \phi_a \left(2(t - \frac{i}{2}) \right),$$

Since $S_a^\infty \delta = S_a^\infty P^1$, we have

$$\phi_a(t) = \sum_i a_i \phi_a(2t - i).$$

This is a refinement equation. Any basic limit function of a convergent subdivision scheme satisfies a refinement equation (two-scale equation) with the coefficients of the mask.

A.3 The Symbol

The symbol of a mask $\mathbf{a} = \{a_i | i \in \sigma(\mathbf{a})\}$ is the Laurent Polynomial

$$a(z) = \sum_{i \in \sigma(\mathbf{a})} a_i z^i.$$

Necessary conditions for convergence of the subdivision scheme S_a is

$$\sum_{2i \in \sigma(\mathbf{a})} a_{2i} = \sum_{1+2i \in \sigma(\mathbf{a})} a_{1+2i} = 1,$$

or in terms of the symbol

$$a(1) = 2, \quad a(-1) = 0.$$

A.4 The Formalism of Laurent Polynomial

We define generating function of control point P^k as

$$F^k(z) = \sum_i P_i^k z^i.$$

The refinement rule $P_i^{k+1} = \sum_j a_{i-2j} P_j^k$ implies the formal equation since the coefficient of z^i in $F^{k+1}(z)$ is P_i^{k+1} and the coefficient of z^i in $a(z)F^k(z^2)$ is $\sum_j a_{i-2j} P_j^k$

$$F^{k+1}(z) = a(z) F^k(z^2).$$

For a convergent scheme S_a, since $a(1) = 2, a(-1) = 0$, we have

$$a(z) = (1+z)q(z), \quad q(1) = 1,$$

where $q(z) := \sum_i q_i z^i$.

How S_q is related to S_a? We have the following equation from the fact $F^{k+1}(z) = a(z)F^k(z^2)$ and $a(z) = (1+z)q(z)$

$$\begin{aligned}
(1-z)F^{k+1}(z) &= q(z)[(1-z^2)F^k(z^2)] \\
(1-z)F^{k+1}(z) &= \sum_i (P_i^{k+1} - P_{i-1}^{k+1})z^i = \sum_i (\Delta P^{k+1})_i z^i \\
(1-z^2)F^k(z^2) &= \sum_i (P_i^k - P_{i-1}^k)z^{2i} = \sum_i (\Delta P^k)_i z^{2i}.
\end{aligned}$$

Therefore we can conclude

$$(\Delta P^{k+1})_i = \sum_j q_{i-2j}(\Delta P^k)_j,$$

that is

$$\Delta S_a P^k = S_q \Delta P^k.$$

A.5 Convergence Analysis

- A scheme S_a with a symbol $a(z)$ can not be convergent if $a(1) \neq 2$ or $a(-1) \neq 0$.
- A scheme S_a with a symbol $a(z) = (1+z)q(z)$ is convergent $\Longleftrightarrow$ S_q is contractive.
- A scheme S_q is contractive if $S_q^\infty P^0 = 0$ for all initial data P^0.
- S_q is contractive if and only if for some positive integer L, $\|S_q^L\|_\infty < 1$.

The symbol of $S_a^L, a^{[L]}(z)$ satisfies

$$\begin{aligned}
a^{[L]}(z) &= a(z)a^{[L-1]}(z^2), \\
F^{k+L}(z) &= a^{[L]}(z)F^k(z^{2^L})
\end{aligned}$$

Exercise 54 *Prove the above equation by induction.*

Exercise 55 *Find an explicit form of $a^{[L]}(z)$ in terms of $a(z)$.*

A.6 What is the Norm of S_a^L?

From the refinement rule of S_a (2 refinement rule)

$$P_i^{k+1} = \sum_j a_{i-2j}P_j^k,$$

we have

$$\|P_i^{k+1}\| \le \left(\sum_j |a_{i-2j}|\right) \max_j \|P_j^k\|$$

and we can calculate the norm of S_a:

$$\|S_a\|_\infty = \max\left\{\sum_j |a_{2j}|, \sum_j |a_{1+2j}|\right\}.$$

Let $a^{[L]}(z) = \sum_i a_i^{[L]} z^i, \quad F^{k+L}(z) = \sum_i P_i^{k+L} z^i$ and well-known fact $F^{k+L}(z) = a^{[L]}(z) F^k(z^{2^L})$, we have the 2^L refinement rules and norm of S_a^L:

$$\begin{aligned} P_i^{k+L} &= \sum_i a_{i-2^L j}^{[L]} P_j^k, \\ \|S_a^L\|_\infty &= \max\left\{\sum_j |a_{i-2^L j}^{[L]}|, i = 0, 1, \ldots, 2^L - 1\right\}. \end{aligned}$$

A.7 The Algebra of Symbols

If $b(z), c(z)$ be symbols of convergent subdivision schemes S_b and S_c, then $a(z) = \frac{b(z)c(z)}{2}$ is a symbol of a convergent scheme S_a with

$$\phi_a = \phi_b * \phi_c = \int_R \phi_b(\cdot - t)\phi_c(t)dt = \int_R \phi_b(t)\phi_c(\cdot - t)dt.$$

A.8 The Symbol $(1 + z)$

Consider the symbol $b(z) = 1 + z, \quad b_0 = b_1 = 1$. It is easy to verify that $(S_b\delta)_i = 0, i \neq 0, 1, \quad (S_b\delta)_i = 1, i = 0, 1$ and

$$S_b^\infty \delta = \chi_{[0,1)}.$$

A.9 A Smoothing Factor

If $a(z) = \frac{(1+z)}{2} c(z)$, then we have $\phi_a = \phi_c * \chi_{[0,1)}$ and

$$\phi_a(x) = \int_R \phi_c(t)\chi_{[0,1)}(x - t)dt = \int_{x-1}^x \phi_c(t)dt,$$

From this fact we get

$$\phi_a'(x) = \phi_c(x) - \phi_c(x - 1).$$

Conclusion: If ϕ_c has l continuous derivatives then ϕ_a has $(l + 1)$ continuous derivatives. Therefore $\frac{1+z}{2}$ is a smoothing factor.

Since

$$F^{k+1}(z) = a(z)F^k(z^2) = \frac{1+z}{2}c(z)F^k(z^2),$$

and

$$F^k(z) = \sum_i (S_a^k P^0)_i z^i = \sum_i P_i^k z^i,$$

and

$$c(z)F^k(z^2) = \sum_i (S_c(S_a^k P^0))_i z^i = \sum_i (S_c P^k)_i z^i,$$

we have

$$F^{k+1}(z) = \frac{1+z}{2}[c(z)F^k(z^2)] = \sum_i \frac{1}{2}\left[(S_c P^k)_i + (S_c P^k)_{i-1}\right] z^i$$

Two-stage refinement rules:

- Perform the refinement rules of S_c.
- Average the refined control polygon.

The limit of two-stage refinement rules has one more continuous derivatives than the limit of S_c. We can see that m repetitions of average-stage correspond to the symbol

$$a(z) = \left(\frac{1+z}{2}\right)^m c(z)$$

Exercise 56 *Prove that B-spline with the symbol $a(z) = 2^{-m}(1+z)^{m+1}$ belongs to C^{m-1}. (hint: $a(z) = \left(\frac{1+z}{2}\right)^{m-1}\frac{(1+z)^2}{2}$. And $c(z)/z$ is the symbol of the two-point scheme which generates $C^0(\mathbb{R})$).*

A.10 Analysis of Bivariate Scheme

We can easily see that two polynomials $(1-z_1)a(z_1,z_2)$ and $(1-z_2)a(z_1,z_2)$ are zeros at $(-1,1),(1,-1),(-1,-1)$ and $(1,1)$. Therefore, the following decomposition can be performed (but, not uniquely)

$$\begin{aligned}(1-z_1)a(z_1,z_2) &= b_{11}(z_1,z_2)(1-z_1^2) + b_{12}(z_1,z_2)(1-z_2^2),\\ (1-z_2)a(z_1,z_2) &= b_{21}(z_1,z_2)(1-z_1^2) + b_{22}(z_1,z_2)(1-z_2^2).\end{aligned}$$

We define the bivariate first order difference operator Δ as

$$(\Delta f)_{i,j} = \begin{bmatrix} f_{i,j} - f_{i-1,j} \\ f_{i,j} - f_{i,j-1} \end{bmatrix}.$$

We denote by F^k the generating function of f^k. We can see that

$$\Delta F^k(z_1,z_2) = \begin{bmatrix} 1-z_1 \\ 1-z_2 \end{bmatrix} F^k(z_1,z_2).$$

Using the refinement relation $F^{k+1}(z_1, z_2) = a(z_1, z_2)F^k(z_1^2, z_2^2)$, we have

$$\Delta F^{k+1}(z_1, z_2) = \begin{bmatrix} b_{11}(z_1, z_2) & b_{12}(z_1, z_2) \\ b_{21}(z_1, z_2) & b_{22}(z_1, z_2) \end{bmatrix} \begin{bmatrix} 1 - z_1^2 \\ 1 - z_2^2 \end{bmatrix} F^k(z_1^2, z_2^2).$$

And since

$$\begin{bmatrix} 1 - z_1^2 \\ 1 - z_2^2 \end{bmatrix} F^k(z_1^2, z_2^2) = (\Delta F^k)(z_1^2, z_2^2),$$

we get

$$\Delta F^{k+1}(z_1, z_2) = B(z_1, z_2)(\Delta F^k)(z_1^2, z_2^2),$$

where $B(z_1, z_2)$ is a matrix bivariate scheme

$$B(z_1, z_2) = \begin{bmatrix} b_{11}(z_1, z_2) & b_{12}(z_1, z_2) \\ b_{21}(z_1, z_2) & b_{22}(z_1, z_2) \end{bmatrix},$$

which corresponds to the matrix refinement rule

$$(S_B f)_i = \sum_{j \in \mathbb{Z}^2} B_{i-2j} f_j.$$

As in the univariate case, the scheme S_a is convergent if S_B is contractive. The contractivity of scheme can be verified by checking that $\|S_B^L\|_\infty$ is less than 1. Since

$$B^L(z_1, z_2) = B(z_1, z_2)B(z_1^2, z_2^2) \cdots B(z_1^{2^{L-1}}, z_2^{2^{L-1}}),$$

yields

$$\|S_B^L\|_\infty = \max \left\| \sum_{j \in \mathbb{Z}^2} |B_{i-j2^L}| \right\|_\infty, \quad i = 0, 1, \ldots, 2^L - 1,$$

where $|A|$ denotes the matrix having elements that are the absolute value of the elements of A, and $\|A\|_\infty$ denotes the usual matrix infinite norm.

Appendix B

Joint Spectral Radius Analysis

We already know the formalism of Laurent polynomial of binary SS:

$$F^{m+1}(z) = a(z)F^m(z^2).$$

To check if the scheme generates a continuous limit curve, that is C^0, we have to check if the difference scheme $b(z) = a(z)/(1+z)$ is contractive. We herby present the **joint spectral radius** analysis of schemes, and examine the relation between the two methods.
Let us consider a univariate scheme with support $[0, s]$.

$$f_j^{m+1} = \sum_{i\in\mathbb{Z}} a_{j-2i} f_i^m, \quad a(z) = \sum_{i=0}^{s} a_i z^i.$$

Since the support of the basic limit function ϕ is also $[0, s]$, and the limit function is

$$f(x) = \sum_i f_i^0 \phi(x - i),$$

the values of f on $[0, 1]$ are fully determined by $u^0 = (f_{-s+1}^0, \ldots, f_0^0)^t$.
The values on $[0, 1/2]$ are determined by $u^{0,0} = (f_{-s+1}^1, \ldots, f_0^1)^t$.
The values on $[1/2, 1]$ are determined by $u^{0,1} = (f_{-s+2}^1, \ldots, f_1^1)^t$.
For example, we present a mask with support $[0, 4]$, i.e. the scheme coefficients are a_0, a_1, a_2, a_3 and a_4. The symbol of the scheme is $a(z) = \sum_{i=0}^4 a_i z^i$ and the scheme is:

$$\begin{aligned} f_{2j}^{m+1} &= a_0 f_j^m + a_2 f_{j-1}^m + a_4 f_{j-2}^m, \\ f_{2j+1}^{m+1} &= a_1 f_j^m + a_3 f_{j-1}^m. \end{aligned}$$

It is obvious that the subdivision matrix S is two-slanted bi-infinite matrix. Necessary and sufficient conditions for the convergence of scheme can be formulated in terms of properties of two matrices which are sections of the infinite matrix S.
Consider the two sub-matrix

$$A_0 : (f_{-s+1}^0, \ldots, f_0^0)^t \to (f_{-s+1}^1, \ldots, f_0^1)^t$$

and

$$A_1 : (f^0_{-s+1}, \dots, f^0_0)^t \to (f^1_{-s+2}, \dots, f^1_1)^t$$

such that

$$\begin{bmatrix} f^{m+1}_{-3} \\ f^{m+1}_{-2} \\ f^{m+1}_{-1} \\ f^{m+1}_{0} \end{bmatrix} = \begin{bmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \end{bmatrix} \begin{bmatrix} f^{m}_{-3} \\ f^{m}_{-2} \\ f^{m}_{-1} \\ f^{m}_{0} \end{bmatrix},$$

and

$$\begin{bmatrix} f^{m+1}_{-2} \\ f^{m+1}_{-1} \\ f^{m+1}_{0} \\ f^{m+1}_{1} \end{bmatrix} = \begin{bmatrix} a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \\ 0 & 0 & a_1 & a_3 \end{bmatrix} \begin{bmatrix} f^{m}_{-3} \\ f^{m}_{-2} \\ f^{m}_{-1} \\ f^{m}_{0} \end{bmatrix}.$$

The vector of initial values $u^0 = (f^0_{-3}, f^0_{-2}, f^0_{-1}, f^0_0)^t$ determines the limit values at $[0, 1]$. $A_0 u^0$ determines the values at $[0, 1/2]$ while $A_1 u^0$ determines the values at $[1/2, 1]$. We can infer that while $u^m = (f^m_{-3}, f^m_{-2}, f^m_{-1}, f^m_0)^t$ determines the limit function on the interval $I = [j2^{-m}, (j + 1)/2^{-m}]$, $A_0 u^m, A_1 u^m$ determine the limit function on the left half and the right half of I respectively. At any point x in $[0, 1]$ can be expressed by diadic expansion.

$$x = \sum_{j=1}^{\infty} i_j 2^{-j}, \quad i_j \in \{0, 1\}.$$

The limit values of a convergent subdivision, starting with initial values, is given by

$$(f(x), f(x), f(x), f(x))^t = \cdots A_{i_3} A_{i_2} A_{i_1} u^0.$$

Note: C^0 scheme must reproduce constants, therefore A_0 and A_1 have eigenvector $(1 \quad 1 \quad 1 \quad 1)^t$ with eigenvalue 1.

The idea is to represent the operators A_0 and A_1 in another basis, the vectors comprising this basis are the columns of the following matrix:

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

In the new basis, the two operators are:

$$B_0 = V^{-1} A_0 V = \begin{bmatrix} 1 & * & * & * \\ 0 & & & \\ 0 & & Q_0 & \\ 0 & & & \end{bmatrix}$$

and

$$B_1 = V^{-1} A_1 V = \begin{bmatrix} 1 & * & * & * \\ 0 & & & \\ 0 & & Q_1 & \\ 0 & & & \end{bmatrix}.$$

Note that only Q_0 and Q_1 determine the behavior of the last 3 components in the new basis. Recalling the products $\ldots A_{i_3}A_{i_2}A_{i_1}u^0$, which in the new basis appear as

$$\ldots B_{i_3}B_{i_2}B_{i_1}v^0, \quad v^0 = V^{-1}u^0$$

The last three coordinates of this product will converges to zero if the joint spectral radius $\rho(Q_0, Q_1)$ is less than 1. Each matrix Q_0, Q_1 has its own spectral radius. The joint spectral radius is defined as:

$$\rho(Q_0, Q_1) = \limsup_{m\to\infty} \left(\max\{\|Q_{i_1}Q_{i_2}\ldots Q_{i_m}\| : i_j \in \{0,1\}\}\right)^{1/m}.$$

If $\rho(Q_0, Q_1) = \mu < 1$, then the last three components of any vector of the form $v^0 = V^{-1}u^0$ will tend to zero.

We already know that a scheme is convergent to a C^0 limit if the differences of the generated values tend to zero. In the new basis, the columns of V, it is clear that differences within the first column are zero. The differences within the other columns tend to zero simply because the corresponding coefficients tend to zero.

The number $\rho(Q_0, Q_1)$ tells us how fast the differences are decaying to zero, and from it we can derive the Hölder exponent of the limit function:

$$\nu = -\log_2 \rho(Q_0, Q_1).$$

Note that with the specific choice of V, the two matrices Q_0 and Q_1 are just two matrices generating the limit values on $[0,1]$ for the differences scheme related with the Laurent polynomial $b(z) = a(z)/(1+z)$.

Exercise 57 *Construct the mask, the Laurent polynomial, and the local subdivision matrices A_0 and A_1 of the interpolating 6-point scheme.*

B.1 Limit Values at Grid Points

Using the fact

$$\begin{aligned}
f(x) &= \sum_i f_i^0 \phi(x-i), \\
f(j) &= \sum_i f_i^0 \phi(j-i), \quad j \in \mathbb{Z} \\
f^{(l)}(j) &= \sum_i f_i^0 \phi^{(l)}(j-i), \quad j \in \mathbb{Z}, 0 \le l \le k.
\end{aligned}$$

It is enough to know that the values of the basic limit function and its derivatives at the integers. Consider the following refinement equation:

$$\begin{aligned}
\phi(x) &= \sum_j a_j \phi(2x-j), \quad supp(\phi) = [0,4] \\
\phi^{(l)}(i) &= 2^l \sum_j a_j \phi^{(l)}(2i-j), \quad 0 \le i \le 4, 0 \le l \le k.
\end{aligned}$$

We can expressed in the matrix form

$$2^{-l}\begin{bmatrix}\phi^{(l)}(0)\\ \phi^{(l)}(1)\\ \phi^{(l)}(2)\\ \phi^{(l)}(3)\\ \phi^{(l)}(4)\end{bmatrix}=\begin{bmatrix}a_0 & 0 & 0 & 0 & 0\\ a_2 & a_1 & a_0 & 0 & 0\\ a_4 & a_3 & a_2 & a_1 & a_0\\ 0 & 0 & a_4 & a_3 & a_2\\ 0 & 0 & 0 & 0 & a_4\end{bmatrix}\begin{bmatrix}\phi^{(l)}(0)\\ \phi^{(l)}(1)\\ \phi^{(l)}(2)\\ \phi^{(l)}(3)\\ \phi^{(l)}(4)\end{bmatrix},\quad 0\le l\le k.$$

Therefore we can get the limit and the first derivative values of the cubic B-spline with $a(z) = \frac{1}{8}(1+z)^4$.

$$\begin{bmatrix}0\\ \frac{1}{6}\\ \frac{4}{6}\\ \frac{1}{6}\\ 0\end{bmatrix}=\begin{bmatrix}\frac{1}{8} & 0 & 0 & 0 & 0\\ \frac{3}{4} & \frac{1}{2} & \frac{1}{8} & 0 & 0\\ \frac{1}{8} & \frac{1}{2} & \frac{3}{4} & \frac{1}{2} & \frac{1}{8}\\ 0 & 0 & \frac{1}{8} & \frac{1}{2} & \frac{3}{4}\\ 0 & 0 & 0 & 0 & \frac{1}{8}\end{bmatrix}\begin{bmatrix}0\\ \frac{1}{6}\\ \frac{4}{6}\\ \frac{1}{6}\\ 0\end{bmatrix},$$

and

$$\frac{1}{2}\begin{bmatrix}0\\ \frac{1}{2}\\ 0\\ -\frac{1}{2}\\ 0\end{bmatrix}=\begin{bmatrix}\frac{1}{8} & 0 & 0 & 0 & 0\\ \frac{3}{4} & \frac{1}{2} & \frac{1}{8} & 0 & 0\\ \frac{1}{8} & \frac{1}{2} & \frac{3}{4} & \frac{1}{2} & \frac{1}{8}\\ 0 & 0 & \frac{1}{8} & \frac{1}{2} & \frac{3}{4}\\ 0 & 0 & 0 & 0 & \frac{1}{8}\end{bmatrix}\begin{bmatrix}0\\ \frac{1}{2}\\ 0\\ -\frac{1}{2}\\ 0\end{bmatrix}.$$

Note: Using the polynomial reproducing property, we have

$$\sum_j j^l\phi^{(l)}(-j)=l!,\quad 0\le l\le k.$$

Exercise 58 *Verify the last statement.*

Exercise 59 *Develop the formula for the values of the basic limit function and its derivatives at the integers for the quintic B-spline scheme.*

B.2 Hurwitz polynomial

A polynomial $p\in\pi_n$ as given by

$$p(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0,\quad z\in\mathbb{C},$$

with $a_n\neq 0$, is called a Hurwitz polynomial if $z_0\in\mathbb{C}$ is such that $p(z_0)=0$, then $Re(z_0)<0$.

The coefficients of a Hurwitz polynomial p are necessarily of the same sign. If p is a Hurwitz polynomial with $p(1)>0$, then $a_j>0, j=0,1,\cdots,n$. It should be noted that the converse of this result is not necessarily true.

Appendix C

Applications

We can investigate the regularity of an approximation to the solution of the integral equation. Consider the following integral equation

$$f(\frac{x}{2}) = 2\int_{x-1}^{x} f(t)dt, \quad x \in \mathbb{R}.$$

Using the trapezoidal rule for integral $\int_a^b y(x)dx$, we have an approximation of integral:

$$\frac{h}{2}\left[y(a) + y(b) + 2\sum_{j=1}^{n-1} y(t_j)\right],$$

where $h = \frac{b-a}{n}, t_j = a + jh, j = 0, 1, \cdots, n$. We consider an approximating solution, in the sense that g satisfies the equation

$$g(\frac{x}{2}) = \frac{1}{n}\left[g(x) + g(x-1) + 2\sum_{j=1}^{n-1} g(x - \frac{j}{n})\right], \quad x \in \mathbb{R}.$$

Setting $x = \frac{t}{n}$ and $\phi(t) = g(\frac{t}{n})$, we get

$$\phi(\frac{t}{2}) = \frac{1}{n}\left[\phi(t) + \phi(x-n) + 2\sum_{j=1}^{n-1} \phi(x-j)\right], \quad t \in \mathbb{R}.$$

We find that this is a refinement equation with mask coefficients given by

$$a_0 = a_n = \frac{1}{n},$$

and

$$a_j = \frac{2}{n}, \quad j = 1, 2, \cdots, n-1.$$

Hence, the corresponding symbol $a(z)$ is given by

$$\begin{aligned} a(z) &= \frac{1}{n}[(1+z^n) + 2\sum_{j=1}^{n-1} z^j] \\ &= \frac{1}{n}\frac{(1+z)(1-z^n)}{1-z}. \end{aligned}$$

In the special case $n = 2^k$, we have

$$\begin{aligned} a(z) &= \frac{1}{2^k}\frac{(1+z)(1-z^{2^k})}{1-z} \\ &= \frac{1}{2^k}(1+z)^2(1+z^2)(1+z^4)\cdots(1+z^{2^{k-1}}) \\ &= \frac{1}{2^{k-1}}(1+z)\prod_{r=1}^{k-1}(1+z^r)(\frac{1+z}{2}). \end{aligned}$$

We find that the associated refinable function ϕ_k satisfies $\phi_k \in C^{k-1}(\mathbb{R})$.

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