Application of degree reduction of polynomial Bézier curves to rational case

Yunbeom Park
Seowon University
ybpark@seowon.ac.kr

- Polynomial Bézier curve of degree n

$$\mathbf{b}^n(t) = \sum_{k=0}^{n} \mathbf{b}_k \mathsf{B}_k^n(t) \ , \qquad 0 \leq t \leq 1,$$

  where $\mathsf{B}_k^n(t)$ are Bernstein polynomials of degree n, and $\mathbf{b}_k (k = 0, \cdots, n)$ are control points of $\mathbf{b}^n(t)$.

- Degree reduction of polynomial Bézier curves is overdetermined problem and it invokes approximation problem.

• Many efforts and proposals for dealing with problems for polynomial cases have been made in the last two decades.

1. M. A. Lachance, Chebyshev economization for parametric surface, *Comput. Aided Geometric Des.* 5:195–208 (1988)

2. M. Watkins and A. Worsey, Degree reduction of curves, *Comput. Aided Des.* 20:398–405 (1988)

3. M. Eck, Degree reduction of Bézier curves, *Comput. Aided Geometric Des.* 10:237–251 (1993)

4. G. Brunnett, T. Schreiber and J. Braun, The geometry of optimal degree reduction of Bézier curves, *Comput. Aided Geometric Des.* 13:773–7882 (1996)

5. H. O. Kim, J. H. Kim and S. Y. Moon, Degree reduction of Bézier curves and filter banks, *Comput. Math. Appl.* 31:23–30 (1996)

6. H. O. Kim and S. Y. Moon, Degree reduction of Bézier curves by L^1–approximation with endpoints interpolation, *Comput. Math. Appl.* 33:67–77 (1997)

7. B. G. Lee and Y. Park, Distance for Bézier curves and degree reduction, *Bull. Australian Math. Soc.* 56:507–515 (1997)

8. D. Lutterkort, J. Peters and U. Reif, Polynomial degree reduction in the L_2–norm equals best Euclidean approximation of Bézier coefficients, *Comput. Aided Geometric Des.* 16:607–612 (1999)

9. J. Peters and U. Reif, Least square approximation of Bézier coefficients provides best degree reduction in the L_2–norm, *J. Approx. Theory* 104:90–97 (2000)

10. B. G. Lee, Y. Park and J. Yoo, Application of Legendre–Bernstein basis transformations to degree elevation and reduction, *Comput. Aided Geometric Des.* 19:709–718 (2002)

- If the error is larger than prespecified tolerance, then subdivision schemes are needed.

- Often the best degree–reduced Bézier curves are not smooth enough at the subdivision points.

- In case of CAD/CAM system, it is required that the curves and surfaces are continuous of order $\alpha \geq 1$.

• The constrained degree reduction of polynomial Bézier curves with $C^{\alpha-1}$–continuity at both endpoints has been developed and discussed in many literatures.

1. P. Bogaki, S. E. Weinstein and Y. Xu, Degree reduction of Bézier curves by uniform approximation with endpoint interpolation, *Comput. Aided Geometric Des.* 27:651–662 (1995)

2. M. Eck, Least square degree reduction of Bézier curves, *Comput. Aided Des.* 27:845–851 (1995)

3. M. A. Lachance, Approximation by constrained parametric polynomials, *Rocky Mountain J. Math.* 21:473–488 (1991)

4. H. J. Kim and Y. J. Ahn, Good degree reduction of Bézier curves using Jacobi polynomials, *Comput. Math. Appl.* 40:1205–1215 (2000)

5. G. Chen and G. Wang, Optimal multi–degree reduction of Bézier curves with constraints of endpoints continuity, *Comput. Aided Geometric Des.* 19:365–377 (2002)

6. Y. J. Ahn, Degree reduction of Bézier curves with C^k–continuity using Jacobi polynomials, *Comput. Aided Geometric Des.* 20:423–434 (2003)

7. Y. J. Ahn, B. G. Lee, Y. Park and J. Yoo, Constrained polynomial degree reduction in the L_2–norm equals best weighted

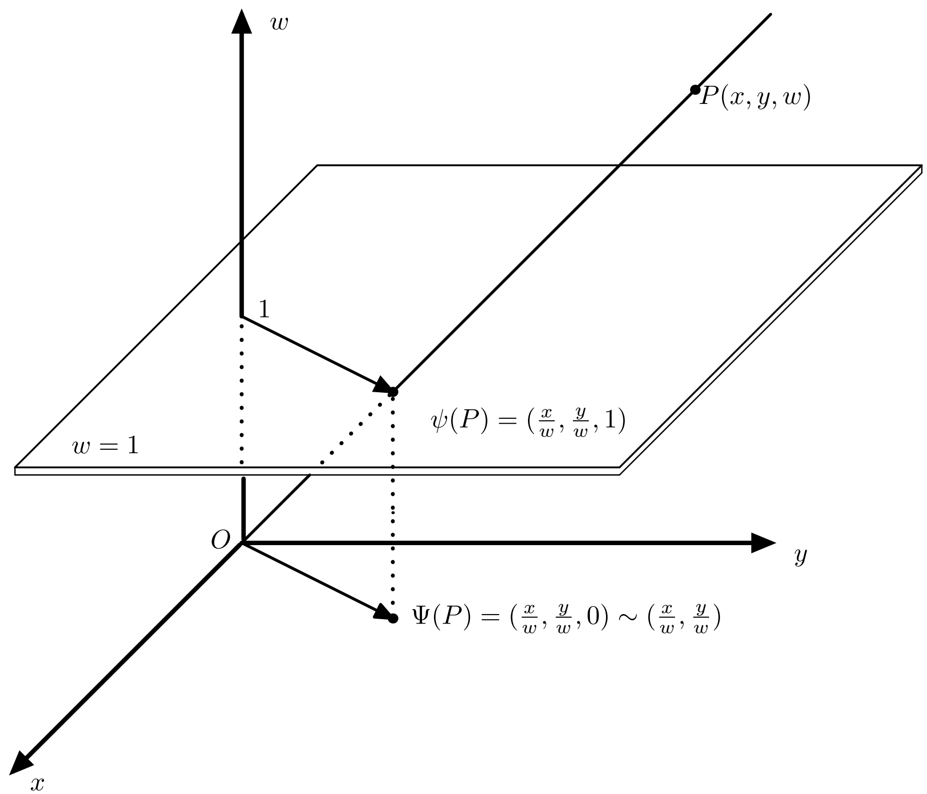
Euclidean approximation of Bézier coefficients, To appeared in *Comput. Aided Geometric Des.* (2003)

• For rational Bézier curve case, algorithms to approximate rational curves with polynomial curves are presented.

1. N. M. Patrikalakis, Approximate conversion of rational splines, *Comput. Aided Geometric Des.* 6:155–165 (1989)

2. B. G. Lee and Y. Park, Approximate conversion of rational Bézier curves, *J. KSIAM* 2:88–93 (1998)

• Rational Bézier curves are defined as the image of a polynomial Bézier curve.



Euclidean model of the projective plane.

• Now, we define a n–th degree two dimensional rational Bézier curve as the image under Ψ of a polynomial Bézier curve:

$$\mathbf{r}^n(t) = \frac{\sum_{k=0}^{n} w_k \mathbf{r}_k \mathsf{B}_k^n(t)}{\sum_{k=0}^{n} w_k \mathsf{B}_k^n(t)} \ , \qquad 0 \le t \le 1,$$

where the $\mathbf{r}_k = \Psi(\mathbf{b}_k) = (x_k, y_k)$. The w_k is weights of the control points $\mathbf{r}_k$.

• We obtain the polynomial case by setting all w_k's are same values.

• A rational Bézier curve with positive weights may be reparametrized so that the end weights are unity.

$$\hat{t} = \frac{ct}{1 + (c-1)t} \qquad \text{and} \qquad c = \sqrt[n]{\frac{w_n}{w_0}}.$$

• Curves in this form are said to be in standard form. We may assume that $w_0 = w_n = 1$.

• Also, we may perform degree elevation by degree elevating the 3–dimensional polygon with control vertices $[w_k\mathbf{r}_k \ w_k]^t$ and projecting the resulting control vertices into the hyperplane $w = 1$.

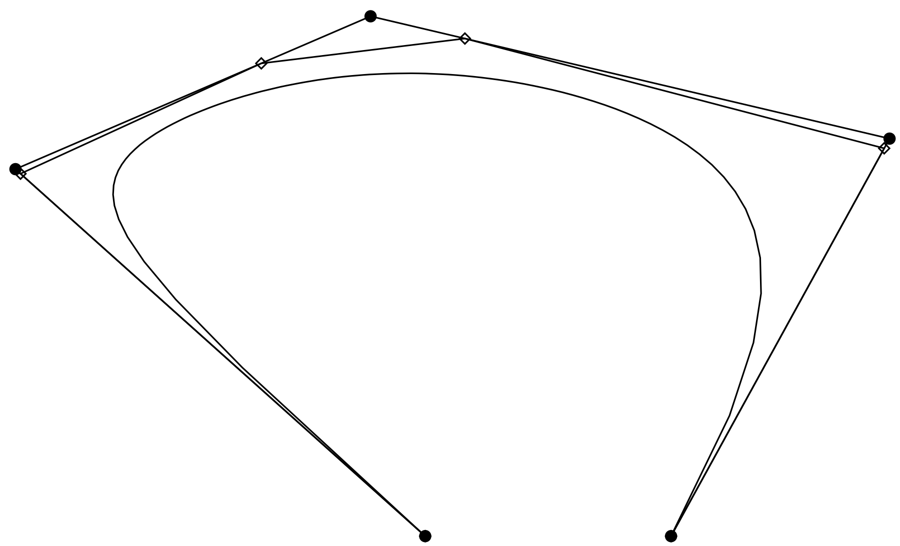
$$\mathbf{r}_k^{(1)} = \frac{w_{k-1}\alpha_k\mathbf{r}_{k-1} + w_k(1-\alpha_k)\mathbf{r}_k}{w_{k-1}\alpha_k + w_k(1-\alpha_k)} \quad \text{for} \quad k = 0, \cdots, n+1$$

where $\alpha_k = 1/(n+1)$.

• The weights $w_k^{(1)}$ of the new control vertices are given by

$$w_k^{(1)} = w_{k-1}\alpha_k + w_k(1-\alpha_k) \quad \text{for} \quad k = 0, \cdots, n+1$$

- Rational degree elevation



Rational degree elevation from degree 4 to degree 5.

- Degree reduction procedure for rational case

(a) convert the rational Bézier curve to polynomial Bézier curve by using homogenous coordinates

(b) reduce the degree of polynomial Bézier curve

(c) determine weights of degree reduced curve

(d) convert the Bézier curve obtained through step (b) to rational Bézier curve with weights in step (c)

• Rational Bézier curve $\mathbf{r}^n$ and $\mathbf{b}^n$ be a polynomial Bézier curve such that $\mathbf{r}^n = \Psi(\mathbf{b}^n)$.

$$\mathbf{r}^n(t) = \left(\frac{\sum_{k=0}^{n} w_k x_k \mathsf{B}_k^n(t)}{\sum_{k=0}^{n} w_k \mathsf{B}_k^n(t)}, \frac{\sum_{k=0}^{n} w_k y_k \mathsf{B}_k^n(t)}{\sum_{k=0}^{n} w_k \mathsf{B}_k^n(t)} \right).$$

$$\mathbf{b}^n(t) = \left(\sum_{k=0}^{n} w_k x_k \mathsf{B}_k^n(t), \sum_{k=0}^{n} w_k y_k \mathsf{B}_k^n(t), \sum_{k=0}^{n} w_k \mathsf{B}_k^n(t) \right).$$

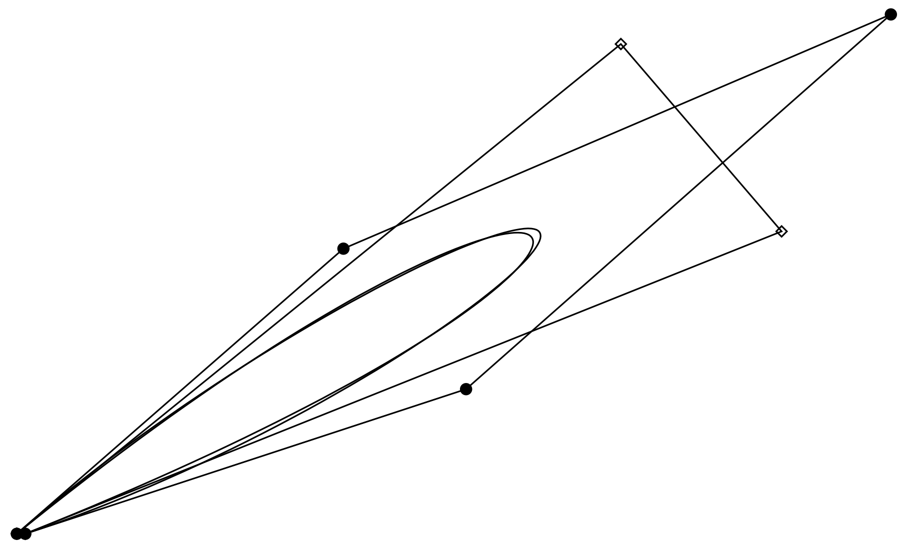
$$\bar{\mathbf{b}}^n(t) = \left(\sum_{k=0}^{n} w_k x_k \mathsf{B}_k^n(t), \sum_{k=0}^{n} w_k y_k \mathsf{B}_k^n(t), 1 \right).$$

- We can find a degree reduced curve

$$\bar{\mathbf{b}}^{n-1}(t) = \left(\sum_{k=0}^{n} X_k \mathsf{B}_k^n(t), \sum_{k=0}^{n} Y_k \mathsf{B}_k^n(t), 1 \right)$$

of $\bar{\mathbf{b}}^n$ by using previously developed method for polynomial Bézier curves.

• An example for degree reduction of polynomial Bézier curve by using the least square method with endpoints interpolation.



Degree reduced curve $\bar{\mathbf{b}}^{n-1}$(diamond) of polynomial curve $\bar{\mathbf{b}}^{n}$(solid circles).

• Now, we have to determine the weights $\bar{w}_k$ of the control points $(X_k, Y_k, 1)$. And then the control points of degree reduced rational Bézier curve $\mathbf{r}^{n-1}$ are given as follows:

$$(\bar{x}_k, \bar{y}_k) = (\frac{X_k}{\bar{w}_k}, \frac{Y_k}{\bar{w}_k}) \quad \text{for} \quad k = 0, \cdots, n-1.$$

• The degree elevation of a polynomial Bézier curve from $n-1$ to n can be written in terms of the control points as

$$\mathbf{c}_k^{(1)} = \frac{k}{n}\mathbf{c}_{k-1} + \frac{n-k}{n}\mathbf{c}_k \quad \text{for} \quad k = 0, \cdots, n.$$

This equation can be used to derive two recursive formulas for the generation of the $\mathbf{c}_k$ from $\mathbf{c}_k^{(1)}$:

$$\mathbf{c}_k^I = \frac{n\mathbf{c}_k^{(1)} - k\mathbf{c}_{k-1}^I}{n-k} \quad \text{for} \quad k = 0, 1, \cdots, n-1.$$

$$\mathbf{c}_{k-1}^{II} = \frac{n\mathbf{c}_k^{(1)} - (n-k)\mathbf{c}_k^{II}}{k} \quad \text{for} \quad k = n, n-1, \cdots, 1.$$

• We observe that the first equation tends to produce approximations near $\mathbf{c}_0^{(1)}$ and that the second equation behave decently near $\mathbf{c}_n^{(1)}$. We may take advantage of this observation by combining both approximations. That is,

$$\bar{\mathbf{c}}_k = (1-\lambda_k)\mathbf{c}_k^I + \lambda_k \mathbf{c}_k^{II} \quad \text{for} \quad k = 0, \cdots, n-1$$

where the unknown factor λ_k were introduced by Eck.

• We may determine the value of λ_k's as follows(Forrest):

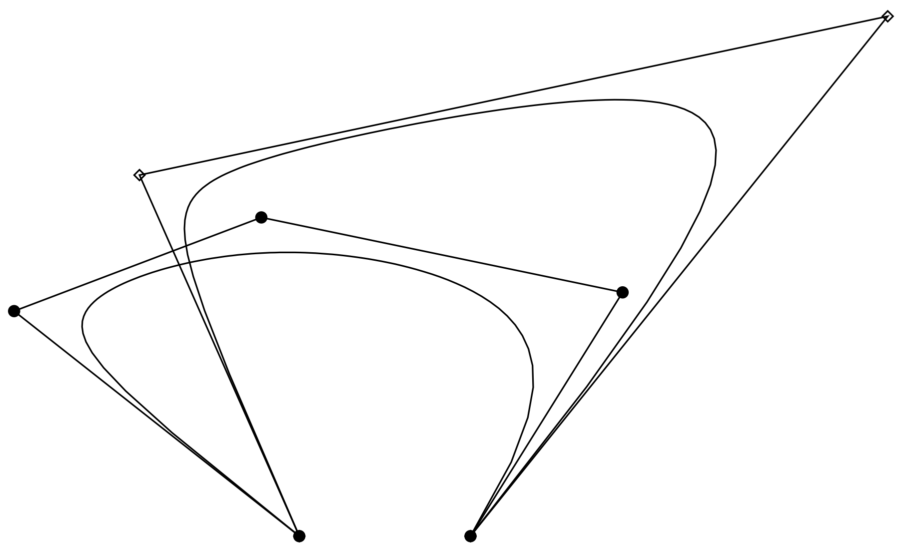
$$\lambda_k = \{ \begin{array}{ll} 1 & \text{for } k < \frac{n}{2}, \\ 0 & \text{for } k > \frac{n}{2}. \end{array}$$

Now, applying this method to determine the weights $\bar{w}_k$,

$$\bar{w}_k = \{ \begin{array}{ll} w_k^I & \text{for } k = 0, 1, \cdots , \\ w_k^{II} & \text{for } k = n-1, n-2, \cdots . \end{array}$$

If n is odd, the weight $\bar{w}_k = \frac{1}{2}(w_k^I + w_k^{II})$ with $k = (n-1)/2$.

- An example of this method

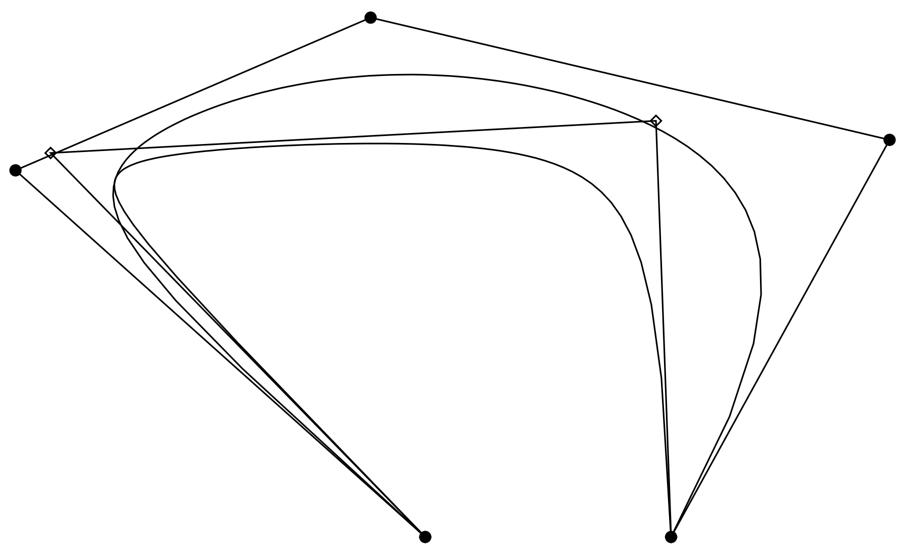


Rational degree reduction from degree 4 to degree 3

- We may set $\lambda_k = k/n$(Farin).

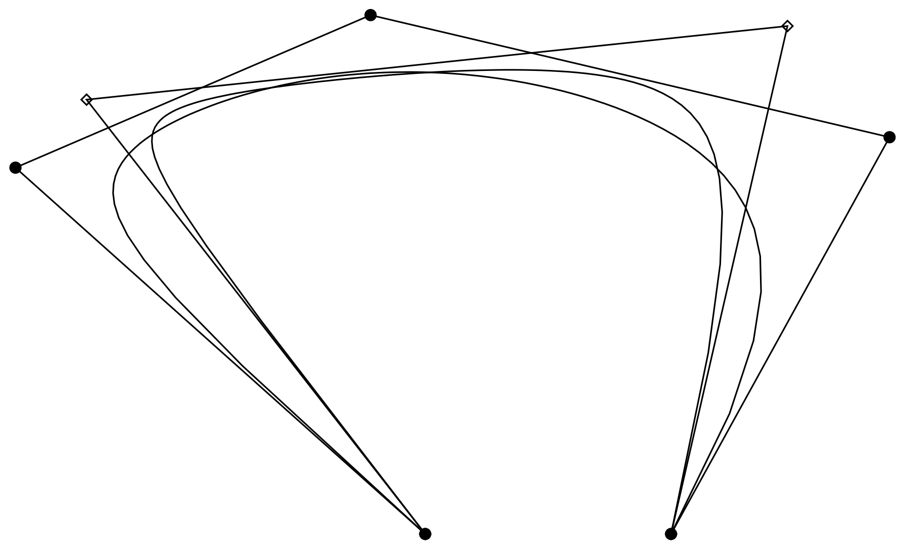
This setting for the λ_k gives us the form of the weights as follows:

$$\bar{w}_k = (1 - \frac{k}{n-1})w_k^I + \frac{k}{n-1}w_k^{II} \quad \text{for} \quad k = 0, \cdots, n-1.$$



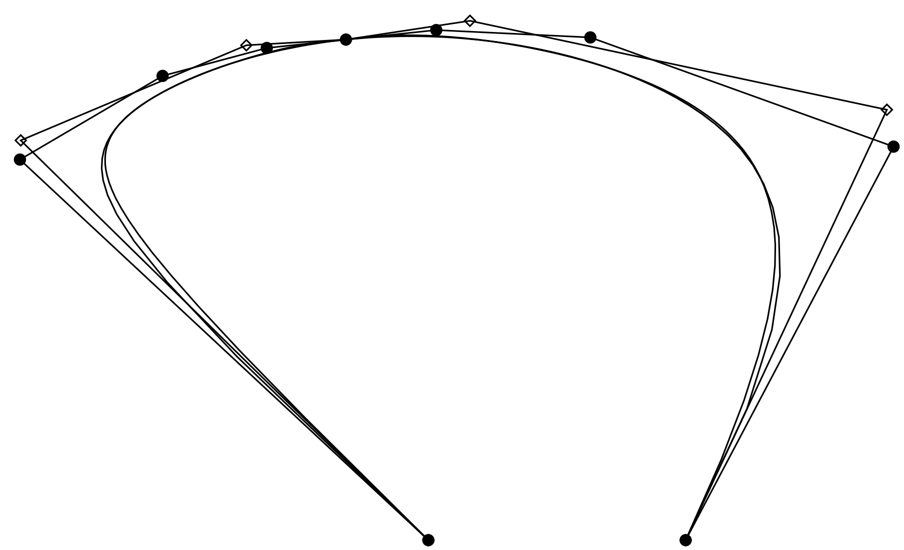
Rational degree reduction from degree 4 to degree 3

• In polynomial degree reduction, the constrained least square method gives us better results(Ahn *et al*).



Rational degree reduction from degree 4 to degree 3 using the least square method.

• For better approximation, we may subdivide the original rational Bézier curve.



Rational degree reduction with subdivision.

References

Y. J. Ahn, Degree reduction of Bézier curves with C^k–continuity using Jacobi polynomials, *Comput. Aided Geometric Des.* 20:423–434 (2003).

Y. J. Ahn, B. G. Lee, Y. Park and J. Yoo, Constrained polynomial degree reduction in the L_2–norm equals best weighted Euclidean approximation of Bézier coefficients, To appeared in *Comput. Aided Geometric Des.* (2003).

P. Bogaki, S. E. Weinstein and Y. Xu, Degree reduction of Bézier curves by uniform approximation with endpoint interpolation, *Comput. Aided Geometric Des.* 27:651–662 (1995).

G. Brunnett, T. Schreiber and J. Braun, The geometry of optimal degree reduction of Bézier curves, *Comput. Aided Geometric Des.* 13:773–7882 (1996).

G. Chen and G. Wang, Optimal multi–degree reduction of Bézier curves with constraints of endpoints continuity, *Comput. Aided Geometric Des.* 19:365–377 (2002).

M. Eck, Degree reduction of Bézier curves, *Comput. Aided Geometric Des.* 10:237–251 (1993).

M. Eck, Least square degree reduction of Bézier curves, *Comput. Aided Des.* 27:845–851 (1995).

G. Farin, Algorithms for rational Bézier curves, *Comput. Aided Des.* 15:73–77 (1983).

G. Farin, *NURBS : from projective geometry to practical use, 2nd Ed.*, A K Peters, Natick, 1999.

G. Farin, *Curves and surfaces for computer aided geometric design : A practical guide, 5th Ed.*, Morgan Kaufmann publishers, San Francisco, 2002.

A. R. Forrest, Interactive interpolation and approximation by Bézier polynomials, *Computer J.* 15:71–79 (1972).

H. O. Kim, J. H. Kim and S. Y. Moon, Degree reduction of Bézier curves and filter banks, *Comput. Math. Appl.* 31:23–30 (1996).

H. O. Kim and S. Y. Moon, Degree reduction of Bézier curves by L^1–approximation with endpoints interpolation, *Comput. Math. Appl.* 33:67–77 (1997).

H. J. Kim and Y. J. Ahn, Good degree reduction of Bézier curves using Jacobi polynomials, *Comput. Math. Appl.* 40:1205–1215 (2000).

M. A. Lachance, Chebyshev economization for parametric surface, *Comput. Aided Geometric Des.* 5:195–208 (1988).

M. A. Lachance, Approximation by constrained parametric polynomials, *Rocky Mountain J. Math.* 21:473–488 (1991).

B. G. Lee and Y. Park, Distance for Bézier curves and degree reduction, *Bull. Australian Math. Soc.* 56:507–515 (1997).

B. G. Lee and Y. Park, Approximate conversion of rational Bézier curves, *J. KSIAM* 2:88–93 (1998).

B. G. Lee, Y. Park and J. Yoo, Application of Legendre–Bernstein basis transformations to degree elevation and reduction, *Comput. Aided Geometric Des.* 19:709–718 (2002).

D. Lutterkort, J. Peters and U. Reif, Polynomial degree reduction in the L_2–norm equals best Euclidean approximation of Bézier coefficients, *Comput. Aided Geometric Des.* 16:607–612 (1999).

N. M. Patrikalakis, Approximate conversion of rational splines, *Comput. Aided Geometric Des.* 6:155–165 (1989).

J. Peters and U. Reif, Least square approximation of Bézier coefficients provides best degree reduction in the L_2–norm, *J. Approx. Theory* 104:90–97 (2000).

L. Piegl and W. Tiller, *The NURBS Book, 2nd Ed.*, Springer–Verlag, Berlin, 1997.

M. Watkins and A. Worsey, Degree reduction of curves, *Comput. Aided Des.* 20:398–405 (1988).

J. Yong, S. Hu, J. Sun and X. Tan, Degree reduction of B–spline curves, *Comput. Aided Geometric Des.* 18:117–127 (2001).