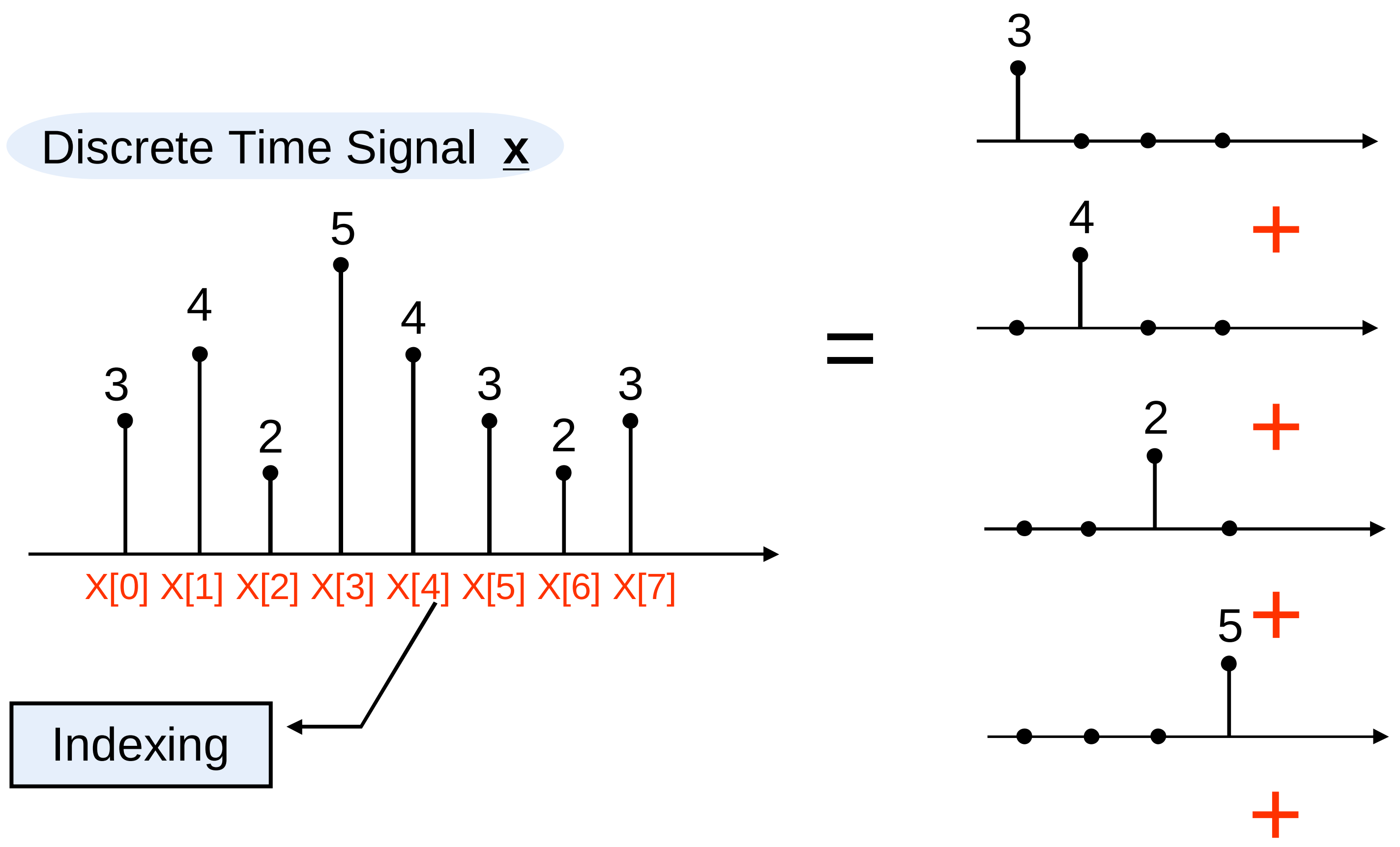
Transforms

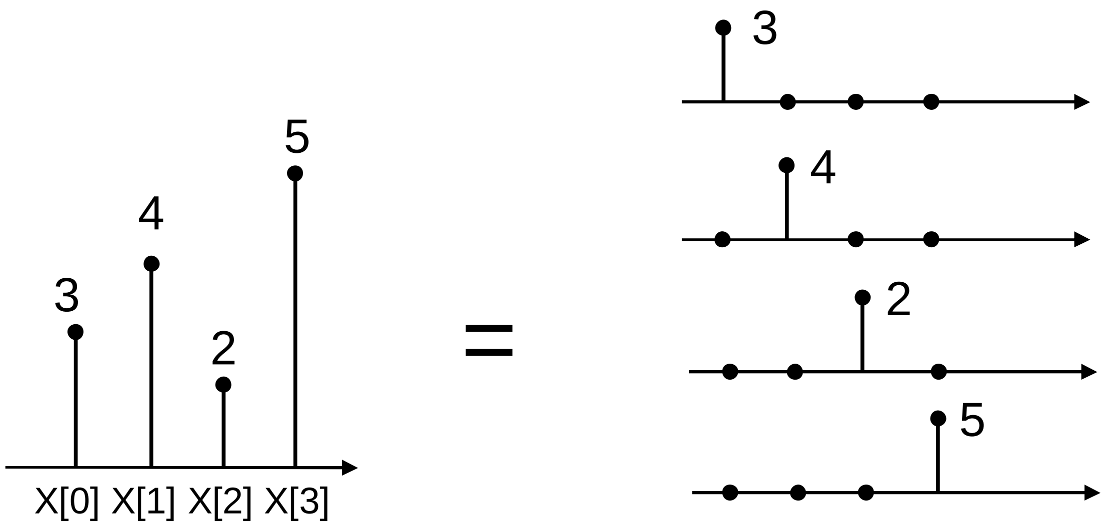
DSU Dongseo University Hoon Yoo, Ph.D.

Signal Analysis

- Mathematical expression of discrete time signal



Signal Analysis

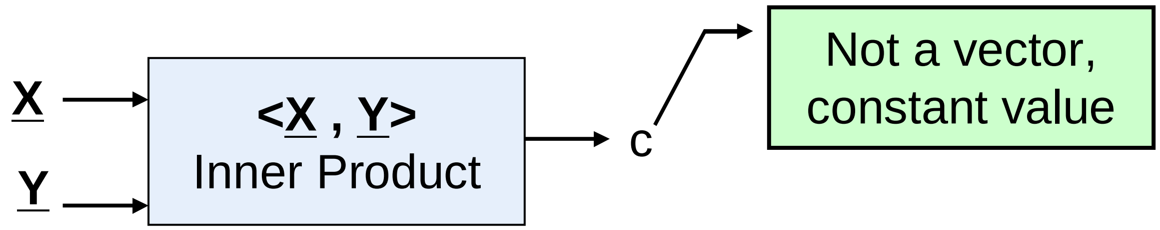


Linear Combination

$$\underline{X} = \{3, 4, 2, 5\} = \begin{pmatrix} 3 \\ 4 \\ 2 \\ 5 \end{pmatrix} = 3^* \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 4^* \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 2^* \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 5^* \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Basis Vectors

Inner Product



$$\langle \underline{X} , \underline{Y} \rangle = \sum_{i}^{N} x[i]y[i]$$

$$< \begin{pmatrix} 3 \\ 4 \\ 2 \\ 5 \end{pmatrix} , \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} > = 3*1 + 4*2 + 2*2 + 5*1 = 20$$

$$< \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} > = 1*0 + 0*1 + 0*0 + 0*0 = 0$$

Orthogonality

We have vector set, $\{\underline{e}_i\}$, i=0..N-1

$< \underline{e}_i, \underline{e}_j > = 0$, i,j=0..N-1, when $i \neq j$

vector set, $\{\underline{e}_i\}$, is orthogonal

$$\begin{array}{cccc} \underline{e}_0 & \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \begin{bmatrix}1\\0\\0\\0\end{bmatrix} & \begin{bmatrix}0\\1\\0\\0\end{bmatrix} & \begin{bmatrix}0\\0\\1\\0\end{bmatrix} & \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \end{array}$$

$< \underline{e}_0 , \underline{e}_0 > \neq 0$

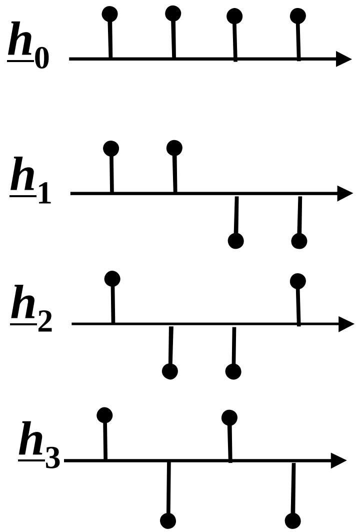
$< \underline{e}_0 , \underline{e}_1 > = 0$ $\quad < \underline{e}_1 , \underline{e}_1 > \neq 0$

$< \underline{e}_0 , \underline{e}_2 > = 0$ $\quad < \underline{e}_1 , \underline{e}_2 > = 0$ $\quad < \underline{e}_2 , \underline{e}_2 > \neq 0$

$< \underline{e}_0 , \underline{e}_3 > = 0$ $\quad < \underline{e}_1 , \underline{e}_3 > = 0$ $\quad < \underline{e}_2 , \underline{e}_3 > = 0$ $\quad < \underline{e}_3 , \underline{e}_3 > \neq 0$

Orthogonality

$$
\begin{array}{cccc}
\underline{h}_0 & \underline{h}_1 & \underline{h}_2 & \underline{h}_3 \\
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} &
\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} &
\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} &
\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}
\end{array}
$$



$< \underline{h}_0 , \underline{h}_0 > = ?$

$< \underline{h}_0 , \underline{h}_1 > = 0$ $< \underline{h}_1 , \underline{h}_1 > = ?$

$< \underline{h}_0 , \underline{h}_2 > = 0$ $< \underline{h}_1 , \underline{h}_2 > = 0$ $< \underline{h}_2 , \underline{h}_2 > = ?$

$< \underline{h}_0 , \underline{h}_3 > = ?$ $< \underline{h}_1 , \underline{h}_3 > = ?$ $< \underline{h}_2 , \underline{h}_3 > = 0$ $< \underline{h}_3 , \underline{h}_3 > = ?$

$\{\underline{h}_i\}$ Orthogonal ?

Change of Basis

$$\underline{X} = \{3, 4, 2, 5\} = \begin{pmatrix} 3 \\ 4 \\ 2 \\ 5 \end{pmatrix} = 3^* \overset{\underline{e}_0}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}} + 4^* \overset{\underline{e}_1}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}} + 2^* \overset{\underline{e}_2}{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}} + 5^* \overset{\underline{e}_3}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$$

Basis

Change of Basis
= Transform

$$\underline{X} = \begin{pmatrix} 3 \\ 4 \\ 2 \\ 5 \end{pmatrix} = ?^* \overset{\underline{h}_0}{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}} + ?^* \overset{\underline{h}_1}{\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}} + ?^* \overset{\underline{h}_2}{\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}} + ?^* \overset{\underline{h}_3}{\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}}$$

$$= \{C_0, C_1, C_2, C_3\} = \underline{Y}$$

Transform Coefficients

$$\underline{X} = \begin{pmatrix} 3 \\ 4 \\ 2 \\ 5 \end{pmatrix} = ?* \overset{\underline{h}_0}{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}} + ?* \overset{\underline{h}_1}{\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}} + ?* \overset{\underline{h}_2}{\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}} + ?* \overset{\underline{h}_3}{\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}}$$

$$= \{C_0, C_1, C_2, C_3\} = \underline{Y}$$

$$< \underline{X}\, , \underline{h}_0 > = \; < C_0 * \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_1 * \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + C_2 * \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + C_3 * \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} , \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} > \; = C_0$$

$$\left.\begin{array}{l} < \underline{X}\, , \underline{h}_1 > = C_1 \\ < \underline{X}\, , \underline{h}_2 > = C_2 \\ < \underline{X}\, , \underline{h}_3 > = C_3 \end{array}\right\} \text{Transform Coefficients}$$

Example

$$\underline{h}_0 = \begin{pmatrix}1\\1\end{pmatrix} \quad \underline{h}_1 = \begin{pmatrix}1\\-1\end{pmatrix}$$

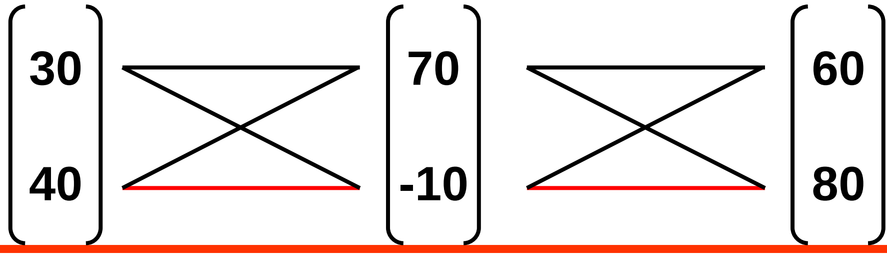
$$\underline{X} = \begin{pmatrix}30\\40\end{pmatrix} \longrightarrow \boxed{\begin{array}{c}\text{Transform}\\\text{(Haar)}\end{array}} \longrightarrow \begin{pmatrix}C_0\\C_1\end{pmatrix}$$

$$C_0 = <\underline{X}, \underline{h}_0> = 30*1 + 40*1 = 70$$

$$C_1 = <\underline{X}, \underline{h}_1> = 30*1 + 40*(-1) = -10$$

$$\begin{pmatrix}C_0\\C_1\end{pmatrix} \longrightarrow \boxed{\begin{array}{c}\text{Inverse}\\\text{Transform}\end{array}} \longrightarrow \begin{pmatrix}60\\80\end{pmatrix} = 2*\underline{X}$$

$$C_0*\underline{h}_0 + C_1*\underline{h}_1 = 70*\begin{pmatrix}1\\1\end{pmatrix} + (-10)*\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}60\\80\end{pmatrix} = 2*\underline{X}$$



Discrete Fourier Transform

Vector set, $\{\boldsymbol{f}_k\}$, k=0..N-1

$$\boldsymbol{f}_k = \frac{1}{\sqrt{N}} e^{jk\frac{2\pi n}{N}}$$

$$\boldsymbol{f}_0 = \begin{pmatrix} ½*\exp(j*0*2\pi 0/4) \\ ½*\exp(j*0*2\pi 1/4) \\ ½*\exp(j*0*2\pi 2/4) \\ ½*\exp(j*0*2\pi 3/4) \end{pmatrix} = \begin{pmatrix} ½ \\ ½ \\ ½ \\ ½ \end{pmatrix}$$

$$\boldsymbol{f}_1 = \begin{pmatrix} ½*\exp(j*1*2\pi 0/4) \\ ½*\exp(j*1*2\pi 1/4) \\ ½*\exp(j*1*2\pi 2/4) \\ ½*\exp(j*1*2\pi 3/4) \end{pmatrix} = \begin{pmatrix} ½ \\ ½e^{j\pi/2} \\ ½e^{j\pi} \\ ½e^{j3\pi/2} \end{pmatrix}$$

$$\vdots$$

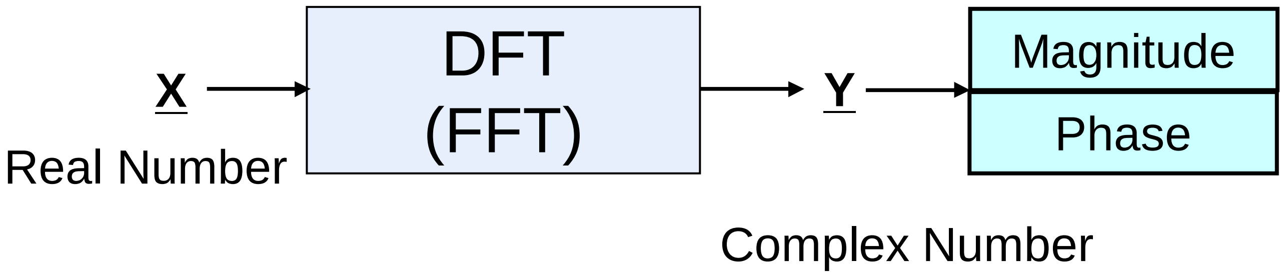
Discrete Fourier Transform

Vector set, $\{\boldsymbol{f}_k\}$, k=0..N-1

$$\boldsymbol{f}_k = \frac{1}{\sqrt{N}} e^{jk\frac{2\pi n}{N}}$$

Note: Complex Number

$$\mathbf{C}_\mathbf{k} = \langle \underline{\mathbf{X}}_k , \boldsymbol{f}_k \rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi n}{N}}$$

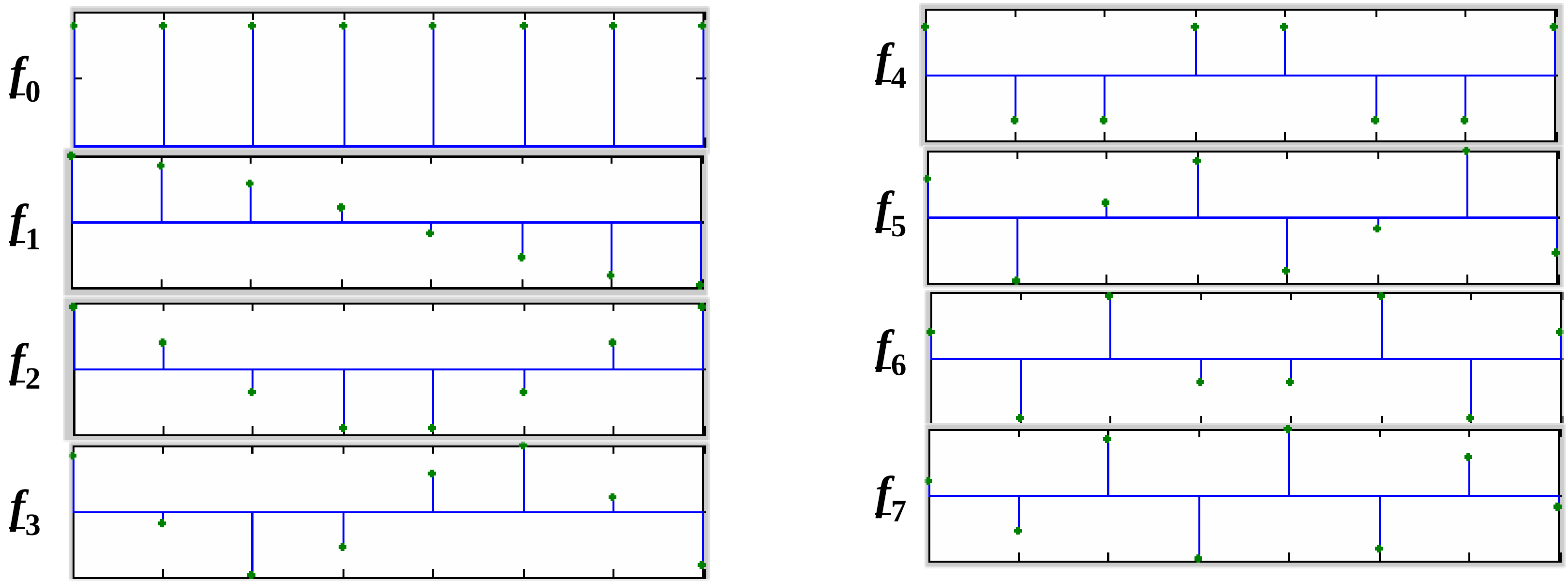


Discrete Cosine Transform

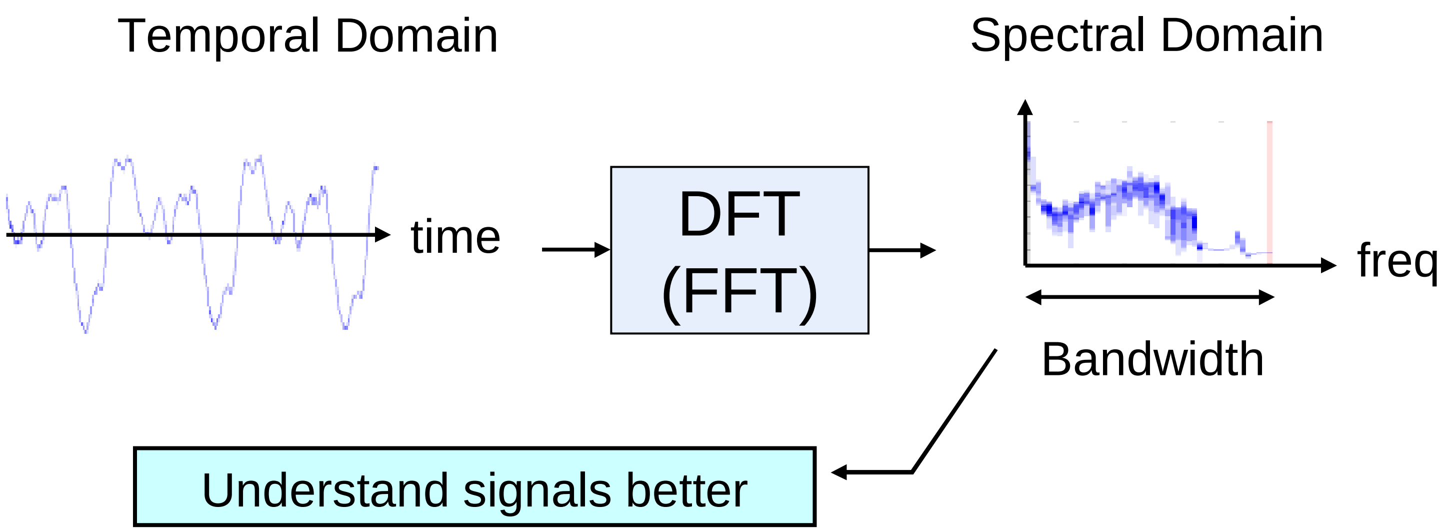
Vector set, $\{\boldsymbol{f}_k\}$, k=0..N-1

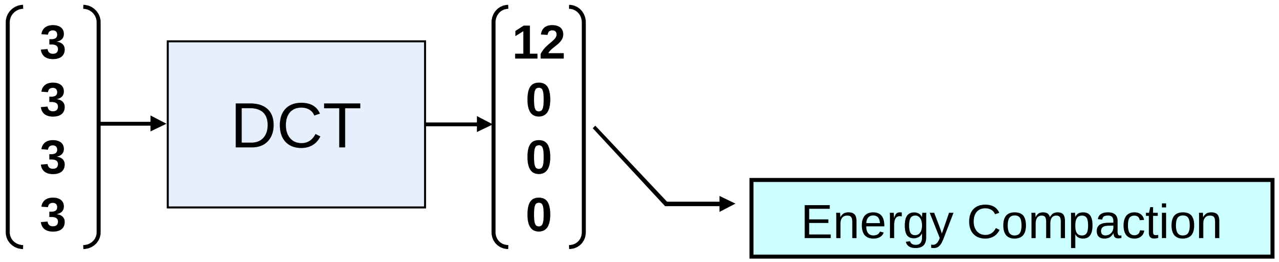
$$\boldsymbol{f}_k = \cos\left[\frac{\pi(2m+1)k}{2N}\right]$$

$$\mathbf{C_k} = < \underline{\mathbf{X}}_k \, , \boldsymbol{f}_k > = a(k)\sum_{m=0}^{N-1} x[n]\cos\left[\frac{\pi(2m+1)k}{2N}\right], \quad 0 \le k \le N-1$$



Why we use transforms





Summary

- Signals are represented by basis vectors
- Transforms is change of basis
  - Hadamard Transform
  - Discrete Fourier Transform (Fast Fourier Transform)
  - Discrete Cosine Transform
- We understand signals better using the DFT
- Energy compaction using the DCT