## We are at square 0

# Three short talks on wavelet representations

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Pusan, Han-Gook, 2007

# Collaborators on this work

1. Part I
   - Yeonhyang Kim
2. Part II
   - Yeongmi Hur
3. Part III
   - Sangnam Nam
   - Vladimir Temlyakov

# Outline

1. **Part I: Time-freq. representations of almost-peridoic functions**
   - Almost perdioic functions
   - Capturing the AP-norm
   - Our results
2. **Part II: L-CAMP – Efficient wavelet represent'n in high D**
   - Pyramidal representations and wavelets
   - Introduction to localness and performance
   - L-CAMP: A bird's view of the CAP methodologies
   - L-CAMP: The algorithms & performance analysis
3. **Part III: L-CAMP representation based on sampling**
   - Motivation
   - Sampling based L-CAMP: the algorithms
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# Definition of almost periodic functions

## Definition

A continuous complex-valued function $f$ on $\mathbb{R}$ is called almost periodic if for every $\varepsilon > 0$ there exists an $l > 0$ such that every interval of length $l$ contains at least one point $\tau$ for which

$$\sup_x |f(x+\tau) - f(x)| < \epsilon.$$

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$\tau$ is called an almost period of $f$ relative to $\varepsilon$.
We denote by $AP$ the set of all almost periodic functions on $\mathbb{R}$.

# The inner product norm

- $AP$ : a non-separable incomplete inner product space

$$\langle f, g\rangle_{AP} := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\bar{g}(x)dx.$$

back to wavelet

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- $\sigma(f) := \{\lambda \in \mathbb{R} \,|\, a(\lambda) \neq 0\}$ countable set.
- $||f||_{AP}^2 = \sum_{\lambda\in\sigma(f)} |a(\lambda)|^2$.

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# The averaging process

## A wavelet system:

back to earlier results

$$\alpha \in \mathbb{N},\ \Psi \subset L_2(\mathbb{R}) \text{ finite},$$

$$X(\Psi,\alpha) := \{\sqrt{\alpha^j}\,\psi_{j,k} \ :\ \psi_{j,k} := \alpha^{-j}\psi(\alpha^{-j}\cdot - k),\ j,k \in \mathbb{Z}\}.$$

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Averaging with wavelet systems

$$\sum_{j\in\mathbb{Z}} \lim_{N\to\infty} \frac{1}{2N} \sum_{k=-N}^{N} \sum_{\psi\in\Psi} |\langle f, \psi_{j,k}\rangle|^2$$

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## A Gabor system:

$$t_0, w_0 > 0,\ K := t_0\mathbb{Z},\ L := w_0\mathbb{Z},\ \Psi \subset L_2(\mathbb{R}) \text{ finite},$$

$$X(\Psi, t_0, w_0) := \{\psi_{k,l} := \psi(\cdot - k)e^{il(\cdot - k)}\ :\ k \in K,\ l \in L,\ \psi \in \Psi\}$$

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$$\sum_{l \in L} \lim_{N \to \infty} \frac{1}{2Nt_0} \sum_{k \in K(N)} \sum_{\psi \in \Psi} |\langle f, \psi_{k,l} \rangle|^2$$

# Earlier results using wavelets

Assumptions:

1. $\Gamma(k) := \sup_{\lambda\in\mathbb{R}} \sum_{j\in\mathbb{Z}} \left|\widehat{\psi}(\alpha^j\lambda)\widehat{\psi}\left(\alpha^j(\lambda - 2\pi k)\right)\right|, \quad k \in \mathbb{Z}.$
2. $\widetilde{A} := \inf_{\lambda\in\mathbb{R}} \sum_{j\in\mathbb{Z}} \left|\widehat{\psi}(\alpha^j\lambda)\right|^2 - \sum_{k\neq 0} (\Gamma(k)\Gamma(-k))^{1/2} > 0$
3. $\widetilde{B} := \sup_{\lambda\in\mathbb{R}} \sum_{j\in\mathbb{Z}} \left|\widehat{\psi}(\alpha^j\lambda)\right|^2 + \sum_{k\neq 0} (\Gamma(k)\Gamma(-k))^{1/2} < \infty$

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**Theorem (F. Galindo, 2004[a])**

[a] following Partington and Ünalmis (2001)

For $f \in AP$ with $\widehat{f}(\{0\}) = 0$,

$$\widetilde{A}||f||_{AP}^2 \leq \sum_{j\in\mathbb{Z}} \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} |\langle f, \psi_{j,k}\rangle|^2 \leq \widetilde{B}||f||_{AP}^2.$$

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Assumptions:

- $\psi$ is bounded and $\psi(t) = O(\frac{1}{t^2})$ as $t \to \pm\infty$.
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for every $f \in AP$.

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# $L_2(\mathbb{R})$-wavelet respresentations

## Definition: Frames

$X \subset L_2(\mathbb{R})$ is a frame iff there exist $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{x \in X} |\langle f, x\rangle|^2 \leq B\|f\|^2, \forall f \in L_2(\mathbb{R}).$$

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Fiberization of wavelets: For an wavelet system $X := X(\Psi, \alpha)$,

$$\widetilde{G}(\lambda) := \left( \sum_{j=\kappa(\lambda-\lambda')}^{\infty} \sum_{\psi \in \Psi} \widehat{\psi}(\alpha^j(\lambda - k))\overline{\widehat{\psi}}(\alpha^j(\lambda - l)) \right)_{(k,l) \in (2\pi\mathbb{Z})^2}.$$

$$\mathcal{G}^* := \mathcal{G}^*_X : \mathbb{R} \to \mathbb{R} : \lambda \mapsto ||\widetilde{G}(\lambda)||.$$
$$\mathcal{G}^{*-} := \mathcal{G}^{*-}_X : \mathbb{R} \to \mathbb{R} : \lambda \mapsto ||\widetilde{G}^{-1}(\lambda)||.$$

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**Theorem, R-Shen, JFA 1997**

$X$ is a fundamental frame for $L_2(\mathbb{R})$
if and only if $\mathcal{G}^*, \mathcal{G}^{*-} \in L_\infty(\mathbb{R})$.

Furthermore, the frame bounds of $X$ are $||\mathcal{G}^*||_{L_\infty(\mathbb{R})}$ and $1/||\mathcal{G}^{*-}||_{L_\infty(\mathbb{R})}$.

# Main result on wavelets

Assumptions:

- $X = X(\Psi, \alpha) \subset L_1(\mathbb{R})$: a wavelet system
- $\sum_{j=\kappa(\gamma)}^{\infty} \widehat{\psi}(\alpha^j \cdot)\overline{\widehat{\psi}}(\alpha^j(\cdot + \gamma))$ is continuous, where $\kappa(\lambda) := \inf\{j \in \mathbb{Z} : \alpha^j \lambda \in 2\pi\mathbb{Z}\},\ \gamma \in \cup_{j\in\mathbb{Z}} 2\pi\mathbb{Z}/\alpha^j,\ \psi \in \Psi$

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**Theorem, Kim-R, CA 200x**

$X$ is an $L_2(\mathbb{R})$ frame with frame bounds $A, B$ iff for any $f \in AP$ with $\widehat{f}(0) = 0$,

$$A \|f\|_{AP}^2 \leq \sum_{j\in\mathbb{Z}} \lim_{N\to\infty} \frac{1}{2N} \sum_{k=-N}^{N} \sum_{\psi\in\Psi} |\langle f, \psi_{j,k}\rangle|^2 \leq B \|f\|_{AP}^2$$

(Note: the sharpest frame bounds are also the sharpest AP bounds).

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- $X = X(\Psi, t_0, w_0) \subset L_1(\mathbb{R})$ a Gabor system
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where $K(N) := \{nt_0 \in K\ :\ -N \leq n \leq N\}$ ($A, B$: optimal).

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# The laplacian (CP) pyramid: I

## Laplacian Pyramids (Burt and Adelson, 1983)

$$\begin{array}{ccccccccccc}
y_0 & \xrightarrow{C} & y_{-1} & \xrightarrow{C} & y_{-2} & \xrightarrow{C} & \cdots & \xrightarrow{C} & y_{-j+1} & \xrightarrow{C} & y_{-j} \\
\downarrow D & & \downarrow D & & \downarrow D & & & & \downarrow D & & \\
\boldsymbol{d}_0 & & \boldsymbol{d}_{-1} & & \boldsymbol{d}_{-2} & & \cdots & & \boldsymbol{d}_{-j+1} & &
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**Definition of the detail map**

$$D = I - PC$$

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$(y_j)_{j=-\infty}^{\infty} \subset \mathbb{C}^{\mathbb{Z}^n}$ s.t:

$$y_{j-1} = Cy_j := (h_c * y_j)_{\downarrow}, \quad \forall j.$$

$C$ is Compression

---

$y_j$ is then **predicted** from $y_{j-1}$ by

$$y_j \approx Py_{j-1} := 2^n \left(h_p * (y_{j-1\uparrow})\right).$$

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$h_c, h_p : \mathbb{Z}^n \to \mathbb{R}$ are symmetric, normalized, lowpass filters
For each $h := h_c$ and $h := h_p$, $h(k) = h(-k)$, $\sum_{k\in\mathbb{Z}^n} h(k) = 1$.

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$\downarrow$, $\uparrow$ are downsampling & upsampling:

$$
\begin{aligned}
y_{\downarrow}(k) &= y(2k), \quad k \in \mathbb{Z}^n. \\
y_{\uparrow}(k) &= \begin{cases} y(k/2), & k \in 2\mathbb{Z}^n, \\ 0, & \text{otherwise}. \end{cases}
\end{aligned}
$$

# Wavelets as a variation of CP

## Derivation of wavelets from CP pyramids

Decompose the detail map $I - PC$: $I - PC = \sum_{i=1}^{r} R_i D_i$

$$D_i : y_j \mapsto (h_i * y_j)_{\downarrow} =: w_{i,j-1}, \quad R_i : y \mapsto 2^n \left(h_i * y_{\uparrow}\right)$$

with $h_i$ a real, symmetric, highpass: $\sum_{k \in \mathbb{Z}^n} h_i(k) = 0$.

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y_0 & \xrightarrow{C} & y_{-1} & \xrightarrow{C} & y_{-2} & \xrightarrow{C} & \cdots & \xrightarrow{C} & y_{-j_0+1} & \xrightarrow{C} & y_{-j_0} \\
 & \searrow^{D} & & \searrow^{D} & & \searrow^{D} & & & & \searrow^{D} & \\
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\end{array}$$

We can recover $y_0$ from $y_{j_0}, w_{1,j_0}, \ldots, w_{r,j_0}, \ldots, w_{1,-1}, \ldots, w_{r,-1}$
since $y_{j_0+1} = \sum_{i=1}^{r} R_i w_{i,j_0} + P y_{j_0}$, $y_{j_0+2} = \sum_{i=1}^{r} R_i w_{i,j_0+1} + P y_{j_0+1}$
and so on.

# Wavelets as a variation of CP: example

## Laplacian pyramid vs. wavelets: Examples in 2D

Burt-Adelson CP: Let $h_c = h_p = \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}.$

There are four (hidden) highpass filters:

$$\begin{bmatrix} +3/4 & -1/4 \\ -1/4 & -1/4 \end{bmatrix}, \begin{bmatrix} -1/4 & +3/4 \\ -1/4 & -1/4 \end{bmatrix}, \begin{bmatrix} -1/4 & -1/4 \\ +3/4 & -1/4 \end{bmatrix}, \begin{bmatrix} -1/4 & -1/4 \\ -1/4 & +3/4 \end{bmatrix}$$

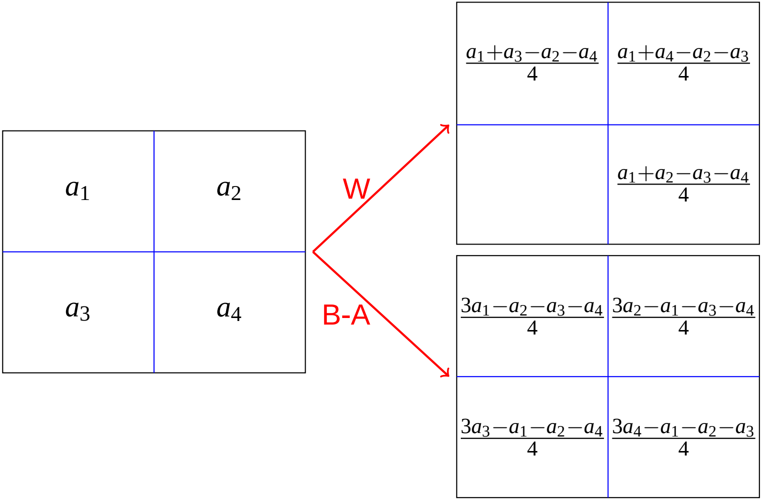
2D Haar wavelets: There are three highpass filters:

$$\begin{bmatrix} +1/4 & -1/4 \\ +1/4 & -1/4 \end{bmatrix}, \begin{bmatrix} +1/4 & -1/4 \\ -1/4 & +1/4 \end{bmatrix}, \begin{bmatrix} +1/4 & +1/4 \\ -1/4 & -1/4 \end{bmatrix}$$

In both cases, average filter size is 4.

back to algorithms

# Laplacian pyramid vs. wavelets: example cont'ed



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# Wavelets: Desired properties
## or, why do we need new constructions?

1. Localness in space

**Quantifying "local":** the number of wavelets within a single resolution whose support contains a given generic point $t \in \mathbb{R}^n$.

Note: this is the same as the total volume of the mother wavelets set $\Psi$:

$$\mathrm{vol}(\Psi) := \sum_{\psi \in \Psi} \mathrm{vol}(\mathrm{supp}\psi).$$

2. Localness in frequency: high performance.
3. Speed: Small constants in the linear complexity of the algorithms.

# Performance I: The analysis operator

$\Psi \subset L_2$ is finite. The wavelet system $X(\Psi)$ is then

$$\psi_{j,k} := 2^{j\frac{n}{2}}\psi(2^j \cdot -k), \quad \psi \in \Psi, j \in \mathbb{Z}, k \in \mathbb{Z}^n$$

The wavelet representation of $f \in L_2$ is then the discrete set of inner products

$$T^*_{X(\Psi)}f := (\langle f, x\rangle)_{x\in X(\Psi)}, \quad \langle f, g\rangle := \int_{\mathbb{R}^n} f(t)\overline{g(t)}\,dt.$$

The wavelet system $X(\Psi)$ is a frame of $L_2$ if

$$\sum_{x\in X(\Psi)} |\langle f, x\rangle|^2 \approx \|f\|^2_{L_2}, \quad \forall f \in L_2.$$

# Performance II: Jackson-type performance

$$W_2^\alpha := \{f \in L_2,\ |f|_{W_2^\alpha} := \|(|\cdot|^\alpha \widehat{f}\,)^\vee\|_{L_2} < \infty\}, \quad \alpha > 0.$$
$$\|c\|^2_{\ell_2(\alpha)} := \sum_{j\in\mathbb{Z},k\in\mathbb{Z}^n} 2^{2j\alpha}|c(j,k)|^2.$$

Jackson-type performance of a frame $X(\Psi)$ :

$$s_J := \sup\{\alpha > 0 :\ X(\Psi) \text{ satisfies (1) for the given } \alpha\},$$

$$\sum_{\psi\in\Psi} \|T^*_{X(\psi)} f\|_{\ell_2(\alpha)} \le A_\alpha |f|_{W_2^\alpha}, \quad \forall f \in L_2. \tag{1}$$

$s_J$ is essentially determined by the vanishing moments of $X(\Psi)$.

# Performance III: Bernstein-type performance

Bernstein-type performance of a frame $X(\Psi)$ :

$$s_B := \sup\{\alpha > 0 : \ X(\Psi) \text{ satisfies (1) and (2) for the given } \alpha\},$$

$$\sum_{\psi\in\Psi} \|T^*_{X(\psi)} f\|_{\ell_2(\alpha)} \geq B_\alpha |f|_{W_2^\alpha}, \quad \forall f \in L_2. \qquad (2)$$

- $s_B \leq s_J$; usually strict inequality holds.
- $s_B$ is not connected directly to any property of the system $X(\Psi)$.
- $s_B$ is essentially determined by $s_J$, and by the smoothness + Strang-Fix order of the dual system.

# Comparison of wavelets to CP

Question: why or why not decomposing the detail map $I - PC$:
$I - PC = \sum_{i=1}^{r} R_i D_i$

Pros

1. Reducing the size of the filters
2. Making it possible to be non-redundant: $r = 2^n - 1$
3. Making it possible to be highly redundant: $r >> 2^n - 1$
   for applications in feature detection and denoising
4. Solid mathematical theory in terms of performance

Cons

1. Non-trivial to do.
   Intrinsic factorizations in high-D are essentially impossible.

Neutral

1. Later: not all wavelet constructions are obtained in this way.

# Wavelets: Challenges in high-D constructions

**or, prevailing approaches go kaput in high-D**

The Laplacian pyramid is challenged since:

1. It becomes immensely non-local.
2. There was no rigorous performance analysis, hence lack of mathematical guidance (not even frame analysis).
3. “Feels not right”: after all, the most general wavelet constructions cannot be associated with such pyramid.

Intrinsic wavelet constructions are challenged since:

1. They are in between very difficult and impossible: In $n$-D, one needs to define $\geq 2^n - 1$ different highpass rules (=mother wavelets).

Simple lifting of univariate wavelets constructions (known as tensor products) are still challenged since:

1. They lead, again, to highly non-local constructs.

# Wavelets: Challenges in high-D cont'ed

## or, tearful moments for wavelet lovers

### Benchmark: Tensor product of biorthogonal 9/7

The tensor biorthogonal 9/7 can analyse $C^{1.70}$-function in $\mathbb{R}^{10}$.
There are 1023 mother wavelets,
each supported in a box of volume.... 562,000,000,
and the total volume is $>$ 575,000,000,000.

# Wavelets: Challenges in high-D cont'ed

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| $s_J$ | 2 | **2** | **2** | 4 | **4** |
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- Step V: bi-orthogonal constructions. Finding a way to remove the redundancy from the CAMP and L-CAMP representation
- Step VI: numerous bi-products. For example, we had to develop new ways for estimating smoothness of refinable functions in high-D.

# L-CAMP: Hallmarks

1. Extreme localness.
2. Works in any spatial dimension.
3. Trivial to construct and implement.
4. Super fast algorithms:
   linear complexity with tiny constants,
   and the constants decay with the dimension!
5. Solid performance theory
   (that shows that, at least in theory, they perform as good as much more complicated wavelets).

# L-CAMP: Extreme localness

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## A competing L-CAMP system

We construct an L-CAMP system such that it analyses $C^2$-function in $\mathbb{R}^{10}$.
There are 1024 mother wavelets,
each supported in a box of average volume.... 0.005857,
and the total volume is $<$ 6.

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# L-CAMP: The algorithms

## Decomposition

**Step I:** choose three lowpass filters

back to haar

$$h_c := 2^{-n} \sum_{\nu \in \{0,1\}^n} \delta_\nu$$ =: compression filter

$h_e :=$ $n$-dimensional enhancement filter

$h :=$ 1-D, supported on the odd integers main filter

# L-CAMP: The algorithms

## Decomposition

**Step I:** choose three lowpass filters

**Step II:** build the MRA

$\downarrow$ is downsampling:

$$y_{\downarrow}(k) = y(2k), \quad k \in \mathbb{Z}^n$$

$(y_j)_{j=-\infty}^{\infty} \subset \mathbb{C}^{\mathbb{Z}^n}$ s.t:

$$y_{j-1} = Cy_j := (h_c * y_j)_{\downarrow}, \quad \forall j.$$

# L-CAMP: The algorithms

## Decomposition

**Step I:** choose three lowpass filters

**Step II:** build the MRA

**Step III:** extract detail coefficients:

(1) For $k \in 2\mathbb{Z}^n$, $d_j(k) := y_j(k) - (h_e * y_{j-1})(k/2)$.
(2) For $\nu \in \{0,1\}^n$, and $k \in \nu + 2\mathbb{Z}^n$,

$$d_j(k) = y_j(k) - (h_{J(\nu)} * y_j)(k).$$

$h_{J(\nu)} =?$ sampling

back to Theorem back to sampling reconstruction

# L-CAMP: The algorithms

## Decomposition

**Step I:** choose three lowpass filters

Examples of $h$:

$$h = [\mathbf{0}, 1], \quad h = [\frac{1}{2}, \mathbf{0}, \frac{1}{2}], \quad h = \frac{1}{16} \times [-1, 0, 9, \mathbf{0}, 9, 0, -1].$$

back to performance

**Step II:** build the MRA

**Step III:** extract detail coefficients:

# L-CAMP: The algorithms

## Reconstruction

Step I: for $k \in 2\mathbb{Z}^n$,

$$y_j(k) := d_j(k) + (h_e * y_{j-1})(k/2).$$

# L-CAMP: The algorithms

## Reconstruction

**Step I:** for $k \in 2\mathbb{Z}^n$,

$$y_j(k) := d_j(k) + (h_e * y_{j-1})(k/2).$$

**Step II:** iteratively, by suitably ordering $\{0,1\}^n \backslash 0$:

$$y_j(k) = d_j(k) + (h_{J(\nu)} * y_j)(k).$$

# L-CAMP: The algorithms

## Complexity

Denote: $h_e$ is $A$-tap, $h$ is $B$-tap

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Decomposition requires for $2^n$ details coefficients:
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Reconstruction requires: $A + 1 + (B + 1) \times (2^n - 1)$.

**Average # of operations per one details coefficient** [a]

[a] per one complete cycle of decom-recon

$2B + 3 + 2^{1-n}(A + 1)$.

# L-CAMP: Performance analysis

## The key components in the L-CAMP performance analysis

- **The accuracy of the main filter $h$:**

$$h * P = P, \quad \forall \text{ univariate polynomial } P \text{ of degree } < s_1$$

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- **The smoothness $s_3$ of the refinable function $\phi^{dual}$** whose mask is

$$\widehat{h}_e \widehat{h}_{tensor},$$

with $h_{tensor}$ the $n$-dimensional tensor-product of $\frac{\delta + h}{2}$.

# L-CAMP: Performance analysis

## L-CAMP based performance results

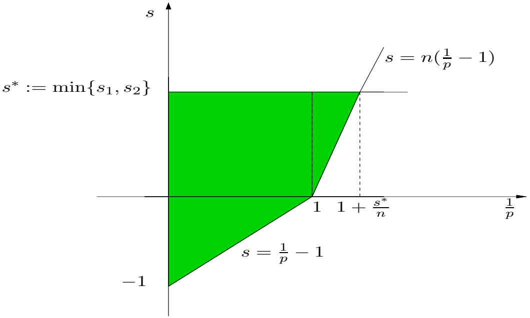
**Theorem (Hur-R, 2005)**

*Assume that we have an L-CAMP system.*
*Let $\Psi$ be the mother wavelet set associated with the highpass filters in L-CAMP Decomposition.*
*Let $\min\{s_1, s_2\} \geq 2$.*
*Let $s_3 > 0$.*
*Then $X(\Psi)$ has $s_J \geq \min\{s_1, s_2\}$ and $s_B \geq \min\{s_1, s_2, s_3\}$.*

# L-CAMP: Performance analysis

## The Jackson-type performance chart of L-CAMP

### performance chart



# L-CAMP: Performance analysis

## Example 1: extremely local MR representations for $C^1$ characterization

$$h := [\frac{1}{2}, \mathbf{0}, \frac{1}{2}], \quad \text{2-tap},$$

$$\widehat{h}_e(\omega) := \frac{3}{4} + \frac{1}{4} e^{-i\mathbf{1}\cdot\omega}, \quad \text{2-tap}.$$

- **The accuracy of the univariate filter $h$:** $s_1 = 2$.
- **The accuracy of the pair $(h_c, h_e)$:** $s_2 = 2$.
- **The smoothness class of the refinable function $\phi^{dual}$ whose mask is $\widehat{h}_e \widehat{h}_{tensor}$ :** $s_3 > 1$ $(s_3 = 1.4)$.

**Average # of operations**: $7 + 3 \cdot 2^{1-n}$.

**Total volume of the wavelets' support**: $< 5$.

# L-CAMP: Performance analysis

## Example 2: extremely local MR representations for $C^2$ characterization

$$h := [\frac{1}{2}, \mathbf{0}, \frac{1}{2}], \quad \text{2-tap},$$

$$\widehat{h}_e(\omega) := \frac{1}{8}e^{i\mathbf{1}\cdot\omega} + \frac{1}{2} + \frac{3}{8}e^{-i\mathbf{1}\cdot\omega}, \quad \text{3-tap}.$$

- **The accuracy of the univariate filter $h$:** $s_1 = 2$.
- **The accuracy of the pair $(h_c, h_e)$:** $s_2 = 2$.
- **The smoothness class of the refinable function $\phi^{dual}$ whose mask is $\widehat{h_e}\widehat{h}_{tensor}$ :** $s_3 > 2$ ($s_3 = 2.4$).

**Average # of operations**: $7 + 4 \cdot 2^{1-n}$.

**Total volume of the wavelets' support**: $< 6$.

# L-CAMP: Performance analysis

## L-CAMP vs. Biorthogonal systems for $n = 3, 4, 5$

We compare the L-CAMP systems with biorthogonal tensor product systems for the spatial dimension $n = 3, 4, 5$.
In the last column, properties of yet another L-CAMP is shown.
In the last 3 rows, the total volume of the mother wavelets is listed for each $n = 3, 4, 5$.

| | 5/3 | **L-CAMP 1** | **L-CAMP 2** | 9/7 | **L-CAMP 3** |
|---|---|---|---|---|---|
| $s_J$ | 2 | **2** | **2** | 4 | **4** |
| $s_B$ | 1 | **1.41** | **2** | 1.70 | **2.02** |
| $n = 3$ | 279 | **4.6** | **5.6** | 2863 | **14.4** |
| $n = 4$ | 2145 | **4.8** | **5.8** | 46529 | **16.7** |
| $n = 5$ | 15783 | **4.9** | **5.9** | 726607 | **18.8** |

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# MRA Pyramids

$y_0$

## Description

- $y_0 : \mathbb{Z}^n \to \mathbb{C}$: initial data set

# MRA Pyramids

$$y_0 \longrightarrow y_{-1} \longrightarrow y_{-2} \longrightarrow \cdots \longrightarrow y_{-j+1} \longrightarrow y_{-j}$$

## Description

- $y_0 : \mathbb{Z}^n \to \mathbb{C}$: initial data set
- $y_{-1}, \ldots, y_{-j}$: coarse resolutions of the data

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$$y_0 \xrightarrow{C} y_{-1} \xrightarrow{C} y_{-2} \xrightarrow{C} \cdots \xrightarrow{C} y_{-j+1} \xrightarrow{C} y_{-j}$$

## Description

- $y_0 : \mathbb{Z}^n \to \mathbb{C}$: initial data set
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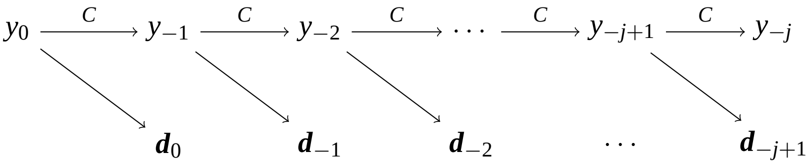
- $C : y \mapsto (h_c * y)_{\downarrow}$, $h_c$: low-pass filter, $\downarrow$: downsampling.

# MRA Pyramids

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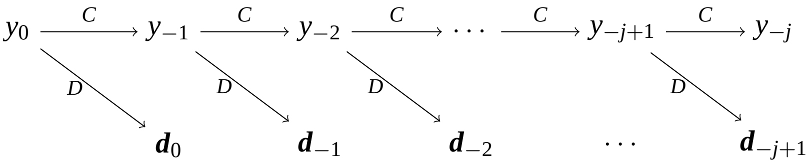


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- $D : y \mapsto ((h_l * y)_\downarrow)_{l=1,\ldots,r}$, $h_l$: high-pass filter

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y_0 & \xrightarrow{C} & y_{-1} & \xrightarrow{C} & y_{-2} & \xrightarrow{C} & \cdots & \xrightarrow{C} & y_{-j+1} & \xrightarrow{C} & y_{-j} \\
& \searrow^{D} & & \searrow^{D} & & \searrow^{D} & & & & \searrow^{D} & \\
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## Concerns with $C : y \mapsto (c * y)_{\downarrow}$

- Computation always required. High overhead when computing detail coefficients selectively.

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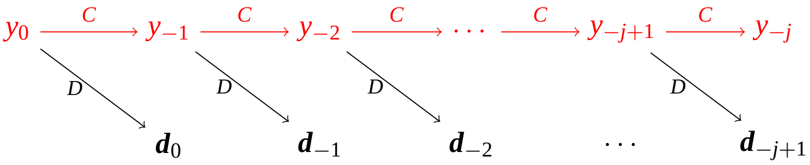



## Concerns with $C : y \mapsto (c * y)_{\downarrow}$

- Computation always required. High overhead when computing detail coefficients selectively.
- Change in resolution of data causes re-computing all coefficients.

# MRA Pyramids

$$
\begin{array}{cccccccccccc}
y_0 & \xrightarrow{C} & y_{-1} & \xrightarrow{C} & y_{-2} & \xrightarrow{C} & \cdots & \xrightarrow{C} & y_{-j+1} & \xrightarrow{C} & y_{-j} \\
 & \searrow^{D} & & \searrow^{D} & & \searrow^{D} & & & & \searrow^{D} & \\
 & & \boldsymbol{d}_0 & & \boldsymbol{d}_{-1} & & \boldsymbol{d}_{-2} & & \cdots & & \boldsymbol{d}_{-j+1}
\end{array}
$$

## Concerns with $C : y \mapsto (c * y)_{\downarrow}$

- Computation always required. High overhead when computing detail coefficients selectively.
- Change in resolution of data causes re-computing all coefficients.
- Excessive blurring.

# MRA Pyramids

$$
\begin{array}{ccccccccccc}
y_0 & \xrightarrow{C} & y_{-1} & \xrightarrow{C} & y_{-2} & \xrightarrow{C} & \cdots & \xrightarrow{C} & y_{-j+1} & \xrightarrow{C} & y_{-j} \\
& \searrow^{D} & & \searrow^{D} & & \searrow^{D} & & & & \searrow^{D} & \\
& & \boldsymbol{d}_0 & & \boldsymbol{d}_{-1} & & \boldsymbol{d}_{-2} & & \cdots & & \boldsymbol{d}_{-j+1}
\end{array}
$$

## Possible Remedy

$$C : y \mapsto (y)_{\downarrow}, \quad \text{i.e.,} \quad c = \delta$$

# MRA Pyramids

$$
\begin{array}{ccccccccccc}
y_0 & \xrightarrow{C} & y_{-1} & \xrightarrow{C} & y_{-2} & \xrightarrow{C} & \cdots & \xrightarrow{C} & y_{-j+1} & \xrightarrow{C} & y_{-j} \\
& \searrow^{D} & & \searrow^{D} & & \searrow^{D} & & & & \searrow^{D} & \\
& & \boldsymbol{d}_0 & & \boldsymbol{d}_{-1} & & \boldsymbol{d}_{-2} & & \cdots & & \boldsymbol{d}_{-j+1}
\end{array}
$$

## Possible Remedy

$$C : y \mapsto (y)_{\downarrow}, \quad \text{i.e.,} \quad c = \delta$$

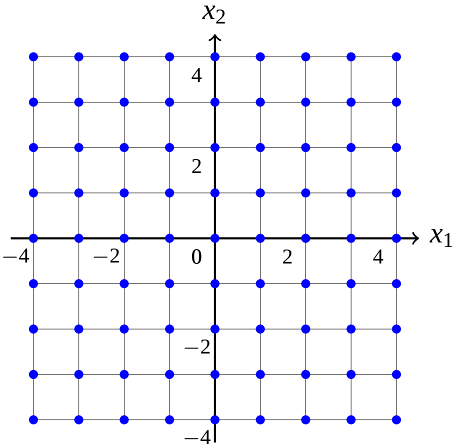
$\Longrightarrow$ Pyramidal Representations based on Sampling

# Outline

1. Part I: Time-freq. representations of almost-peridoic functions
   - Almost perdioic functions
   - Capturing the AP-norm
   - Our results
2. Part II: L-CAMP – Efficient wavelet represent'n in high D
   - Pyramidal representations and wavelets
   - Introduction to localness and performance
   - L-CAMP: A bird's view of the CAP methodologies
   - L-CAMP: The algorithms & performance analysis
3. **Part III: L-CAMP representation based on sampling**
   - Motivation
   - Sampling based L-CAMP: the algorithms
   - Sampling based L-CAMP: performance

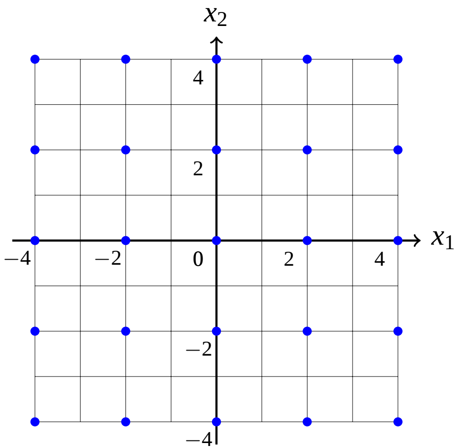
# Sampling based L-CAMP

## Decomposition



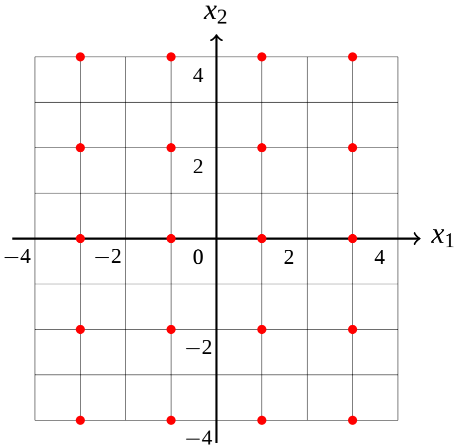

Starting data

# Sampling based L-CAMP
## Decomposition




Coarser resolution

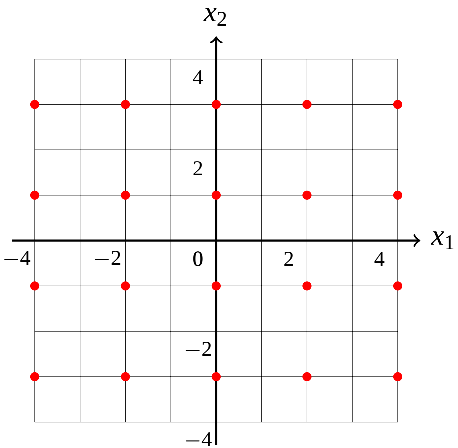
# Sampling based L-CAMP
## Decomposition




Locations of detail coefficients $d_{(1,0)}$

# Sampling based L-CAMP

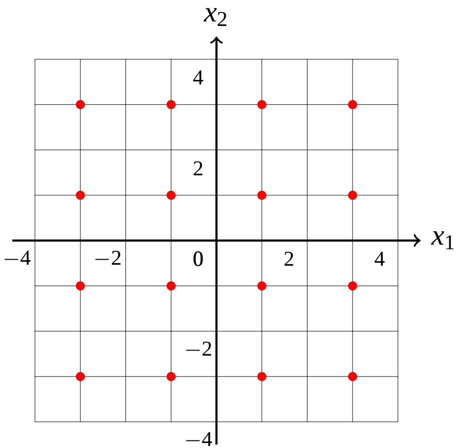
## Decomposition




Locations of detail coefficients $d_{(0,1)}$

# Sampling based L-CAMP

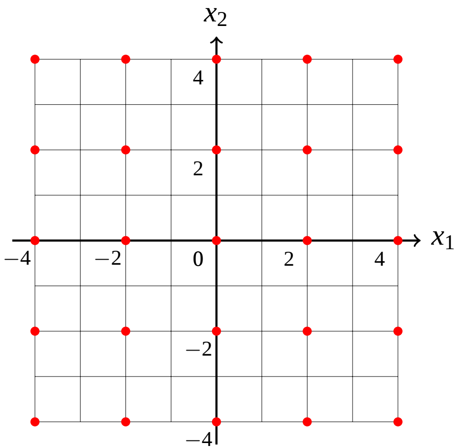
## Decomposition




Locations of detail coefficients $d_{(1,1)}$

# Sampling based L-CAMP

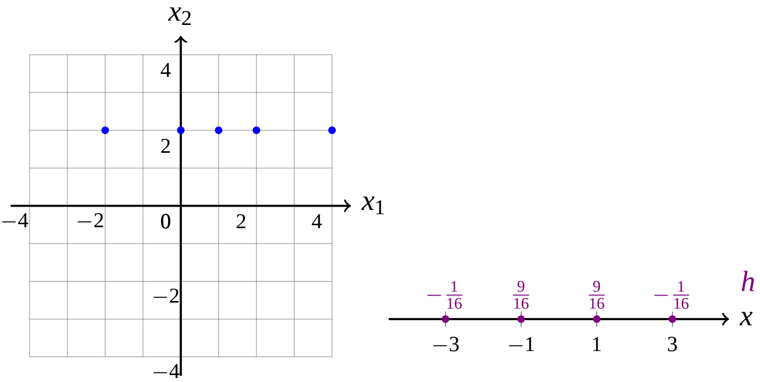
## Decomposition




Locations of detail coefficients $d_{(0,0)}$

# Sampling based L-CAMP

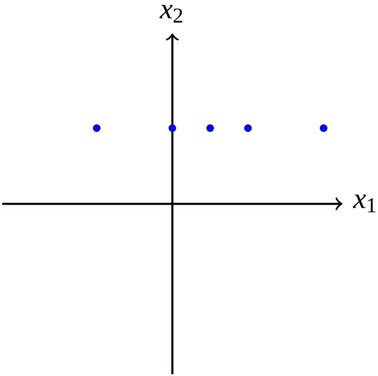
## Decomposition: Computation of $d_{(1,0)}(1,2)$



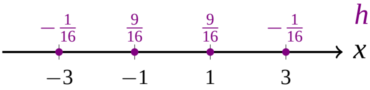

The main filter $h$ (right), and the points involved in calculation (blue)

# Sampling based L-CAMP

## Decomposition: Computation of $d_{(1,0)}(1,2)$



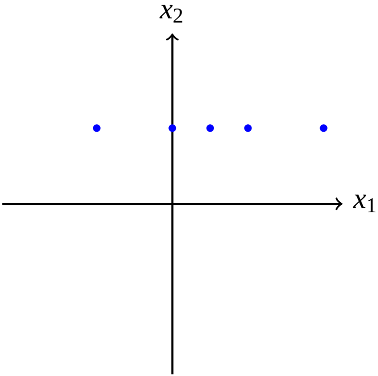

$$h_{(1,0)} = (\delta - h_1)(\cdot - (1,0))$$




High-pass filter associated with $(1,0)$

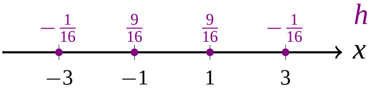
# Sampling based L-CAMP

## Decomposition: Computation of $d_{(1,0)}(1,2)$




$$h_{(1,0)} = (\delta - h_1)(\cdot - (1,0))$$

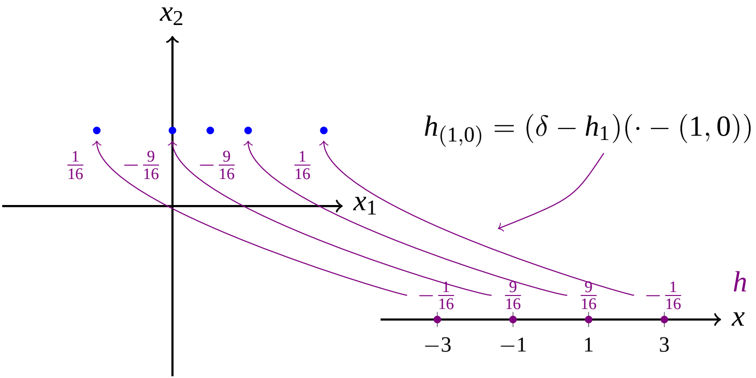
index




Subscript $1$ : index of $1$ in $(1,0)$

# Sampling based L-CAMP

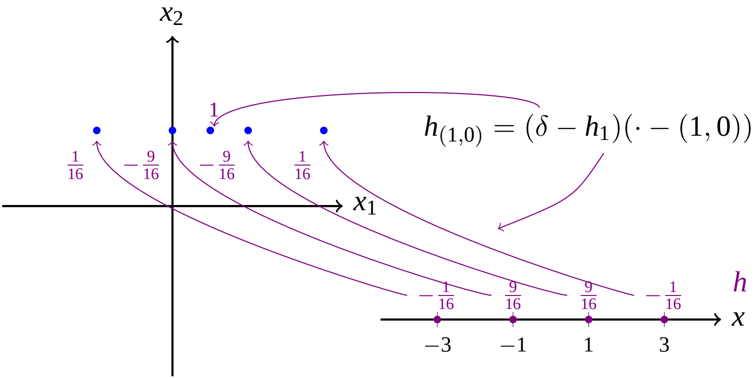
## Decomposition: Computation of $d_{(1,0)}(1,2)$




$h_1$ : $h$ in $x_1$-direction

# Sampling based L-CAMP

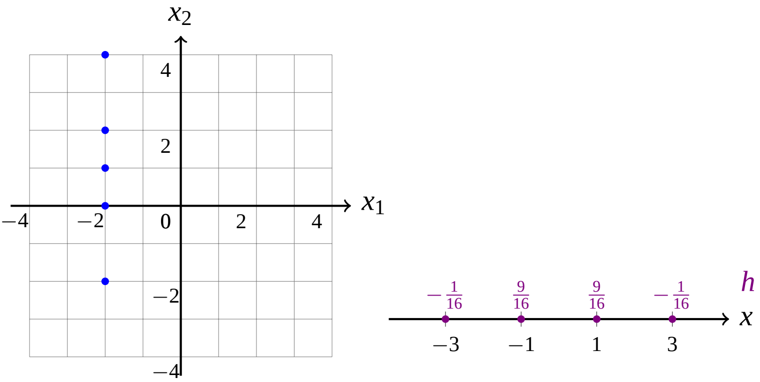
## Decomposition: Computation of $d_{(1,0)}(1,2)$




Weights used in calculation

# Sampling based L-CAMP

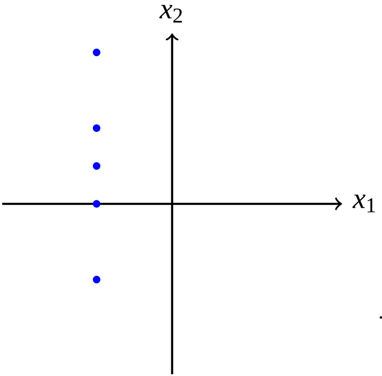
## Decomposition: Computation of $d_{(0,1)}(-2,1)$



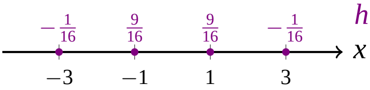

The main filter $h$ (right), and the points involved in calculation (blue)

# Sampling based L-CAMP

## Decomposition: Computation of $d_{(0,1)}(-2,1)$



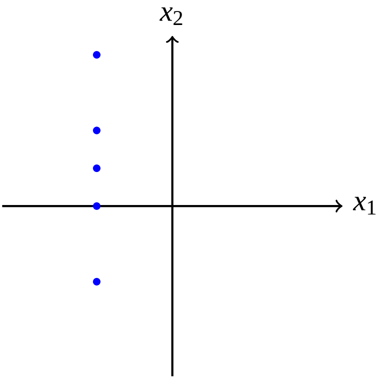

$$h_{(0,1)} = (\delta - h_2)(\cdot - (0,1))$$




High-pass filter associated with $(0, 1)$

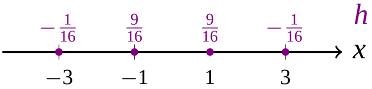
# Sampling based L-CAMP

## Decomposition: Computation of $d_{(0,1)}(-2,1)$




$$h_{(0,1)} = (\delta - h_2)(\cdot - (0,1))$$

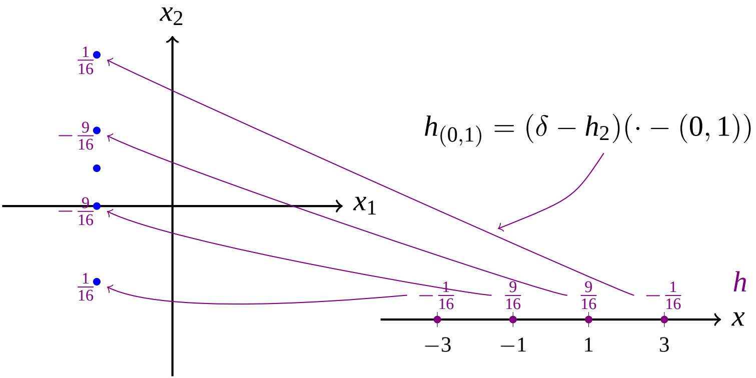
index




Subscript 2 : index of 1 in $(0,1)$

# Sampling based L-CAMP

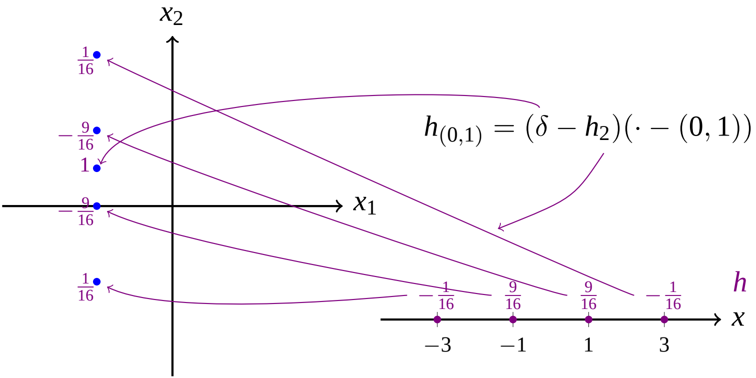
## Decomposition: Computation of $d_{(0,1)}(-2,1)$




$h_2$ : $h$ in $x_2$-direction

# Sampling based L-CAMP

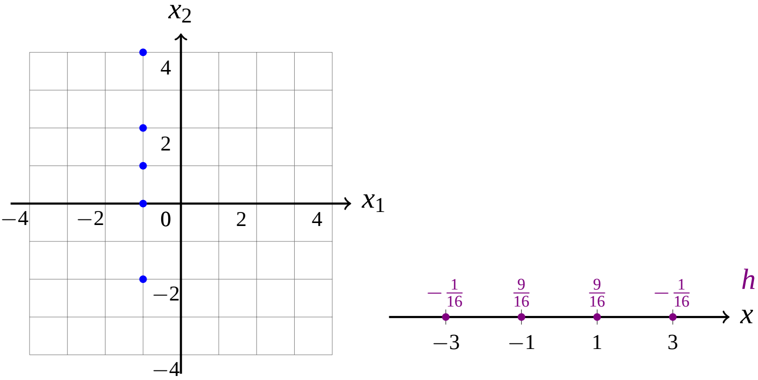
## Decomposition: Computation of $d_{(0,1)}(-2,1)$




Weights used in calculation

# Sampling based L-CAMP

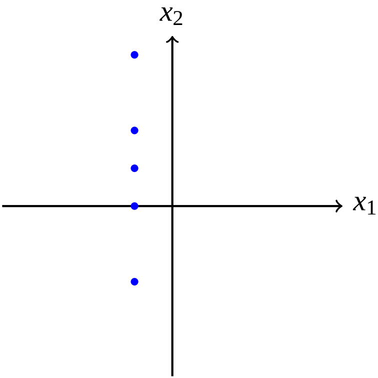
## Decomposition: Computation of $d_{(1,1)}(-1, 1)$



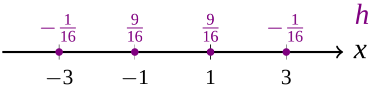

The main filter $h$ (right), and the points involved in calculation (blue)

# Sampling based L-CAMP

## Decomposition: Computation of $d_{(1,1)}(-1,1)$



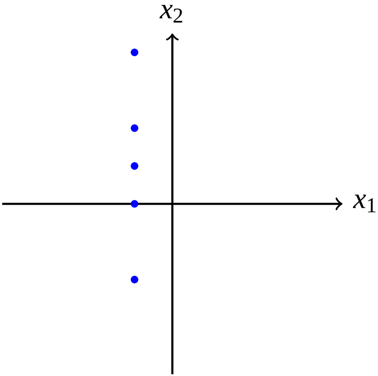

$$h_{(1,1)} = (\delta - h_2)(\cdot - (1,1))$$




High-pass filter associated with $(1,1)$

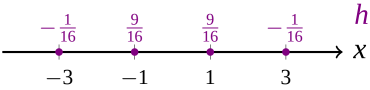
# Sampling based L-CAMP

## Decomposition: Computation of $d_{(1,1)}(-1,1)$




$$h_{(1,1)} = (\delta - h_2)(\cdot - (1,1))$$

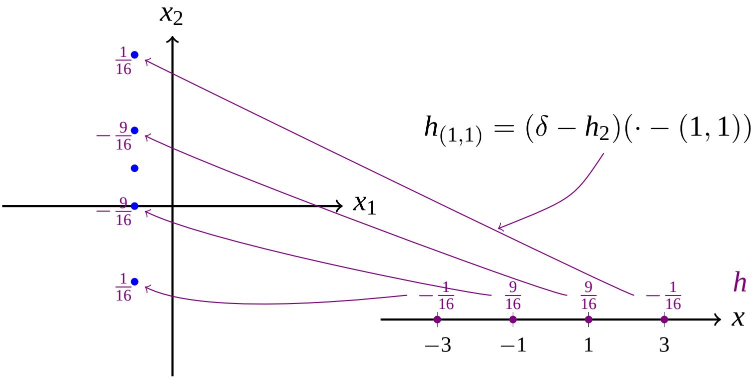
index




Subscript $2$ : index of last $1$ in $(1,1)$

# Sampling based L-CAMP

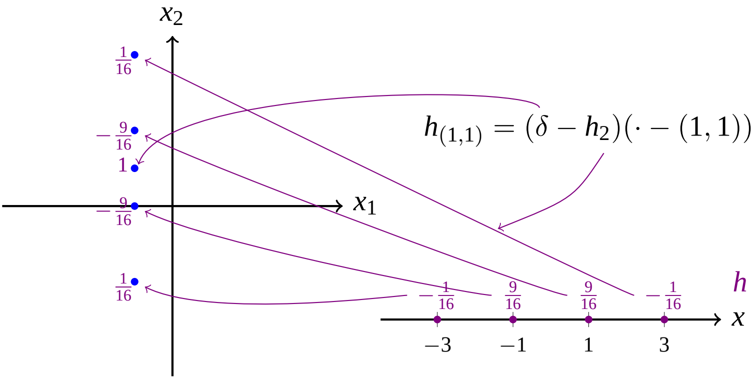
## Decomposition: Computation of $d_{(1,1)}(-1,1)$




$h_2$ : $h$ in $x_2$-direction

# Sampling based L-CAMP

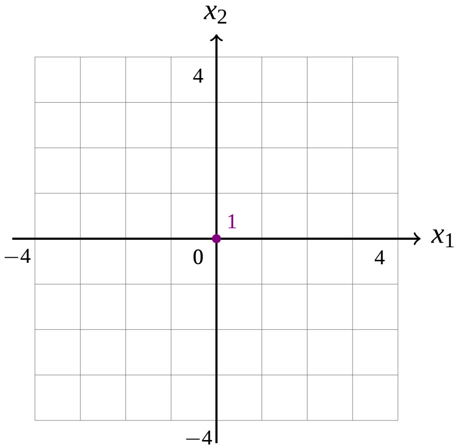
## Decomposition: Computation of $d_{(1,1)}(-1,1)$




Weights used in calculation

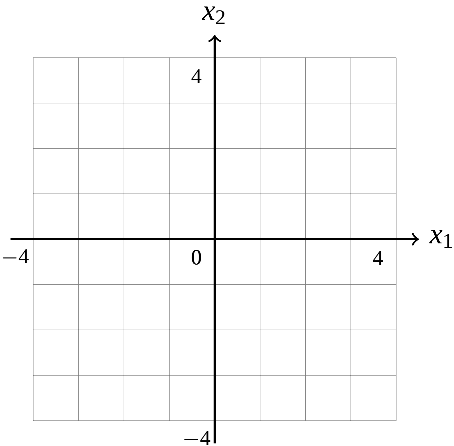
# Sampling based L-CAMP

## Decomposition




Enhancement filter $h_e = \delta$

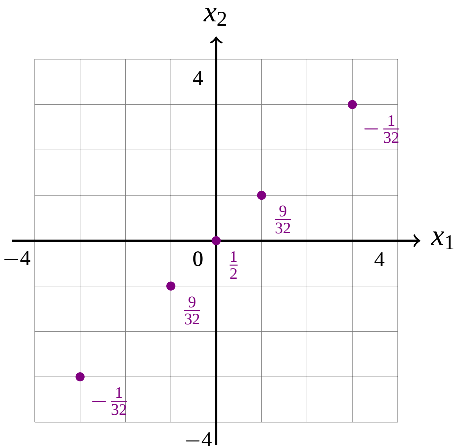
# Sampling based L-CAMP
## Decomposition




$d_{(0,0)}$ is 0 for $h_e = \delta$

# Sampling based L-CAMP

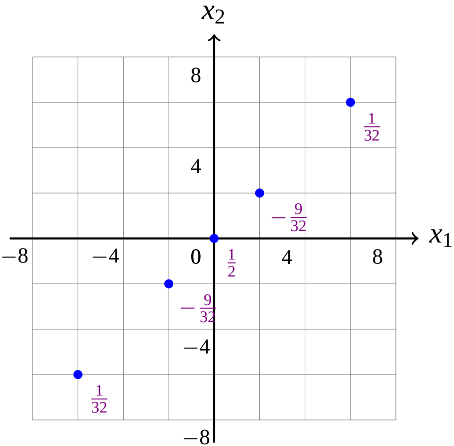
## Decomposition




Enhancement filter $h_e$ (DD4)

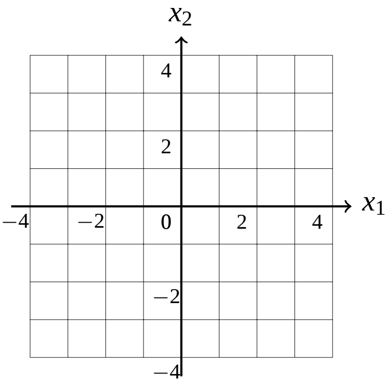
# Sampling based L-CAMP

## Decomposition: Computation of $d_{(0,0)}(0,0)$



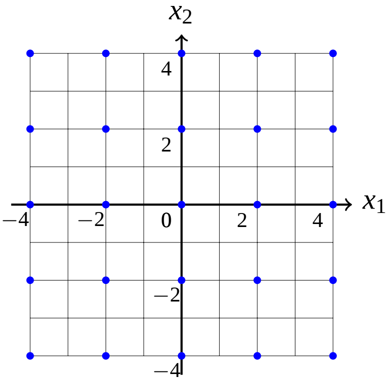

Weights used when $h_e$ is (diagonal) DD4 filter

# Sampling based L-CAMP
## Reconstruction




Data at grid points are to be recovered back to algorithms

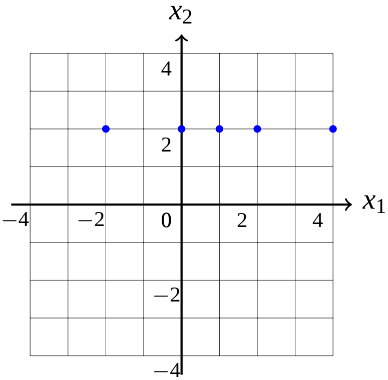
# Sampling based L-CAMP
## Reconstruction




back to algorithms Recovered points (from coarse representation)

# Sampling based L-CAMP

## Reconstruction

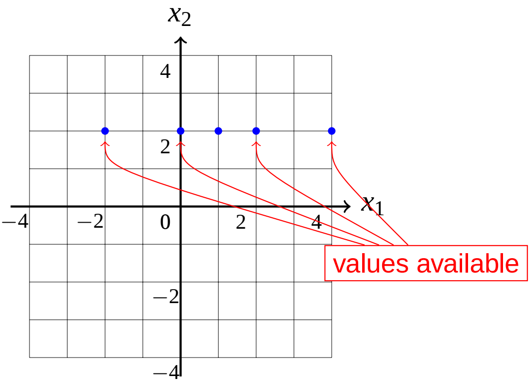



back to algorithms

Points used for computing $d_{(1,0)}(1,2)$

# Sampling based L-CAMP

## Reconstruction

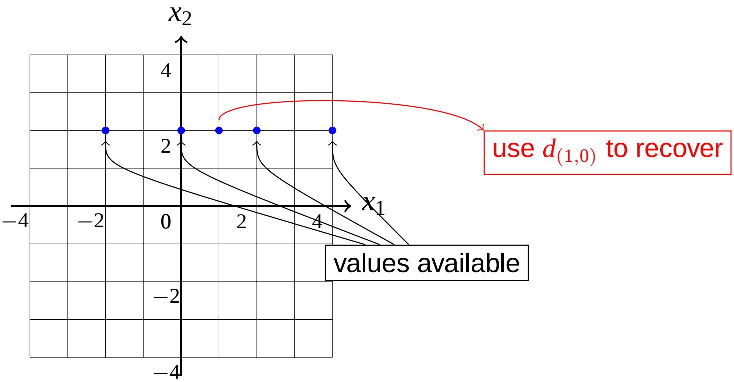



back to algorithms

Four points already recovered

# Sampling based L-CAMP

## Reconstruction

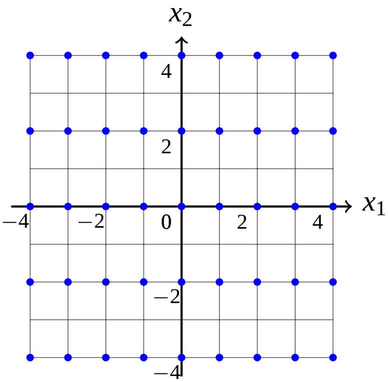



back to algorithms

Recover all $(1,0) + 2\mathbb{Z} \times 2\mathbb{Z}$ this way

# Sampling based L-CAMP

## Reconstruction

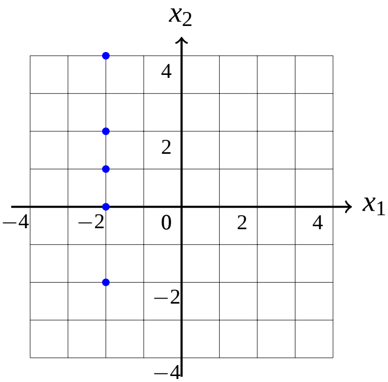



back to algorithms

Recovered points so far

# Sampling based L-CAMP

## Reconstruction

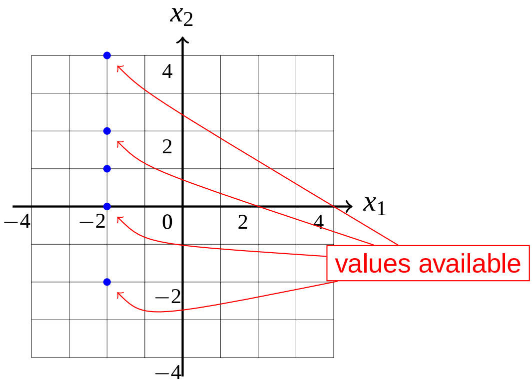



back to algorithms

Points used for computing $d_{(0,1)}(-2, 1)$

# Sampling based L-CAMP

## Reconstruction

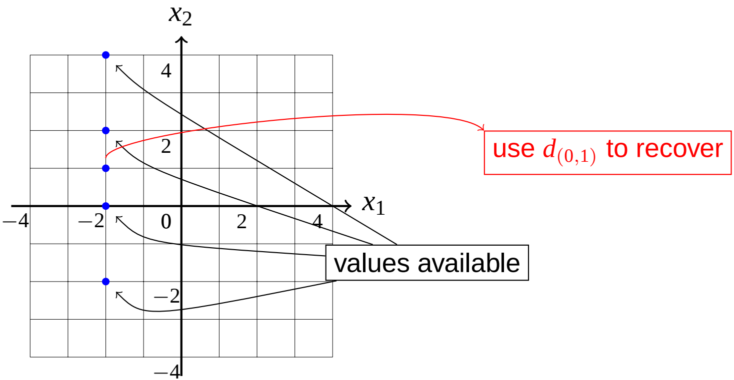



back to algorithms

Four points already recovered

# Sampling based L-CAMP

## Reconstruction

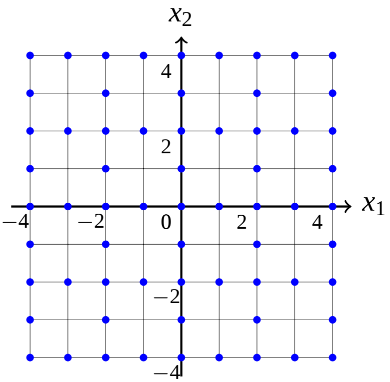



▸ back to algorithms

Recover all $(0,1) + 2\mathbb{Z} \times 2\mathbb{Z}$ this way

# Sampling based L-CAMP

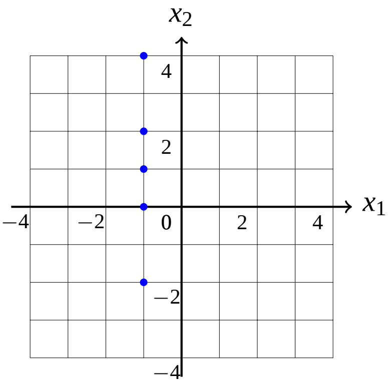
## Reconstruction




back to algorithms

Recovered points so far

# Sampling based L-CAMP
## Reconstruction

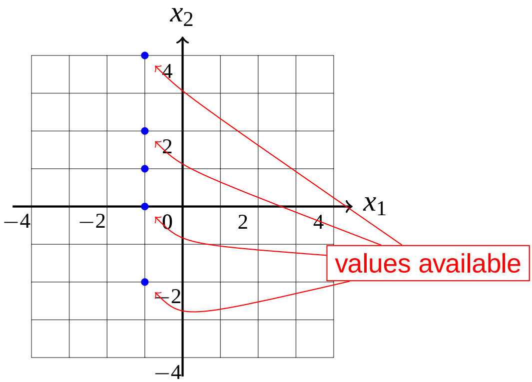



back to algorithms

Points used for computing $d_{(1,1)}(-1,1)$

# Sampling based L-CAMP

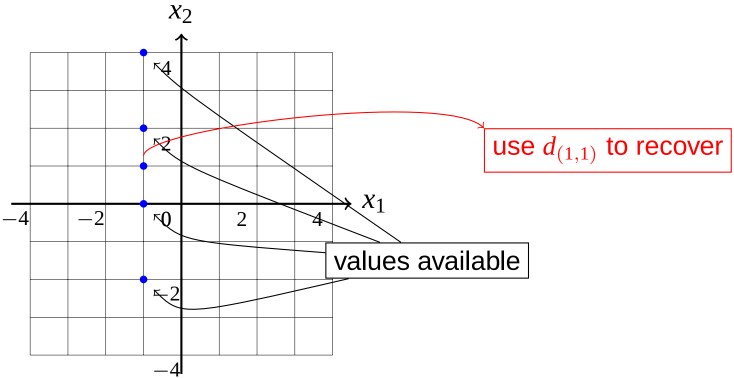
## Reconstruction




back to algorithms

Four points already recovered

# Sampling based L-CAMP
## Reconstruction

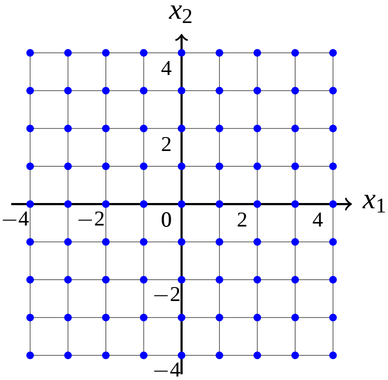



back to algorithms

Recover all $(1,1) + 2\mathbb{Z} \times 2\mathbb{Z}$ this way

# Sampling based L-CAMP

## Reconstruction




back to algorithms

All data recovered!

# Outline

1. **Part I: Time-freq. representations of almost-peridoic functions**
   - Almost perdioic functions
   - Capturing the AP-norm
   - Our results
2. **Part II: L-CAMP – Efficient wavelet represent'n in high D**
   - Pyramidal representations and wavelets
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   - L-CAMP: The algorithms & performance analysis
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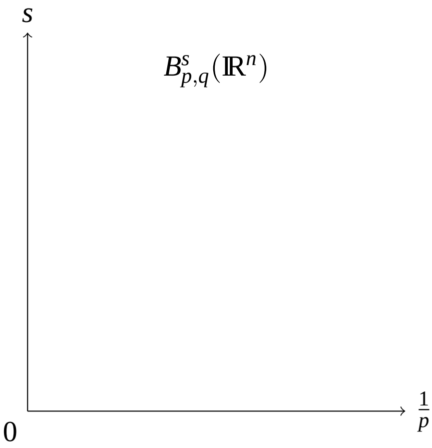
   - Motivation
   - Sampling based L-CAMP: the algorithms
   - Sampling based L-CAMP: performance

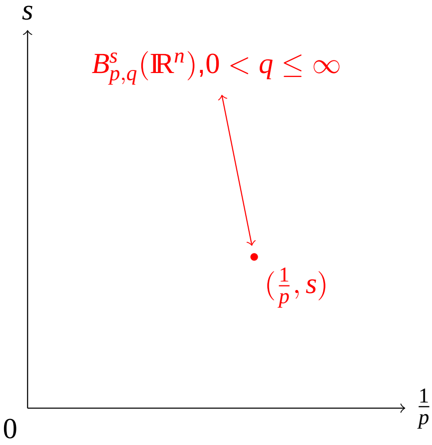
# Sampling based L-CAMP

## Performance Regions



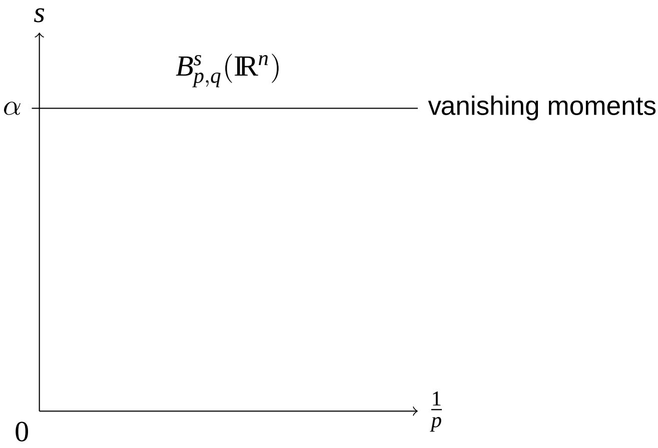
# Sampling based L-CAMP

## Performance Regions



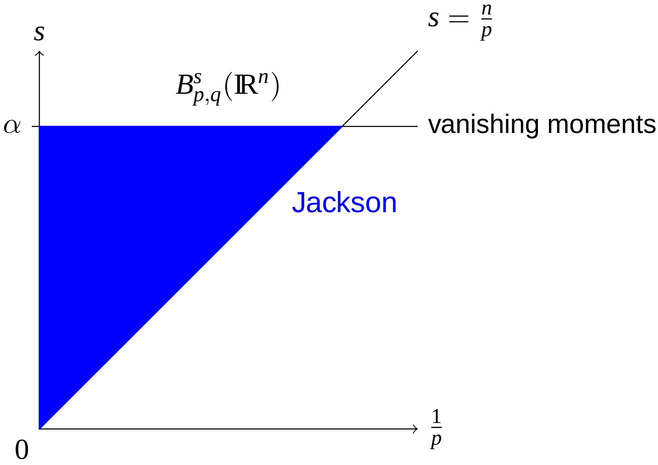
# Sampling based L-CAMP

## Performance Regions



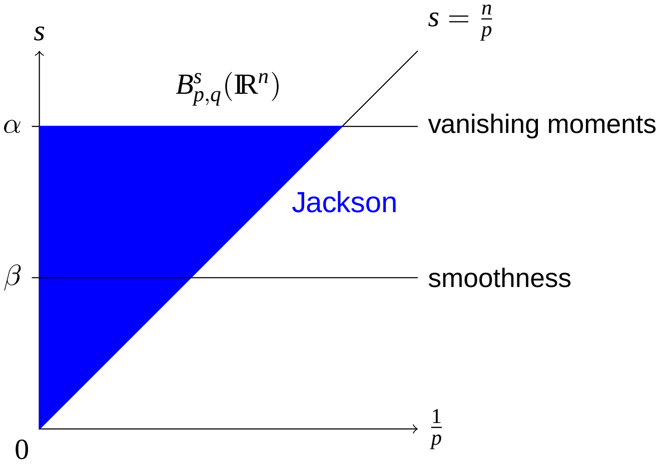
# Sampling based L-CAMP

## Performance Regions



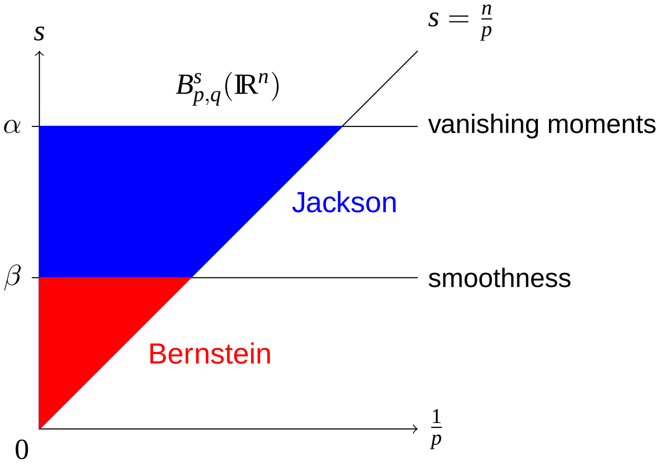
# Sampling based L-CAMP

## Performance Regions

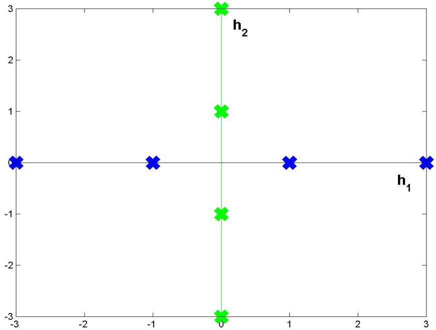


# Sampling based L-CAMP

## Performance Regions



# Orienting the univariate filter

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# Definition of vanishing moments

Wavelet system $X(\Psi)$ has $m$ vanishing moments if

$$\int t^{\beta}\psi(t)dt = 0, \quad \forall 0 \leq |\beta| \leq m-1, \forall \psi \in \Psi.$$

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