Non-uniform Variational Subdivision

Outline:

I. Subdivision Literature and Preliminaries.
II. Trisection and Corner-Cutting Schemes.
III. Four-point interpolatory Schemes.
IV. Global Variational Subdivision.
V. Local Variational Subdivision.
VI. Smoothing Variational Subdivision Surfaces.

Non-uniform Variational Subdivision

Emphasis:

- Uniform and Non-uniform Schemes.
- Effect of Parametrization on Uniformity and Smoothness.
- Smoothness (Discrete Hölder Regularity).
- Approximation Order.
- Localizing Global Variational Schemes.

Part I

Literature and Preliminaries

Subdivision Literature

Corner-Cutting

de Rham, G.	Trisection Algorithms of 1947,’53,’56,’57,’58	1947–
Chaikin, G. M.	An algorithm for high speed curve generation	1974
Riesenfeld, R.	On Chaikin’s Algorithm	1975
Catmull & Clark	Recursively generated B-Spline Surfaces on arbitrary topological meshes	1978
Doo & Sabin	Behavior of recursive division surfaces near extraordinary points	1978
Lane & Riesenfeld	Uniform B-splines of arbitrary order	1980
de Boor	Corner-Cutting always works	1987
Gregory & Qu	Non-uniform corner-cutting	1988
de Boor	Local corner cutting and smoothness of limit curve	1990
Dyn, Gregory & Levin	Analysis of uniform binary subdivision	1991

Subdivision Literature

Interpolatory Subdivision

Dubuc	Interpolation through an iterative scheme.	1986
Dyn, Levin & Gregory	A 4-point interpolatory subdivision scheme	1987
Gregory & Qu	Non-uniform corner-cutting	1988
Deslauries & Dubuc	Symmetric Iterative Refinement	1989
Dyn, Gregory & Levin	Butterfly subdivision for surfaces	1990
Kobbelt	A variational approach to subdivision	1996
Kobbelt	Discrete Fairing (for surfaces)	1997
Kuijt & van Damme	Convexity preserving interpolatory subdivision	1998
Levin	Analysis of non-uniform binary subdivision	1999
Marinov, Dyn & Levin	Geometrically controlled 4-point schemes	2004

Subdivision Literature

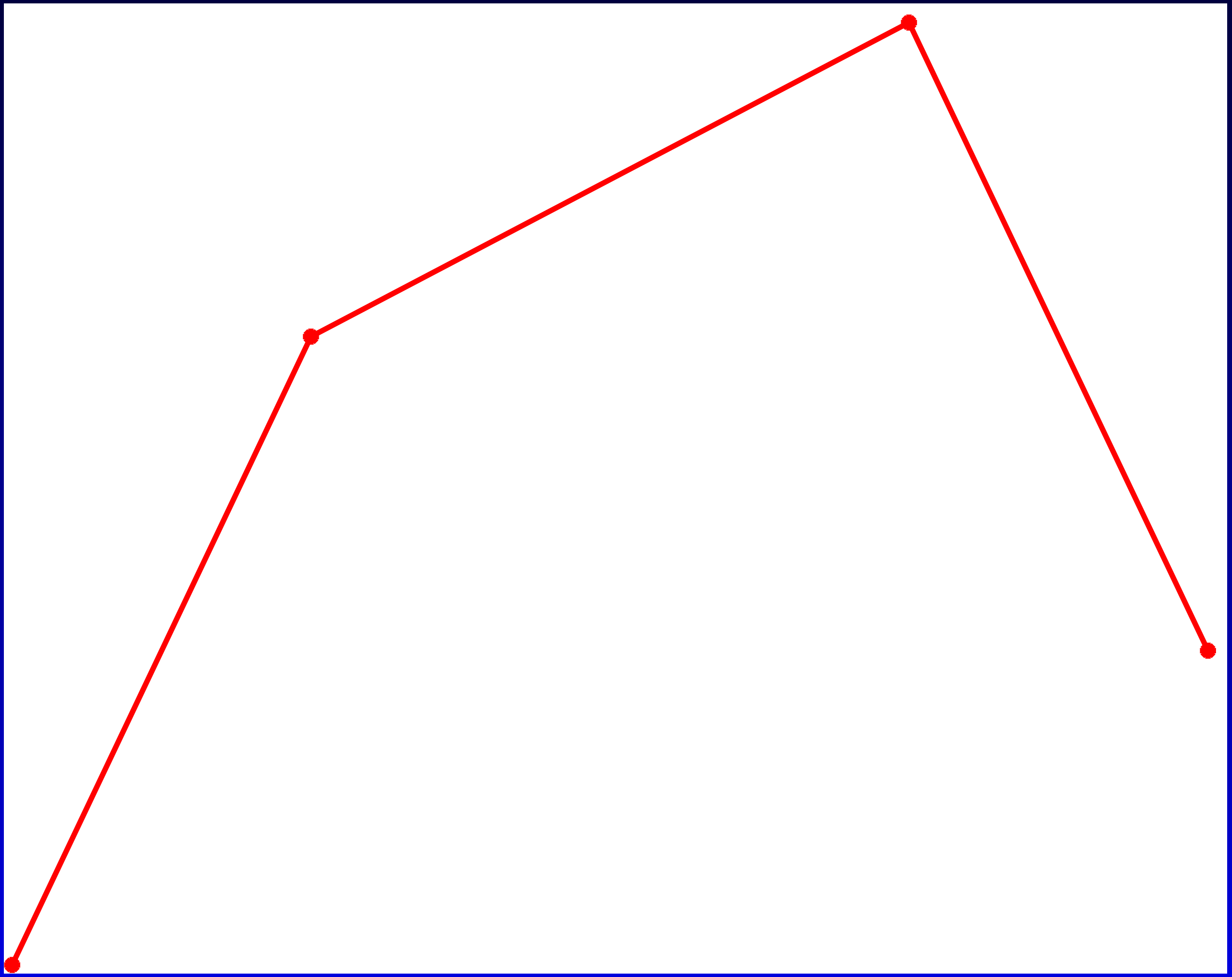
My work

Kersey	Constrained subdivision curves	’03
Kersey	Near-interpolation subdivision surfaces	’04
Kersey	Smoothness of Non-uniform variational subdivision	’04
Kersey	An abstract approach to variational refinement	’04
Kersey	Local Variational subdivision	’05

Current Work:

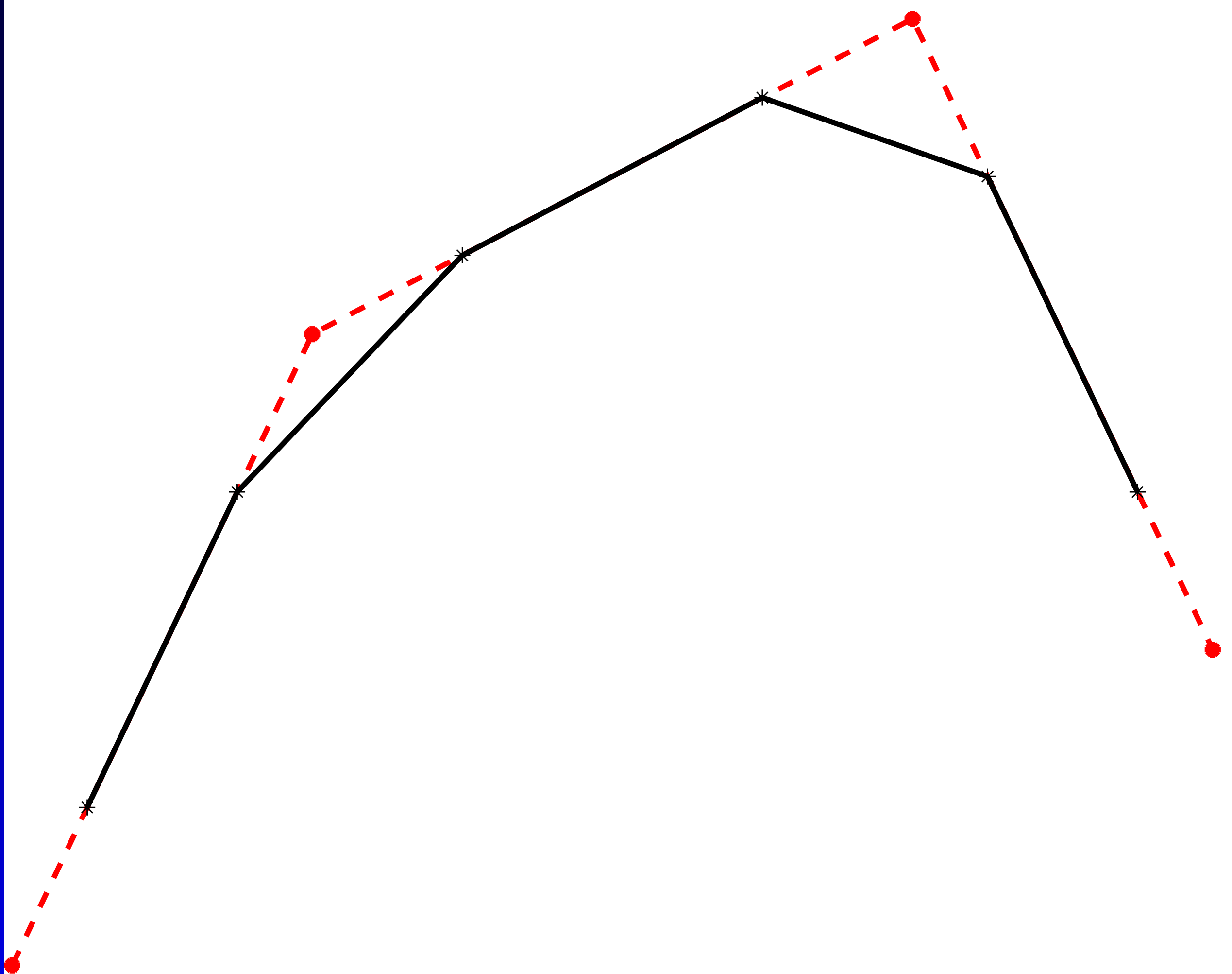
- Approximation order in non-uniform variational Refinement.
- Convergence of local variational subdivision.

Non-Interpolatory



Example: Corner-Cutting (Convex Scheme)

Non-Interpolatory



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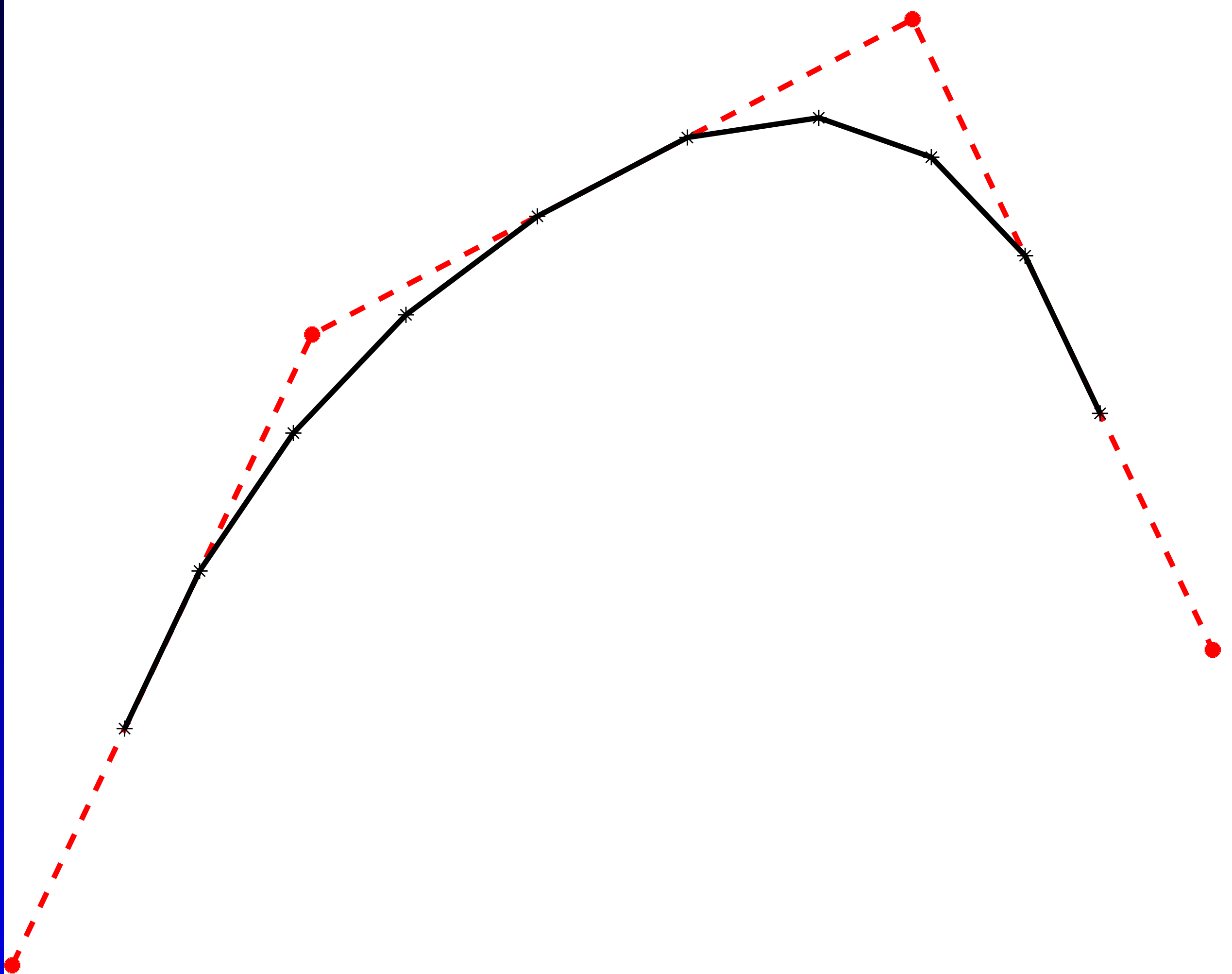

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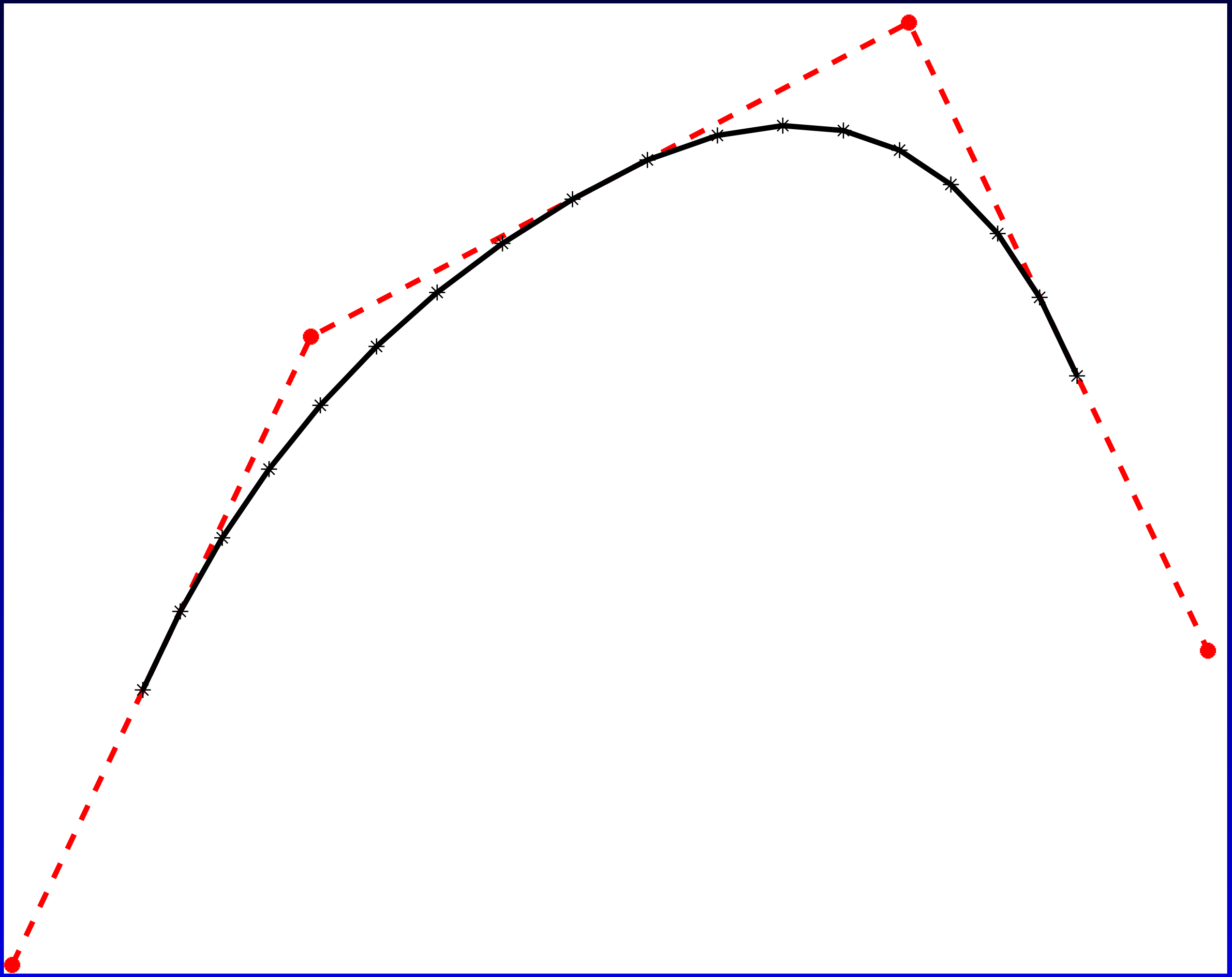


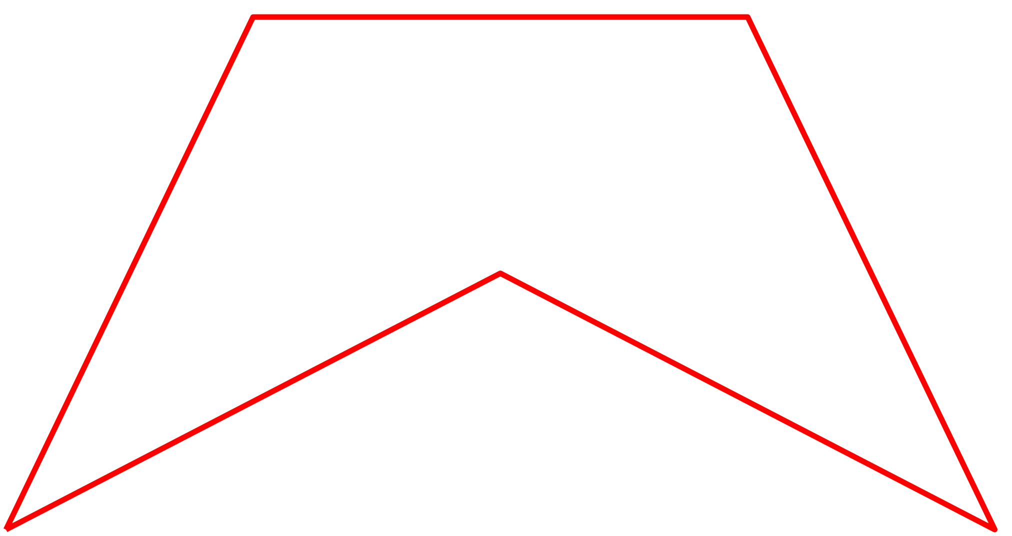
Example: Corner-Cutting (Convex Scheme)

Non-Interpolatory

Non-interpolatory (corner-cutting) schemes

- Trisection: de Rham
- Corner-Cutting: Chaikin
- B-Spline Subdivision: Lane and Riesenfeld
- Corner-Cutting Always works: de Boor
- Non-uniform Corner-cutting: Gregory and Qu

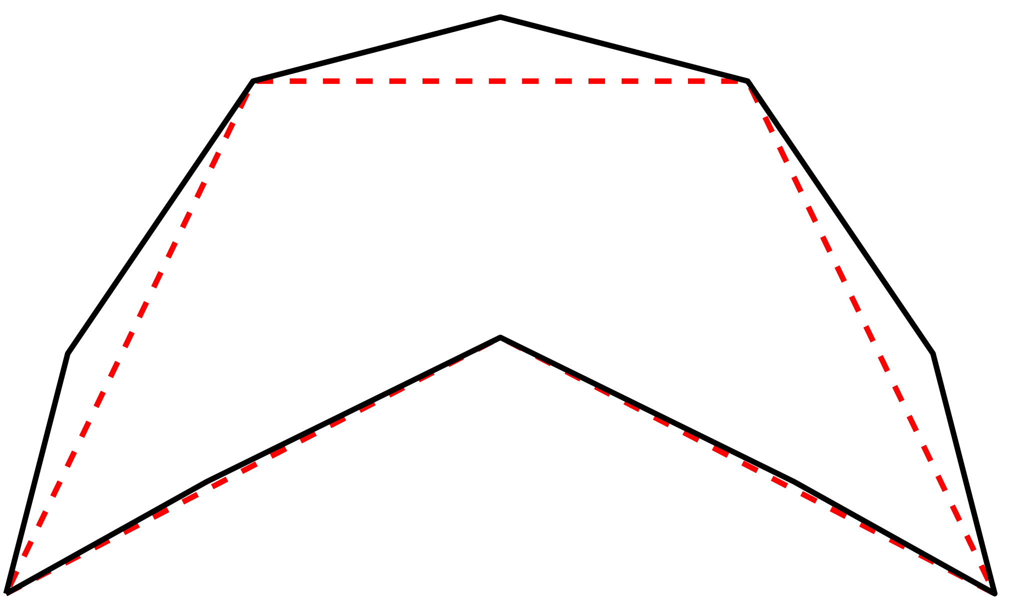
Interpolatory



Example: Four-Point and variational subdivision

- Non-convex

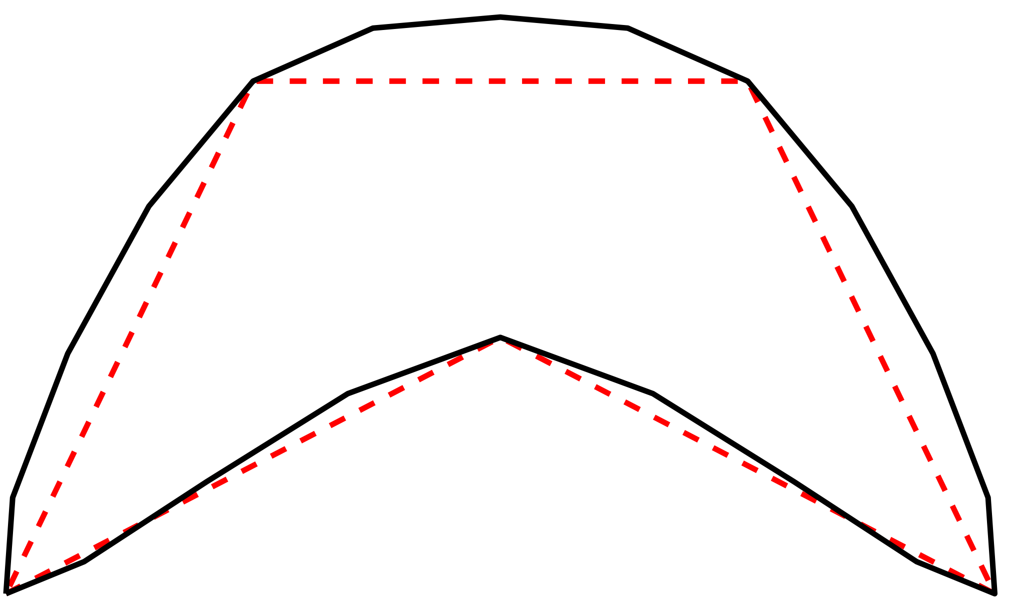
Interpolatory



Example: Four-Point and variational subdivision

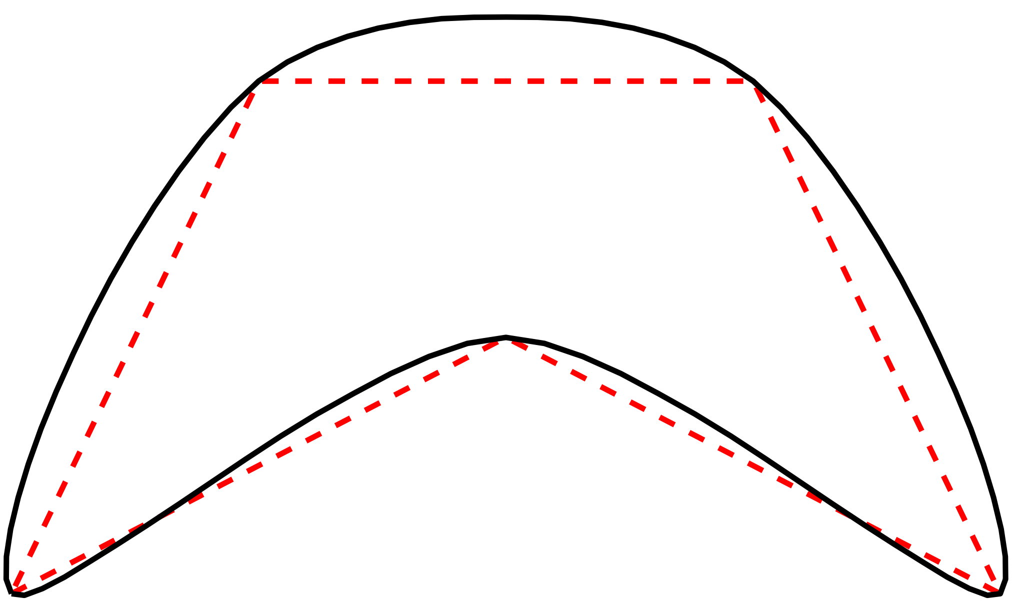
- Non-convex

Interpolatory



Examples: Four-Point and variational subdivision
Non-convex

Interpolatory



Non-convex

Interpolatory

Interpolatory Subdivision

- Local Four-point Subdivision: Dubuc
- Local Four-point Subdivision with Tension: Gregory, Dyn, Levin
- Local n-point Subdivision: Dubuc and Deslauries
- Local Non-uniform Four-point Subdivision: Levin
- Global Uniform Variational Subdivision: Kobbelt
- Global Non-uniform Variational Subdivision: Kersey
- Local Non-uniform Variational Subdivision: Kersey

Representation of PWL Curves

At each stage in the subdivision process the curves are parametric piecewise linear curves.

- Coefficients $f_i \in \mathbb{R}^2$.
- Parameter values (knots) $t_i \in [a, b]$.
- Linear B-splines $N_i(t)$ with $N_i(t_j) = \delta_{ij}$.
- Piecewise linear spline curve: $f(t) = \sum_i f_i N_i(t)$.
- Interpolation: $f(t_i) = f_i$.

General Linear Subdivision

Given:

- Coefficients at “level 0”: $f_i = f_i^0 \in \mathbb{R}^2$.
- Parameters at level 0: $t_i = t_i^0 \in [a, b]$.

Find:

- Coefficients at level k: $f_i^k := \sum_j \alpha_{ij}^k f_i^{k-1}$.
- Parameters at level k: t_i^k.

Note that the coefficients are determined by a linear process: $f^k := A_k\, f_{k-1}$. (Very general nonlinear schemes have been studied by Kuijt, et al.)

Uniform Schemes

With Uniform schemes the knots are equally spaced. That is, $h := h_i$ is constant where $h_i := t_{i+1} - t_i$. In dyadic subdivision $h^k = h^{k-1}/2$.

Examples of uniform schemes:

- de Rham's trisection
- Chaikin's algorithm
- Dubuc's Four-point subdivision
- Gregory's Four-point subdivision
- Kobbelt's variational subdivision

Note that any of these schemes can be made nonuniform under different parametrizations.

Limit Curves

Questions:

- Do these schemes converge, i.e., $f^k(t) \rightarrow f(t)$?
- Under what norms?
- What is the smoothness of $f(t)$?
- Does the scheme approximate given functions?

Corner-cutting schemes do not approximate;
Interpolatory schemes do.

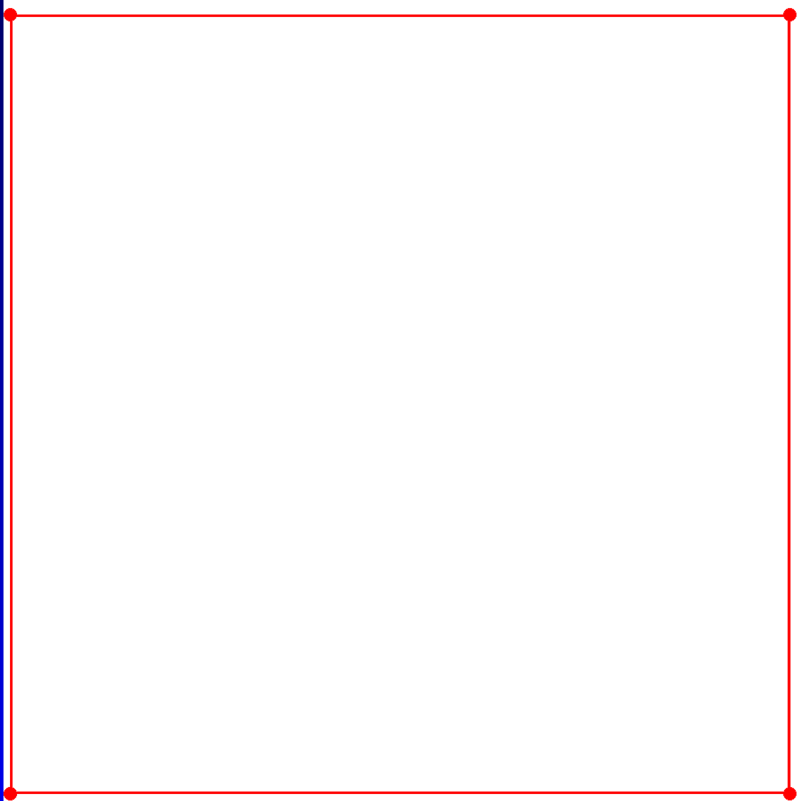
Smoothness and approximation order depend on parametrization (uniformity).

Part II

Trisection and Corner-Cutting

Trisection – de Rham (1947)

The first publication on ‘corner-cutting’ was de Rham’s 1:1:1 trisection algorithm of 1947. Under a uniform parametrization the scheme is C^0, not C^1.



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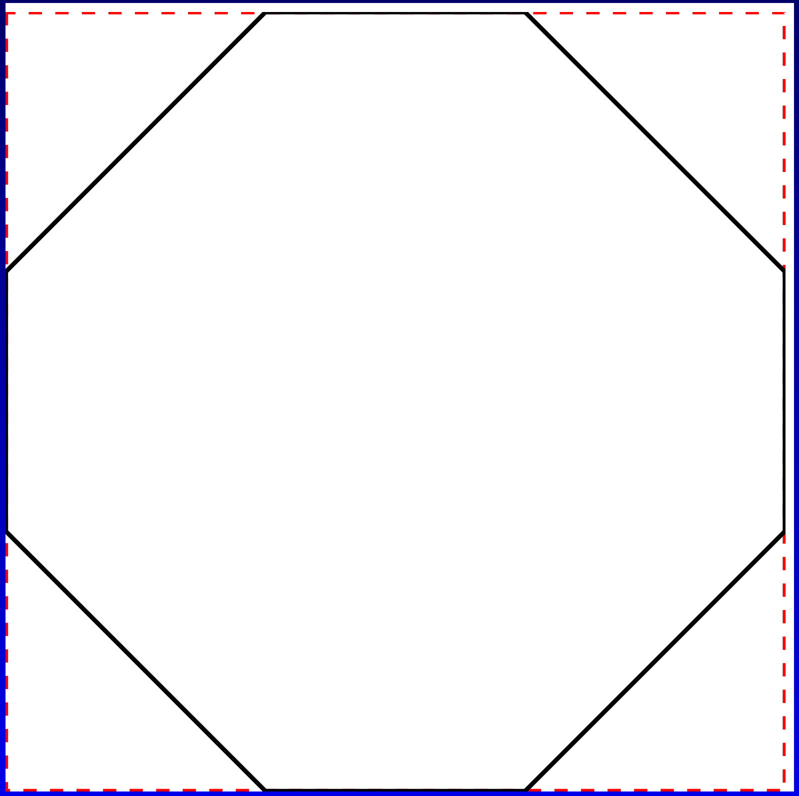
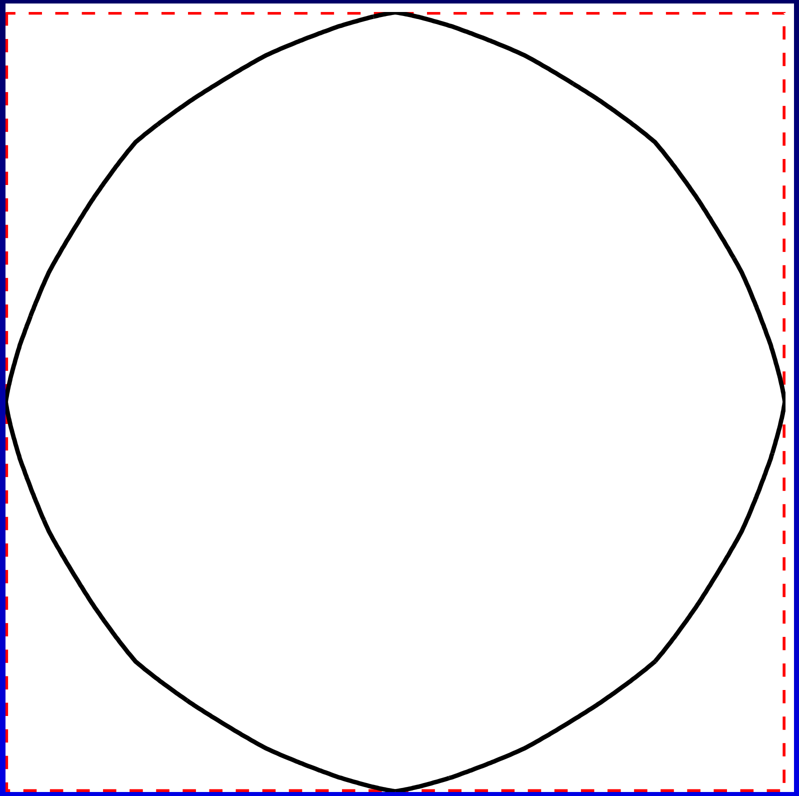


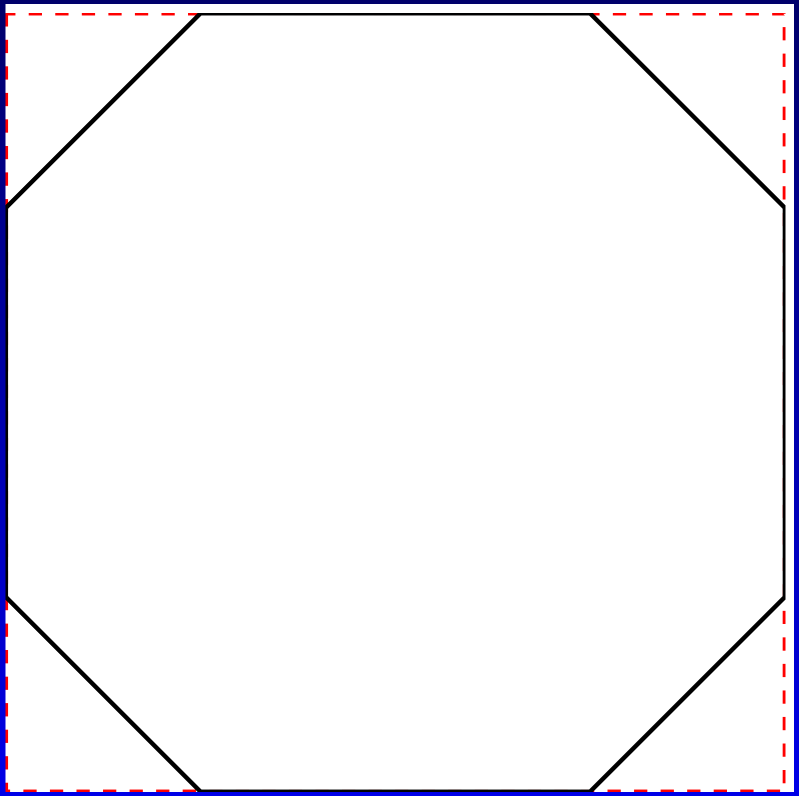
Trisection – de Rham (1947)

But de Rham’s curve (limit of trisection of the square) is geometrically smooth, even though not C^1!!



Trisection – de Rham (1956)

Trisection $w : 1 - 2w : w$ is C^1-scheme under a uniform parametrization iff $w = 1/4$ (i.e., 1:2:1). In modern-day the 1:2:1 trisection is known as Chaikins algorithm.



Trisection – de Rham (1956)

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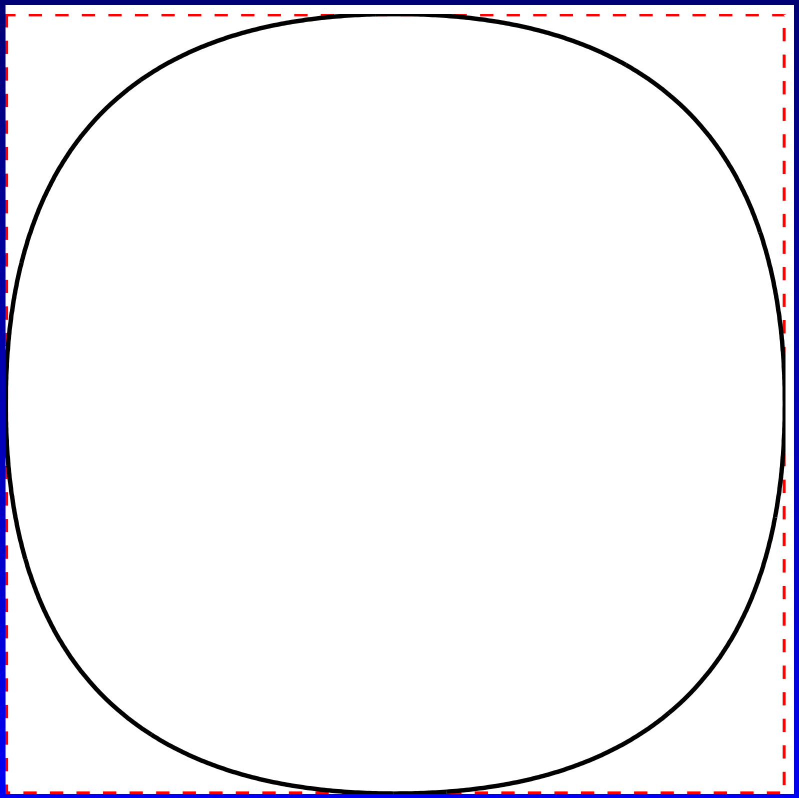
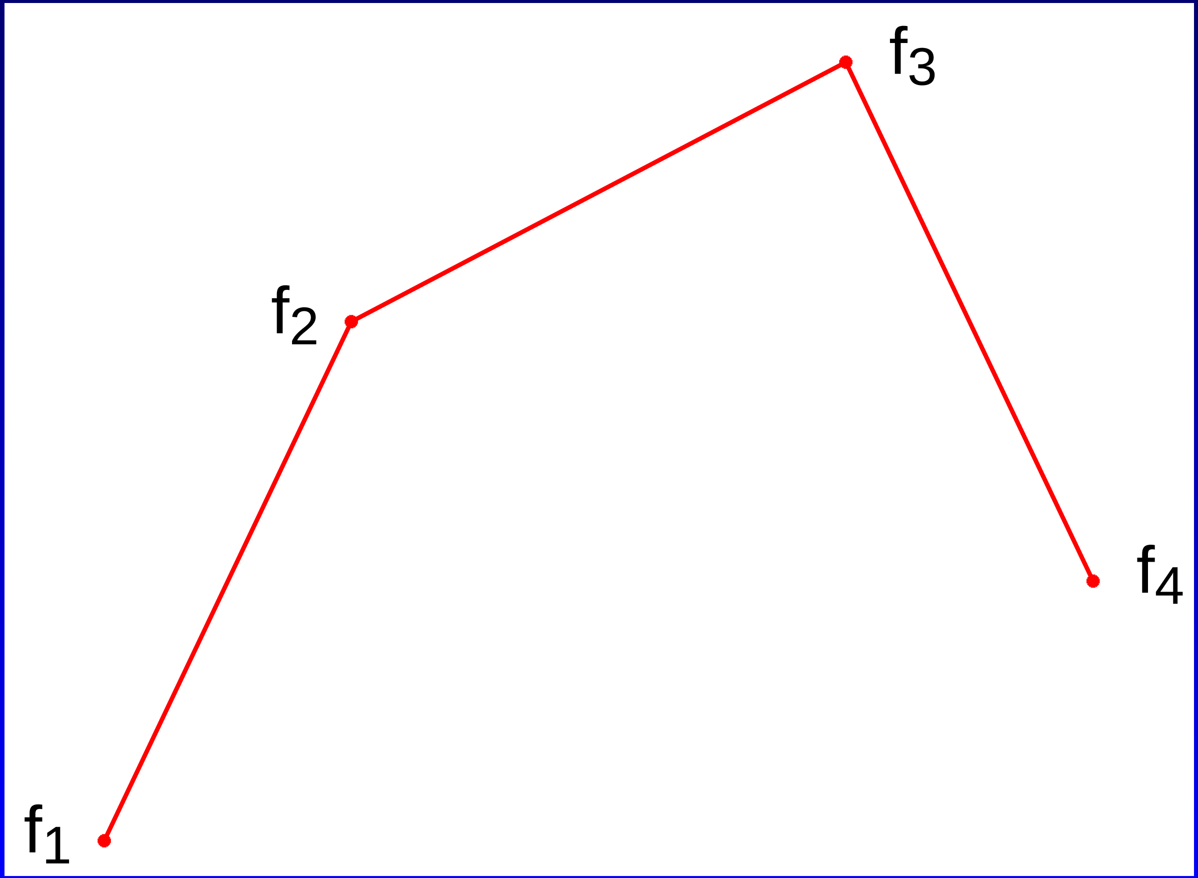


Chaikin’s Algorithm (1974)

$$f_{2i}^{k+1} = \frac{3}{4} f_i^k + \frac{1}{4} f_{i+1}^k, \qquad f_{2i+1}^{k+1} = \frac{1}{4} f_i^k + \frac{3}{4} f_{i+1}^k$$



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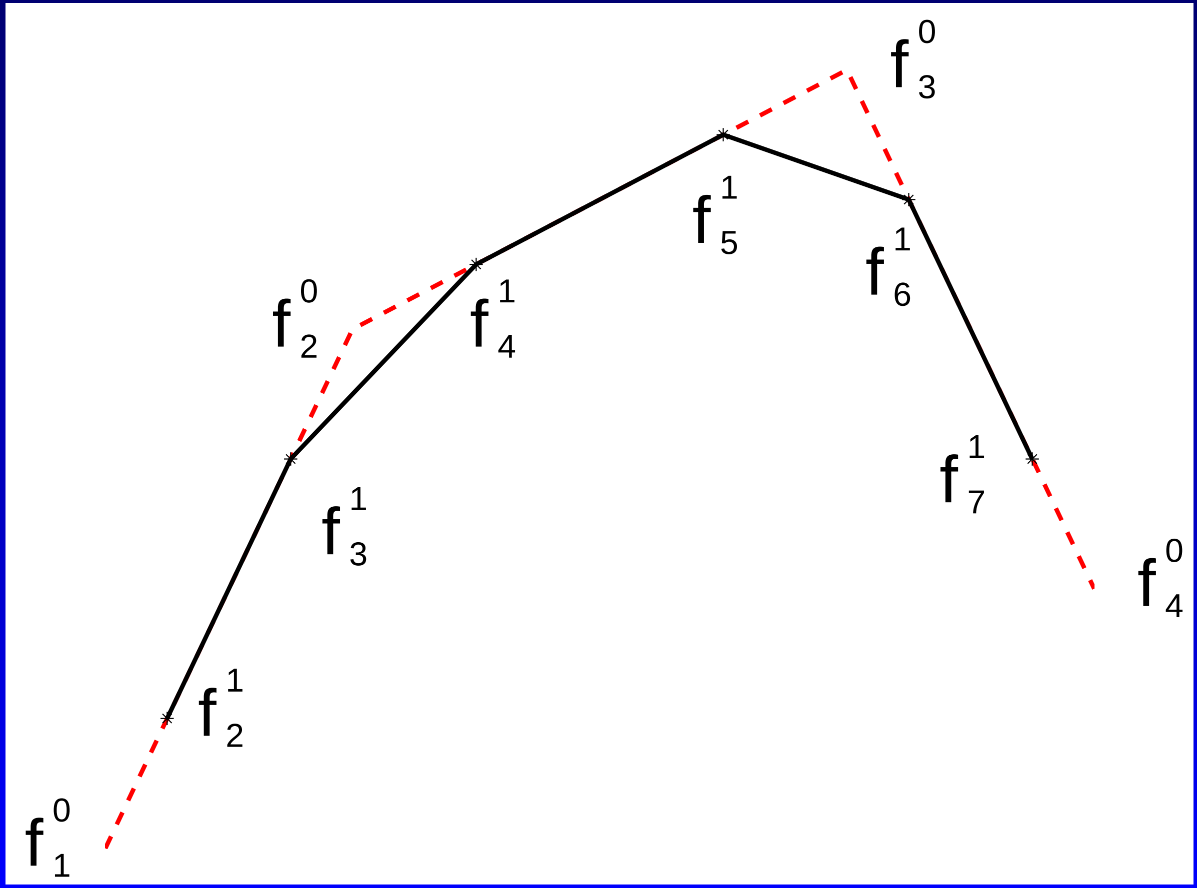

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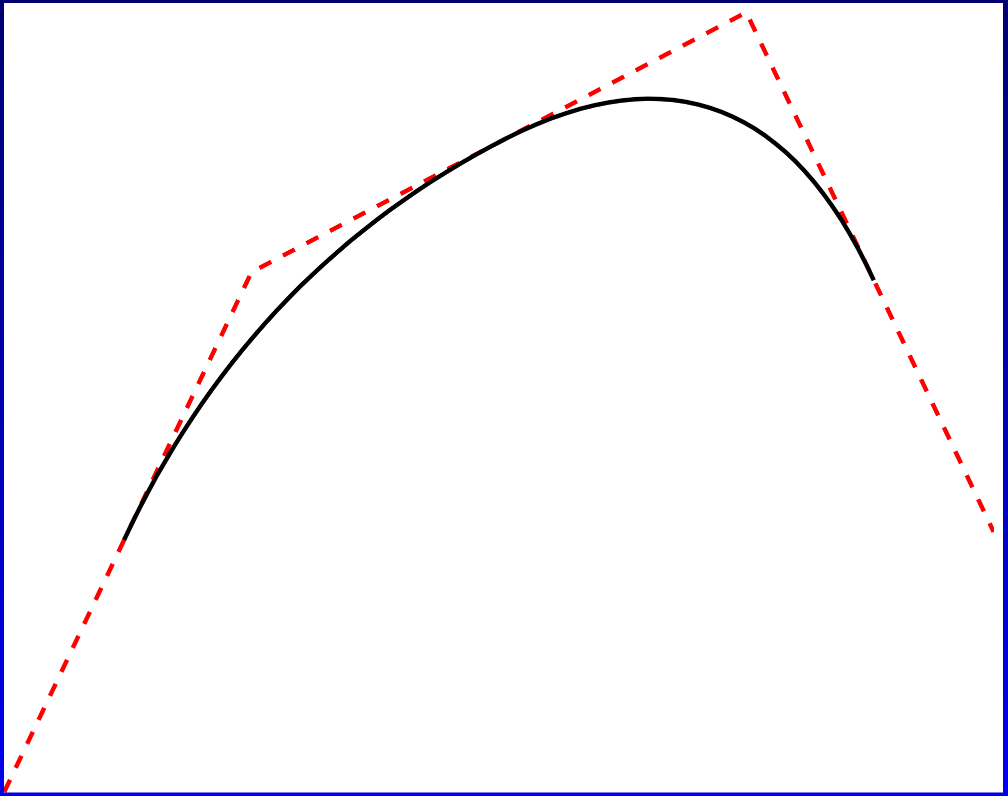


Corner-Cutting Always Works – de Boor (1987)

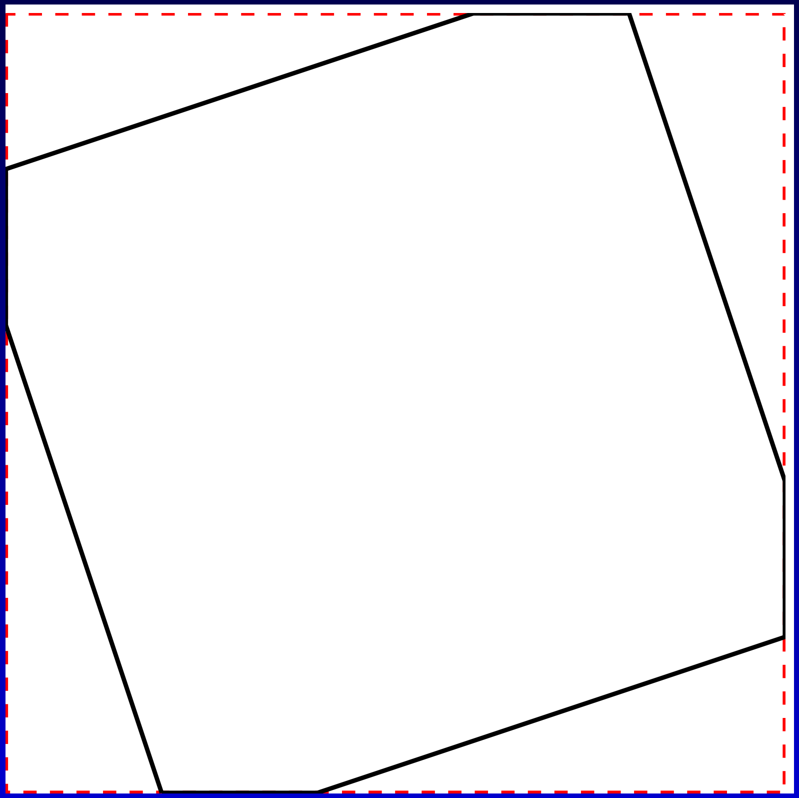
For any corner-cutting scheme (even non-stationary) the limit is always Lipschitz continuous. That is,

$$|f(x) - f(y)| \leq \alpha |x - y|$$

for some α. Such curves are C^1 almost everywhere.

Corner-Cutting Always Works – de Boor (1987)

3:1:1 trisection:



Corner-Cutting Always Works – de Boor (1987)

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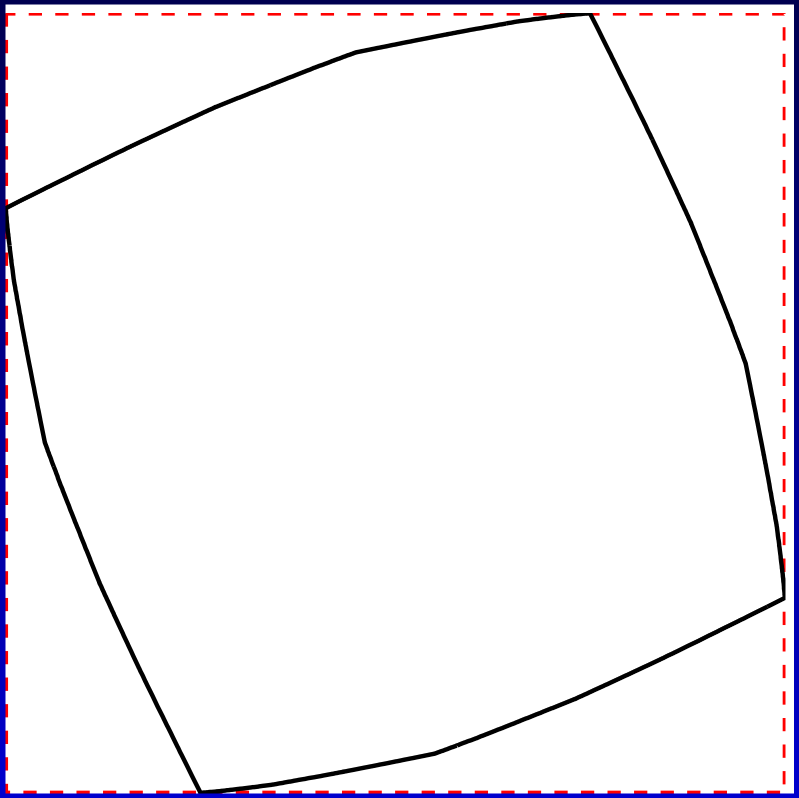


B-spline subdivision – Riesenfeld and Lane

- Showed that Chaikin’s algorithm produces C^1 quadratic uniform B-spline curves.
- Derived subdivision algorithms for uniform B-splines of arbitrary order.

Based on the observation that:

$$f(t) := \sum_i f_i^0 B_m(t-i) = \cdots = \sum_i f_i^k B_m(2^k t - i)$$

For some refined coefficients f_i^k. (B-spline control polygon approaches limit curve.)

Towards a non-uniform scheme – Gregory and Qu (1998)

Subdivision:

$$f_{2i}^{k+1} = (1-\alpha_i^k) f_i^k + \alpha_i^k f_{i+1}^k, \quad t_{2i}^{k+1} = (1-\alpha_i^k) t_i^k + \alpha_i^k t_{i+1}^k$$

$$f_{2i+1}^{k+1} = \beta_i^k f_i^k + (1-\beta_i^k) f_{i+1}^k, \quad t_{2i+1}^{k+1} = \beta_i^k t_i^k + (1-\beta_i^k) t_{i+1}^k$$

- Linear (non-stationary) corner cutting scheme.
- Non-uniform parametrization.
- Subdivision scheme applied to coefficients AND knot sequence.
- Preserves arc length parametrizations.
- Partly non-uniform: parametrization does not affect trace of curve.

Smoothness – GQ98

$$d^k(t) := \sum_i d_i^k \, N_i(t) \quad \text{with} \quad d_i := \frac{f_{i+1} - f_i}{h_i}.$$

Lemma 0.0 *If $d^k \xrightarrow{u} d$ for some $d \in C([a,b])$, then $f^k \xrightarrow{u} f \in C^1([a,b])$ where $f' = d$.*

Theorem 0.0 *The corner-cutting scheme converges to a C^1 curve provided*

$$\underline{\alpha} > 0, \ \underline{\beta} > 0, \ 2\,\overline{\alpha} + \overline{\beta} < 1, \ \overline{\alpha} + 2\,\overline{\beta} < 1.$$

Consequence: For $0 < w < 1/3$, de Rham's w:1-2w:w is C^1 under the above parametrization.

Generalization of GQ

Question: Does their lemma generalize to second (or higher) order derivatives/differences with arbitrary parametrizations and arbitrary subdivision schemes (not necessarily corner-cutting)?

Answer: It does, however not directly because we need ‘locally de£ned’ second divided differences.

I.e.,

$$s_i = \frac{d_{i+1} - d_i}{h_{i,2}} \neq \frac{d_{i+1} - d_i}{h_i},$$

with $h_i := t_{i+1} - t_i$, $h_{i,2} := h_i + h_{i+1} = t_{i+2} - t_i$, and s_i second divided differences.

Generalization of GQ

Symmetric Differences:

$$s^k(t) := \sum_i s_i^k \, N_i(t) \quad \text{with} \quad s_i := \frac{d_{i+1} - d_i}{h_{i,2}}$$

$$\tilde{d}^k(t) := \sum_i \tilde{d}_i^k \, N_i(t) \quad \text{with} \quad \tilde{d}_i := \frac{h_{i-1} d_i + h_i d_{i-1}}{h_{i-1,2}}$$

$$\tilde{s}^k(t) := \sum_i \tilde{s}_i^k \, N_i(t) \;\; \text{with} \;\; \tilde{s}_i := \frac{\tilde{d}_{i+1} - \tilde{d}_i}{h_i} = s_i + s_{i-1}.$$

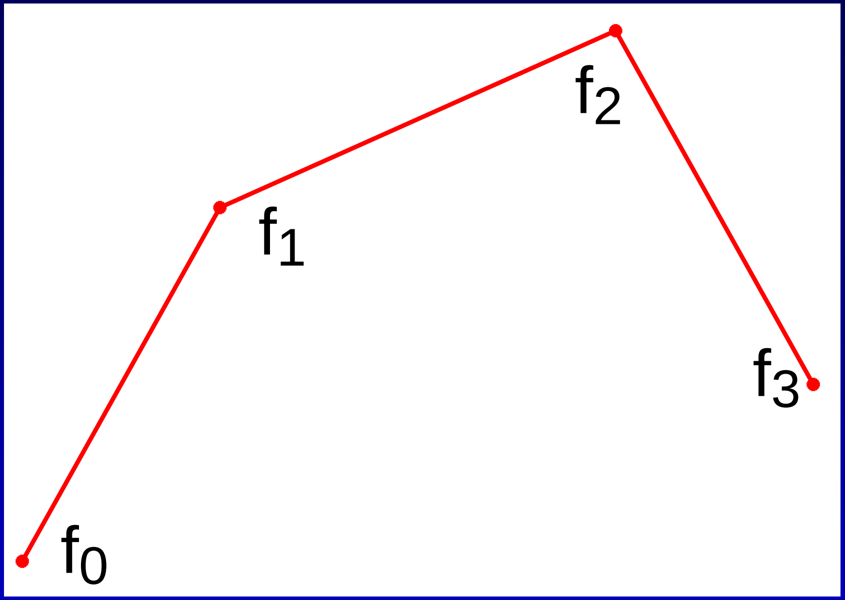
Theorem 0.0 *(Kersey, 2004) Assume that* $\limsup h_i^k \to 0$. *If* $\tilde{s}^k \xrightarrow{u} \tilde{s}$ *for some* $\tilde{s} \in C([a,b])$, *then* $f^k \xrightarrow{u} f \in C^2([a,b])$ *where* $f'' = d' = \tilde{s} = 2\,s$.

Part III

Four-Point Interpolatory Schemes

Uniform Four Point Scheme – Dubuc (1986)

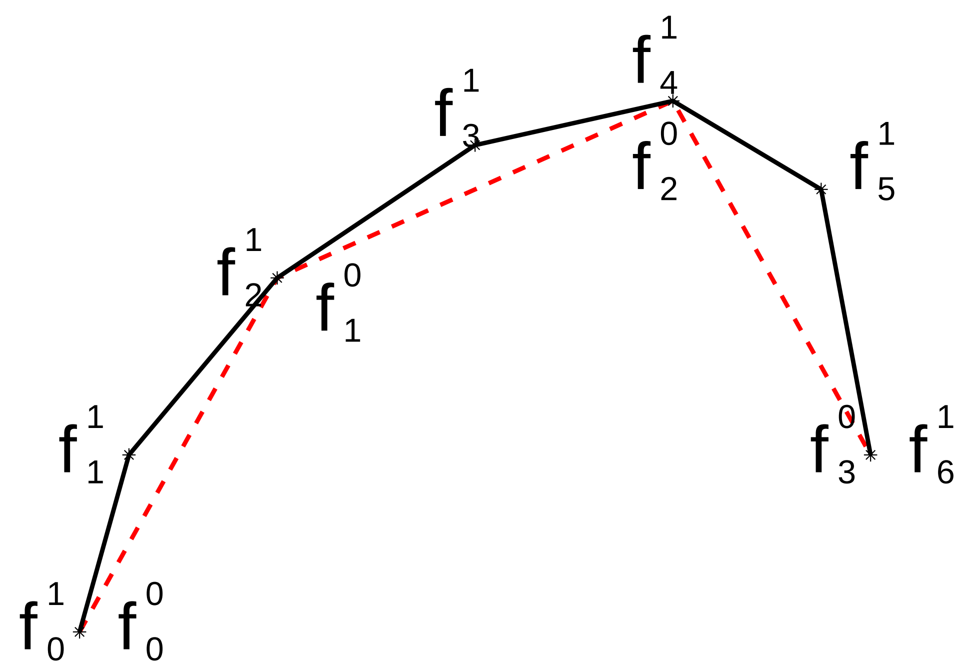
Idea: Add new point between f_1 and f_2 by evaluating on the interpolating cubic polynomial.



Uniform Four Point Scheme – Dubuc (1986)

$$f_{2i}^{k+1} = f_i^k$$

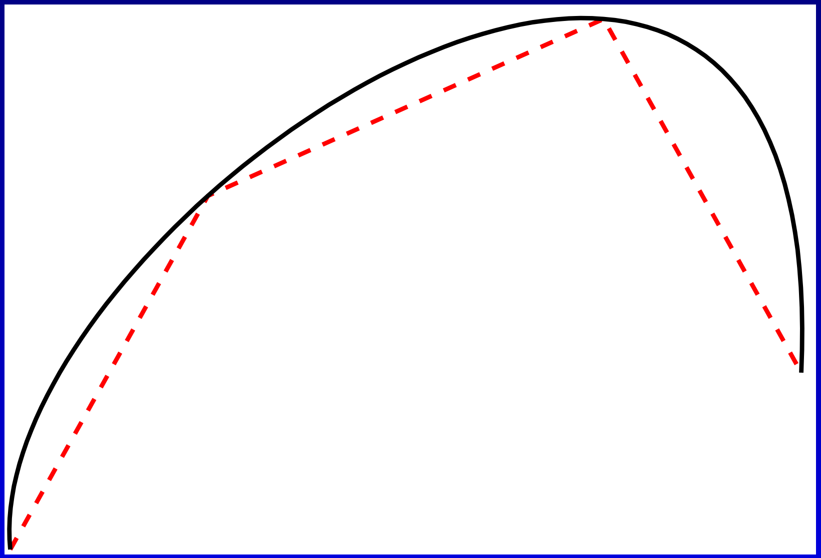
$$f_{2i+1}^{k+1} = -\frac{1}{16}\, f_{i-1}^k + \frac{9}{16}\, f_i^k + \frac{9}{16}\, f_{i+1}^k - \frac{1}{16}\, f_{i+2}^k$$



Uniform Four Point Scheme – Dubuc (1986)

Properties:

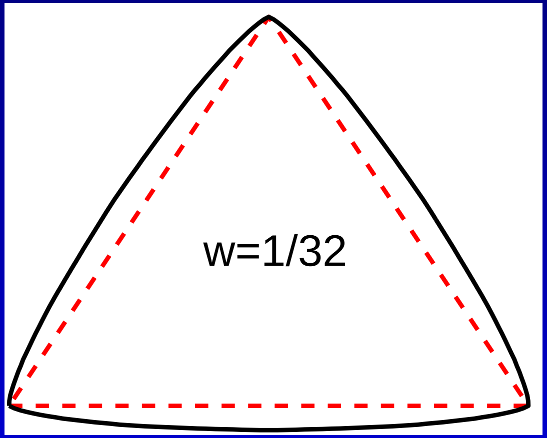
- Cubic polynomial precision.
- Almost C^2 under uniform parametrization – Holder exponent arbitrary close to 2.



Uniform Four Point Scheme with Tension – Dyn, Levin, Gregory (1986)

$$f_{2i} = f_i$$

$$f_{2i+1} = -wf_{i-1} + (\frac{1}{2} + w)f_i + (\frac{1}{2} + w)f_{i+1} - wf_{i+2}$$



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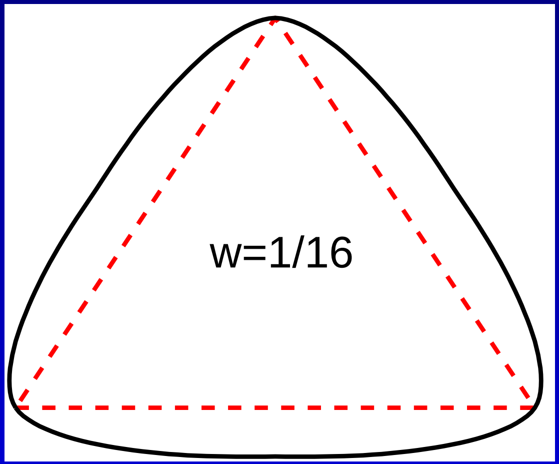

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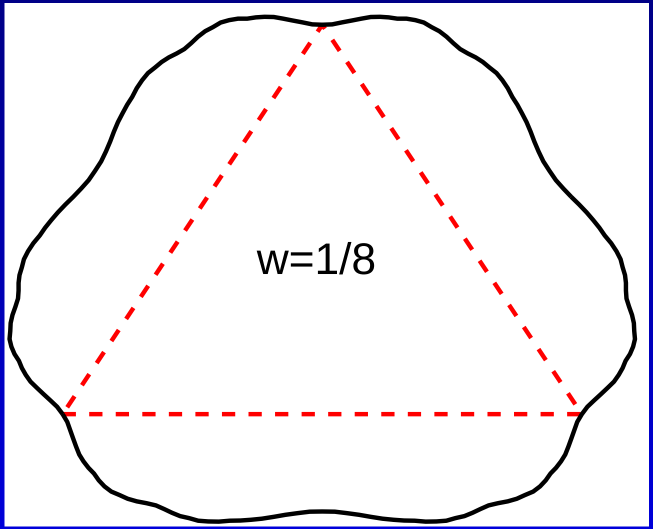


Properties of Weighted Four Point Scheme

- Reduces to Dubuc scheme when $w = \frac{1}{16}$.
- Suf£cient C^0 condition: $0 < w < \frac{1}{4}$.
- Suf£cient C^1 condition: $0 < w < \frac{1}{8}$.
- Deslauries and Dubuc generalizes to $2n$ schemes (by higher degree polynomial interpolation).
- For $|w| < 1/4$ approximation order to $f \in C^2$ is $O(h^2)$, with uniform mesh-spacing h.
- For $w = \frac{1}{16}$ approximation order to $f \in C^4$ is $O(h^4)$.
- Note that there is no approximation order between h^2 and h^4. Reason: Quadratic reproduction requires $w = 1/16$, which reproduces cubic.

Non-Uniform Four Point Scheme

Levin (1999)

$$f_{2i} = f_i$$

$$f_{2i+1} = -w_i^k f_{i-1} + (\frac{1}{2} + w_i^k) f_i + (\frac{1}{2} + w_i^k) f_{i+1} - w_i^k f_{i+2}$$

- Non-uniform scheme over a uniform parametrization.
- Non-stationary.
- Suf£cient C^1 condition: $0 < \epsilon < w_i^k < \frac{1}{8}$.

Non-Uniform Four Point Scheme

Choosing Weights: Marinov, Dyn & Levin (2004) Let $w_i := f(g(i))$ with $e_i := |f_{i+1} - f_i|$,

$$g(i) := \frac{3\,|e_i|}{|e_{i-1}| + |e_i| + |e_{i+1}|} \quad \text{if } |e_{i-1}| + |e_i| + |e_{i+1}| > 0,$$

otherwise g(i)=0, and with

$$f(x) = \left\{ \begin{array}{ll} W\,x, & 0 \le x \le 1, \\ W \text{ or } W\,\frac{3-x}{2}, & 1 < x \le 3, \end{array} \right\}.$$

Non-Uniform Four Point Scheme

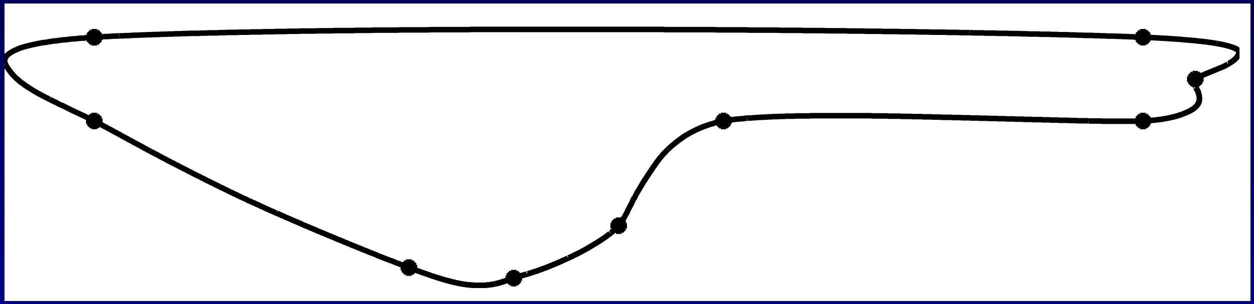
Properties:

- A “Local Chordal Parametrization” for parametric curves.
- Non-Linear.
- “Artifact-Free”
- No “corners” (i.e., no edges of zero length)
- Produces C^0 curves

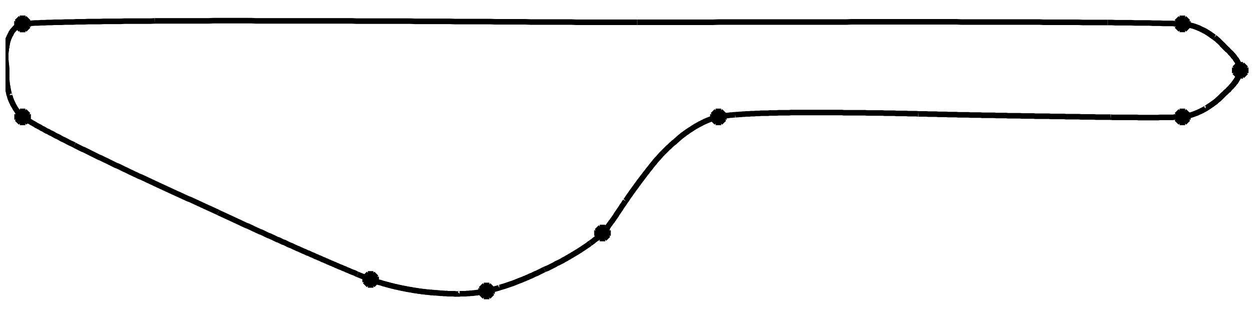
Modifications: A modification is given that will guarantee C^1 limit curves by bounding the weights away from zero, i.e., $w_i \in [\epsilon, \frac{1}{8} - \epsilon]$ for some $\epsilon > 0$. The authors also derive a convexity preserving scheme.

Non-Uniform Four Point Scheme

Uniform [Dubuc 86]:



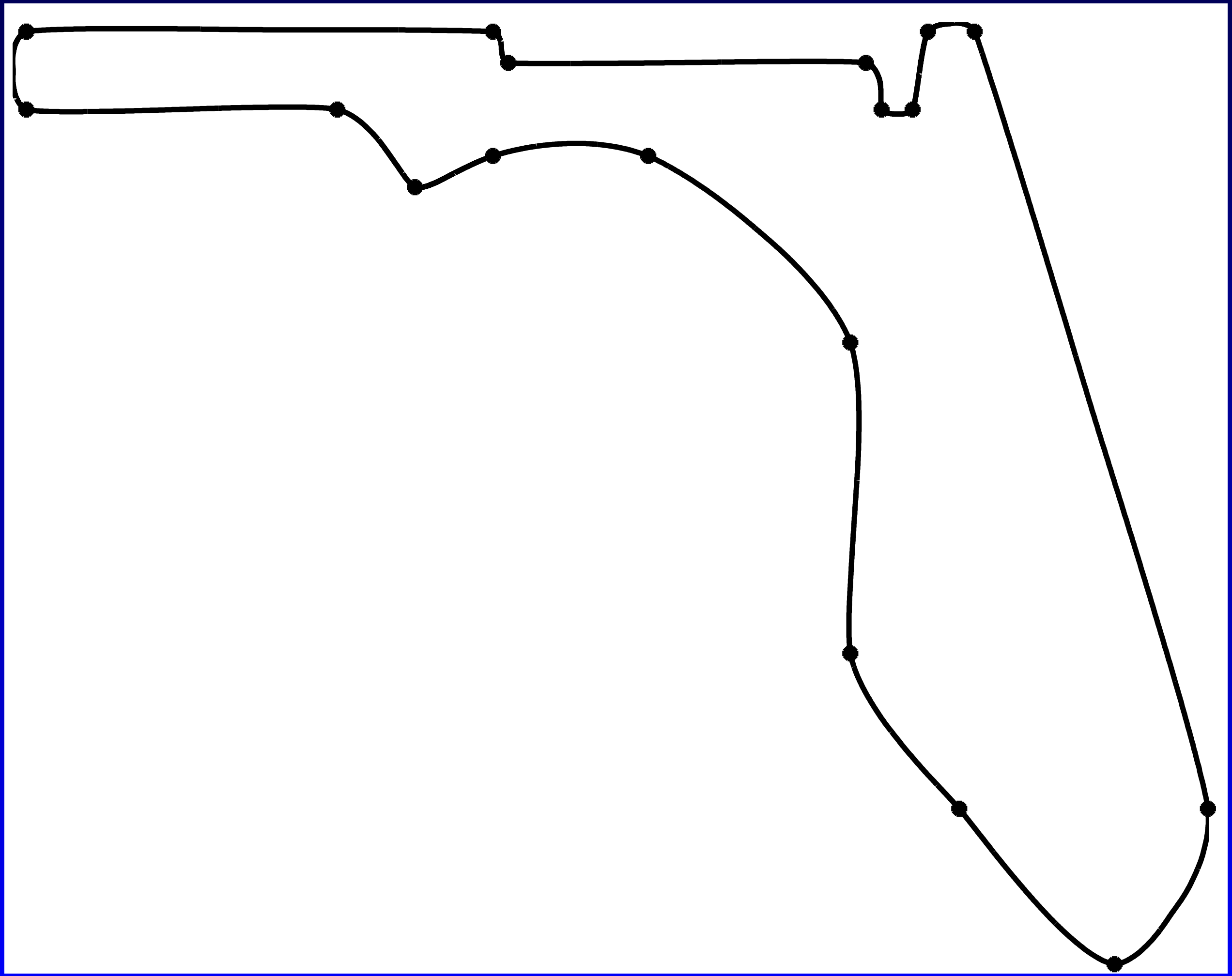
Non-uniform [MDL04]:



(Example from [MDL04]).

Non-Uniform Four Point Scheme

Another example: by [MDL04] method.



Part IV

Global Variational Subdivision

Uniform Variational Re£nement – Kobbelt (1996)

Solve:

$$\min_{f_i} \{E(f) : f_{2i} = f_i\} \quad \text{with } E(f) := \sum_i \Big| \sum_{j=0}^{k} \alpha_i f_{i+j} \Big|^2 .$$

General C^k suf£cient condition.

Theorem 0.1 *If* $\sum_{k=0}^{\infty} |2^{mk} \Delta^{k+l} f^k(t)|_\infty < \infty$ *for some* $l \in \mathbb{N}$ *then the* $f^k \xrightarrow{u} f$ *for some* $f \in C^m$.

Uniform Variational Refinement – Kobbelt (1996)

Forward-difference scheme (Second difference below):

$$E(f) := \sum_i |\Delta_i^2 f|^2.$$

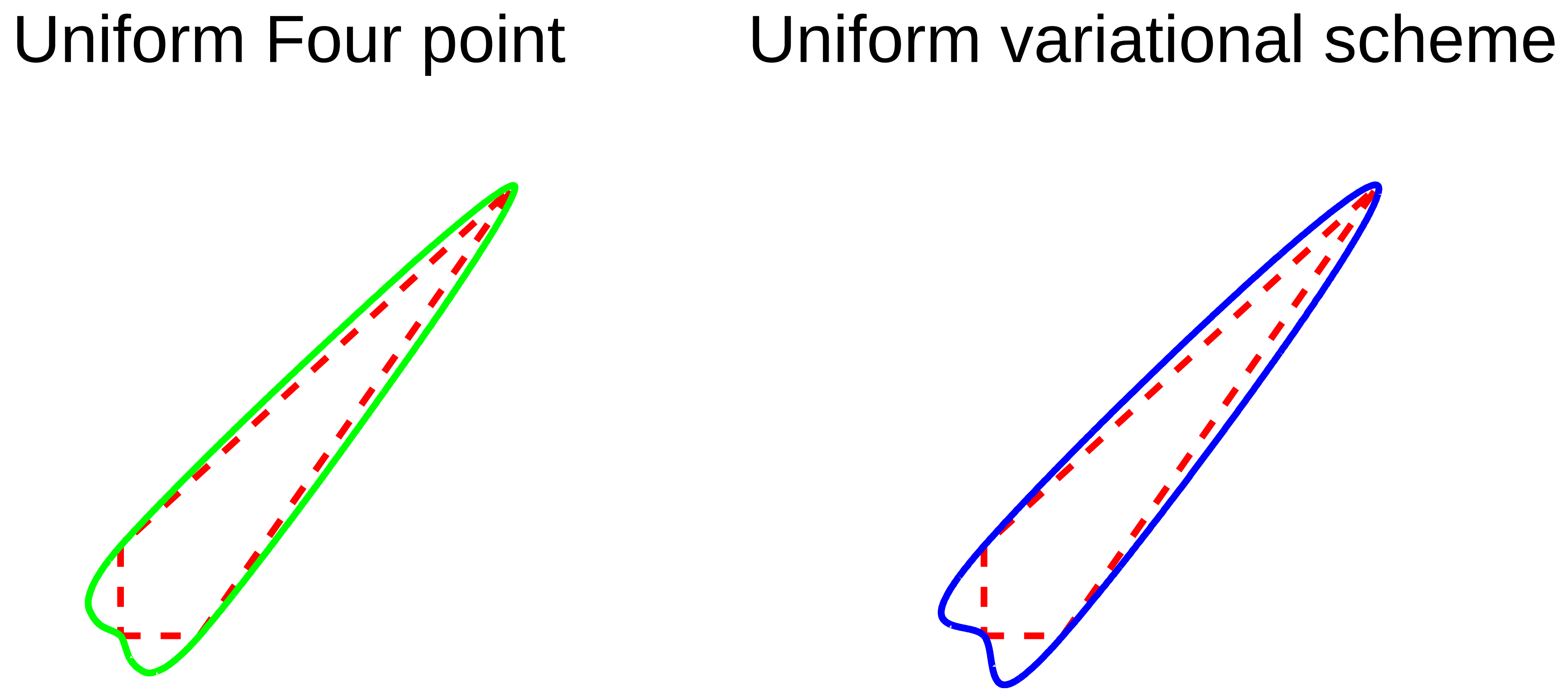
Euler-Lagrange Equation:

$$\Delta^4 f_{2l-1}^{k+1} = 0, \quad l{=}1{:}n.$$

Note: These are not ‘divided differences’. That is, the parameter spacing is not involved, it is assumed to be uniform.

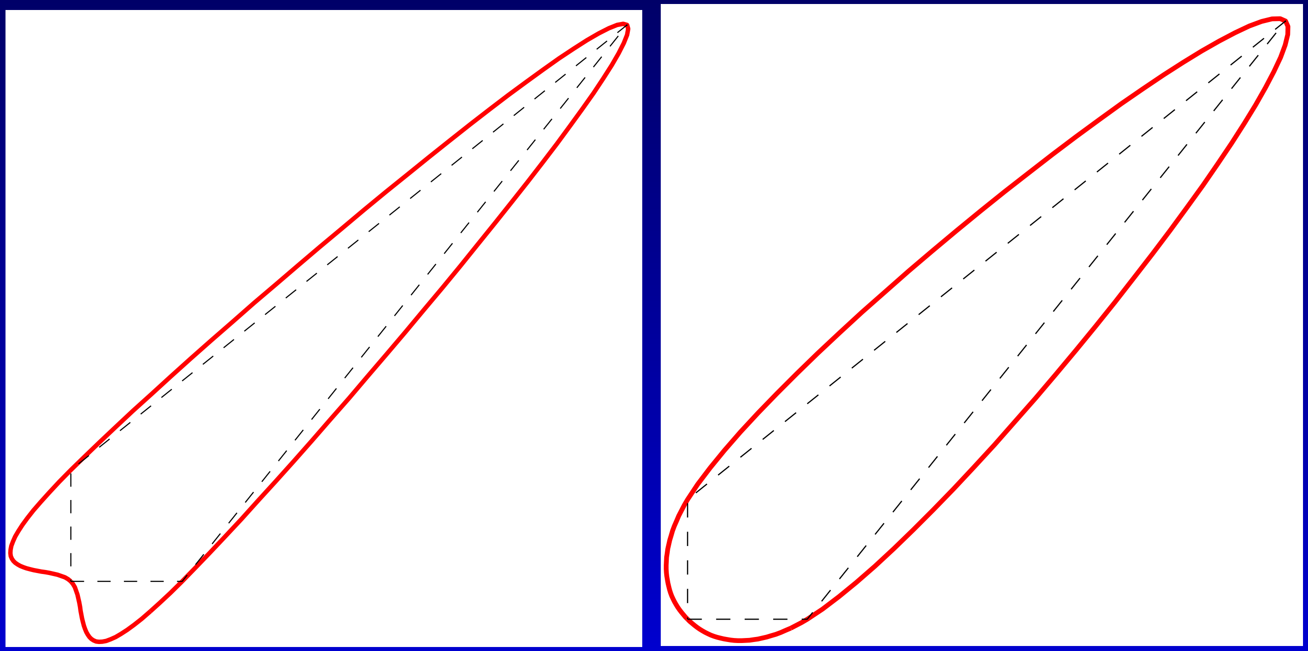
Why Non-Uniform?

Recall the uniform four point scheme of Dubuc with the uniform variational difference scheme of Kobbelt.



Why Non-Uniform?

Now compare the uniform variational subdivision with the non-uniform variational scheme to be described. We can control the shape by the parametrization.



Uniform ($t_i = 0, .25, .5, .75, 1$) and non-uniform ($t_i = 0, .1416, .5, .8584, 1$) parametrizations.

Non-Uniform Variational Refinement – Kersey (2003)

Let $f(t) = \sum_i f_i N_i(t)$ be a piecewise linear spline curve with coefficients f_i and knots $t_1, t_2, \ldots$, such that $f(t_j) = f_j$. Let

$$\begin{aligned} h_i &:= t_{i+1} - t_i \\ h_{i,2} &:= t_{i+2} - t_i = h_i + h_{i+1} \\ h_{i,3} &:= t_{i+3} - t_i = h_i + h_{i+1} + h_{i+2} \end{aligned}$$

Possible choices of t_i:

- Uniform: h_i = constant
- Chordal: $h_i = |f_{i+1} - f_i|$
- Centripetal: $h_i = |f_{i+1} - f_i|^e$

Non-Uniform Variational Refinement – Kersey (2003)

Consider the following discretization of the thin beam functional:

$$E(f) := \sum_i |s_i|^2 h_{i-1,2} \approx \int_a^b |f''(t)|^2 \, \mathrm{d}t$$

Variational Problem:

$$\mathrm{minimize}\{E(f) : f_{2i}^{k+1} = f_i^k\}$$

Optimality condition (jump in third divided difference):

$$\mathrm{jmp}_{t_i}(\Delta^3 f) := \frac{h_{i-1,3}}{h_i} \Delta_{i-1,3} f - \frac{h_{i-2,3}}{h_{i-1}} \Delta_{i-2,3} f = 0.$$

Non-Uniform Variational Refinement – Kersey (2003)

Advantages:

- Parametrization is built in to the scheme.
- Standard parametrizations can be used to control the shape of the curves.

Disadvantages:

- Extra computation needed to compute the ‘masks’.
- The scheme is still global.

Smoothness – Kersey (2004)

Assumptions:

- A1: (f^k) is sequence PWL curves minimizing $E(f)$
- A2: $\max\{h_{2i-1}^{k+1}, h_{2i}^{k+1}\} \leq \alpha h_i^k$ for some $1/2 \leq \alpha < 1$.
- A3: $E(f^k)$ is uniformly bounded as $k \rightarrow \infty$.
- A4: $|s_i^k|^2 h_{i,2}^k$ are uniformly bounded for all i as $k \rightarrow \infty$.
- A5: $|s_i^k|$ are uniformly bounded for all i as $k \rightarrow \infty$.
- A6: $|v_i^k|$ are uniformly bounded for all i as $k \rightarrow \infty$.
- A7: $|\mathrm{jmp}_{t_i}(\Delta^3 f^k)|$ are uniformly bounded for all i as $k \rightarrow \infty$, with $\mathrm{jmp}_{t_i}(\Delta^3 f) := h_{i-1,3}\, v_{i-1}\, / h_i - h_{i-2,3}\, v_{i-2}\, / h_{i-1}$.
- A8: $\dfrac{h_i^k h_{i+1}^k}{h_{i-2}^k h_{i-1}^k} \rightarrow 1$ as $k \rightarrow \infty$.

Smoothness – Kersey (2004)

Corollary 0.1 *Assume A1, A2 and A3. Then* $f^k \xrightarrow{u} f \in C^0([a,b])$.

Corollary 0.1 *Assume A1, A2 and A3. Then* $f^k \xrightarrow{u} f$ *for some* $f \in C^1([a,b])$ *with* $f' = d$.

Corollary 0.1 *Assume either A1, A2 and A6, or A1, A2, A7 and A8. Then,* $f^k \xrightarrow{u} f$ *for some* $f \in C^2$.

- Proofs depend on extension of Gregory and Qu Lemma given earlier.
- Uniform boundedness of s_i causes contraction of $d(t)$ to get C^1.

Approximation Order of Non-uniform Variational Subdivision

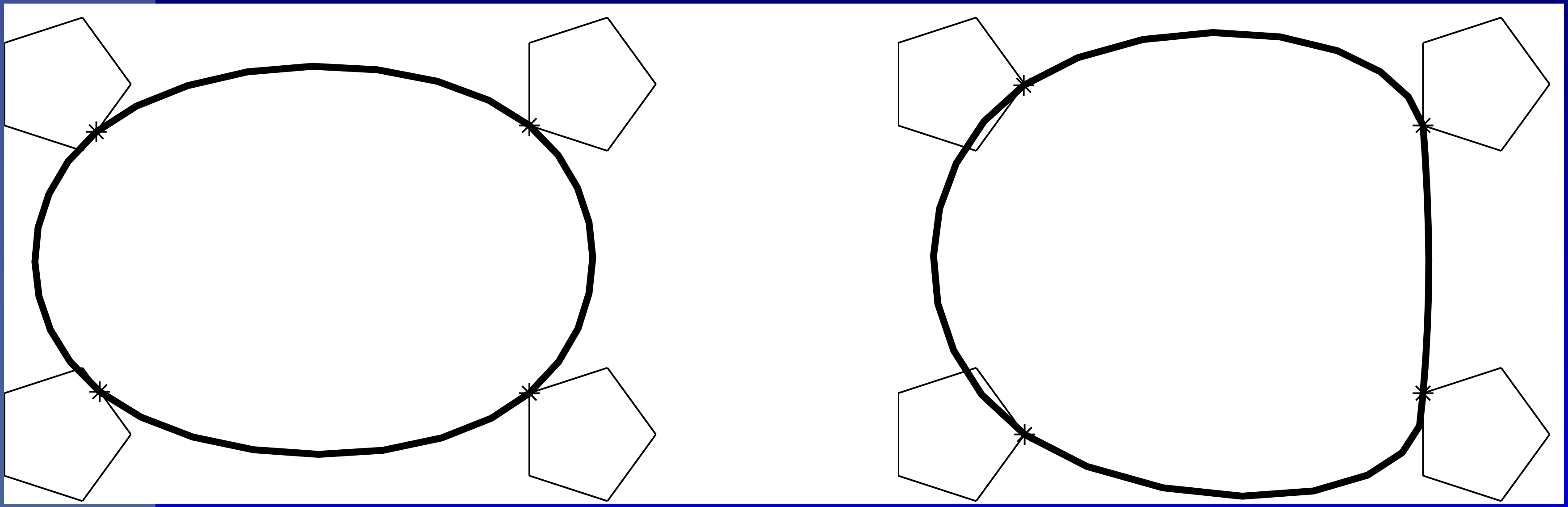
(Current work)

Approximation Order: Suppose that variational scheme reproduces C^k curves for $k \leq 3$. Then the approximation order is $O(h^k)$.

Proof based on linearity of the scheme and reproduction of polynomials.

Near-interpolatory subdivision curve in tension – Kersey (2003)

$$E(f) := \sum_i |\alpha_i s_i|^2 h_{i-1,2} \approx \int_a^b |\alpha(t)\, f''(t)|^2 \,\mathrm{d}t$$

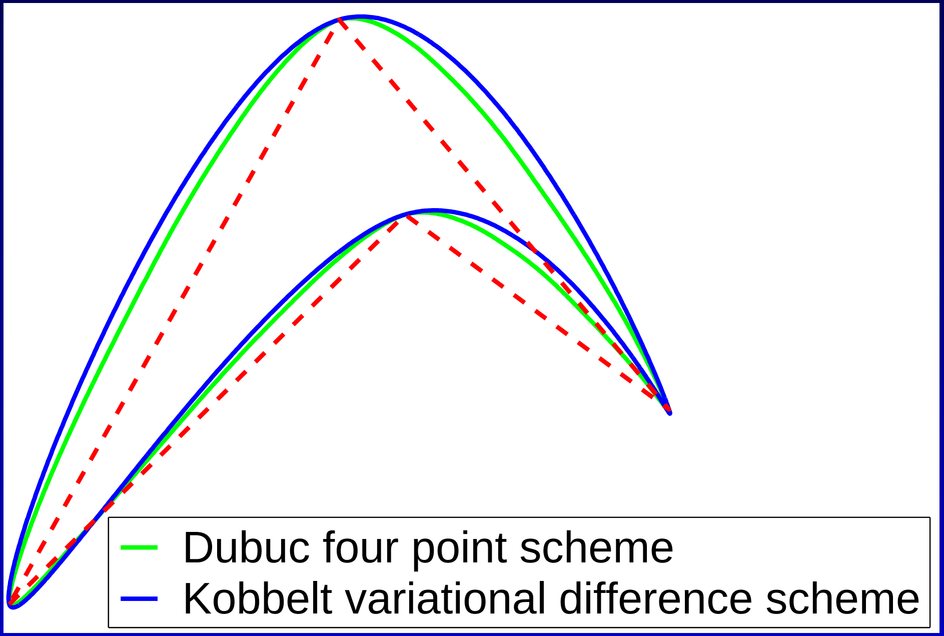


Part V

Local Variational Subdivision

Why Local?

Dubuc's four point scheme and Kobbelt's variational difference scheme.



C^1 non-regular curves, with cusps.

Why Local?

- For both schemes the number of points grows exponentially. That is $n = 2^{level} n_0$.
- Both schemes are $O(n) = O(2^{level} n_0)$. (The variational solves a 5 banded matrix.)
- The variational scheme is global – all vertices are needed to compute the value at one point. The four point scheme is local.
- "Big Advantage": The 4-pt scheme can evaluate at any non-dyadic point to machine tolerance (only a support of 6 points is needed to do this).

Idea: Local variational subdivision (current work).

Local Variational Refinement – (K05)

Ideas (current work):

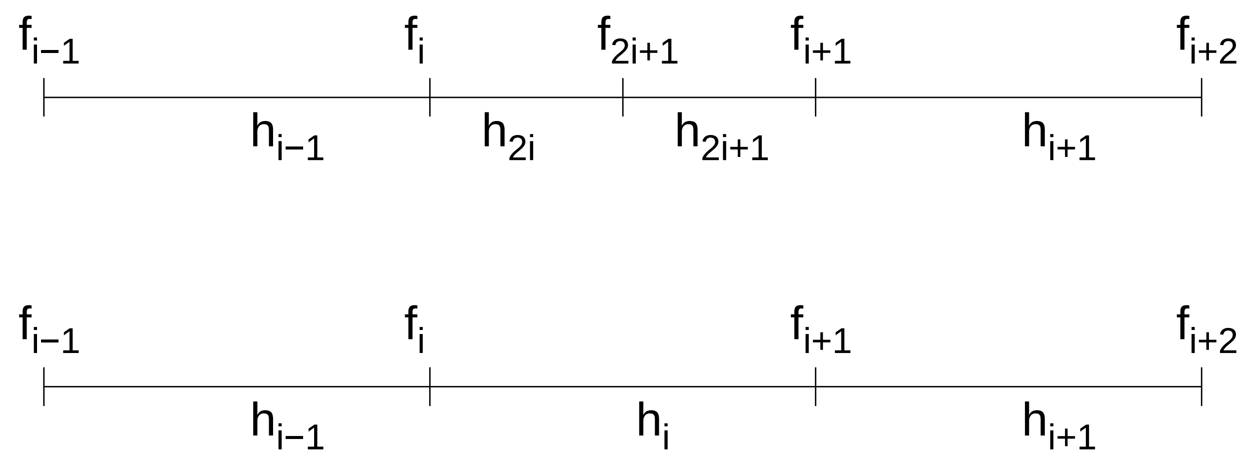
1. Insert each new point f_{2i+1} in the sequence, one at a time, and minimize energy functional $E_i(f)$. This generates a four-point scheme.
2. Insert three new points $f_{2i-1}, f_{2i+1}, f_{2i+3}$ to determine f_{2i+1}. This generates a six-point scheme.
3. To determine f_i minimize the local energy functional

$$E_i(f) := \sum_{i=j-k}^{j+k} |s_i|^2 h_{i-1,2}.$$

As $k \to \infty$ this approximates the global functional.

Variational four-point scheme (K05)

1. Insert each new point f_{2i+1}, one point at a time:



2. For each: minimize$\{E_i(f) := \sum_i |s_i|^2 h_{i-1,2}\}$

3. Optimality:

$$-\frac{h_{i-1,2}}{h_{2i}} f[t_{i-1}, t_i, t_{2i+1}, t_{i+1}] + \frac{h_{i,2}}{h_{2i+1}} f[t_i, t_{2i+1}, t_{i+1}, t_{i+2}] =$$

Variational four-point scheme (K05)

Theorem 0.1 *Suppose that* $\frac{h_{i-1,2}}{h_{i,2}} = \frac{h_{2i+1}}{h_{2i}}$ *for all* i. *If* $\partial_{f_{2i+1}} E_i(f) = 0$, *then*

$$f_{2i+1} = -w_{i-1}\, f_{i-1} (w_{i-1} + \frac{h_{2i+1}}{h_i})\, f_i + (w_i + \frac{h_{2i}}{h_i})\, f_{i+1} - w_i\, f_{i-}$$

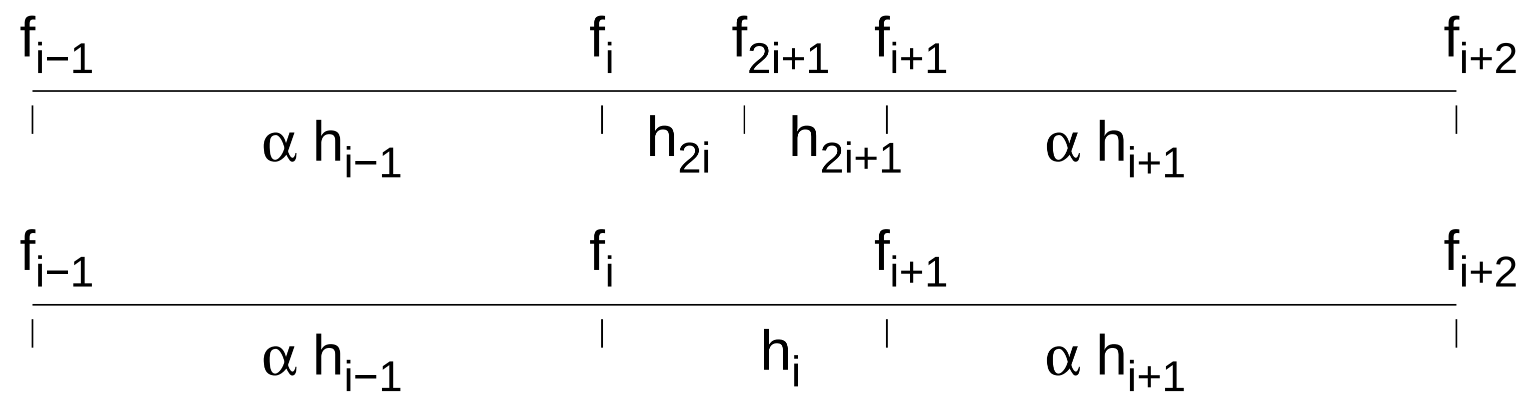
with

$$w_{i-1} = \frac{h_{2i} h_{2i+1}^2}{h_{i-1} h_{i-1,2} h_i} \quad \text{and} \quad w_i = \frac{h_{2i}^2 h_{2i+1}}{h_{i,2} h_i h_{i+1}}$$

Variational four-point scheme (K05)

Stretching (Interval Tension):

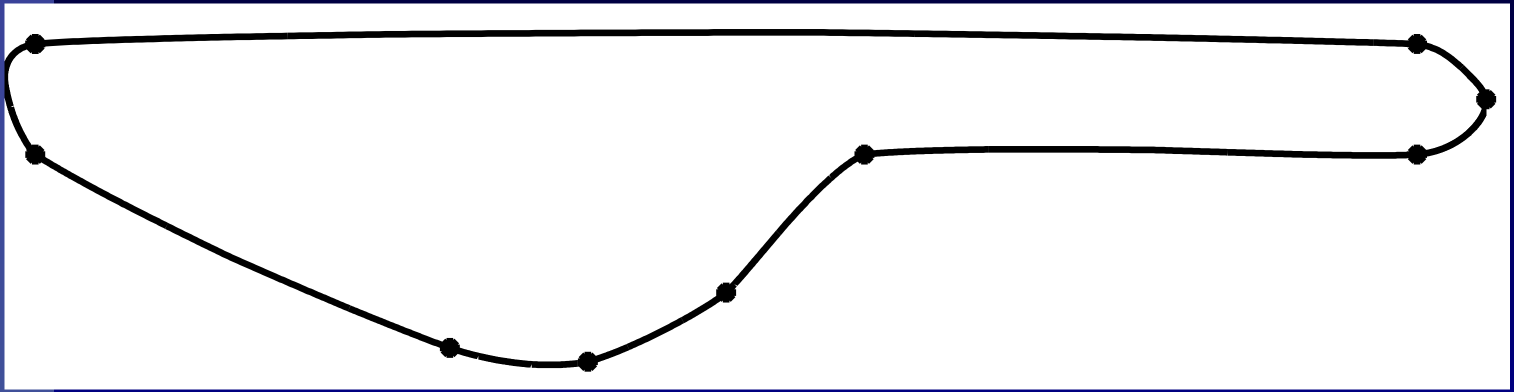
$$h_{i-1} := \alpha_i h_{i-1}, \quad h_{i+1} := \alpha_i h_{i+1}$$



Centripetal Parametrization: $h_i := |f_{i+1} - f_i|^e$

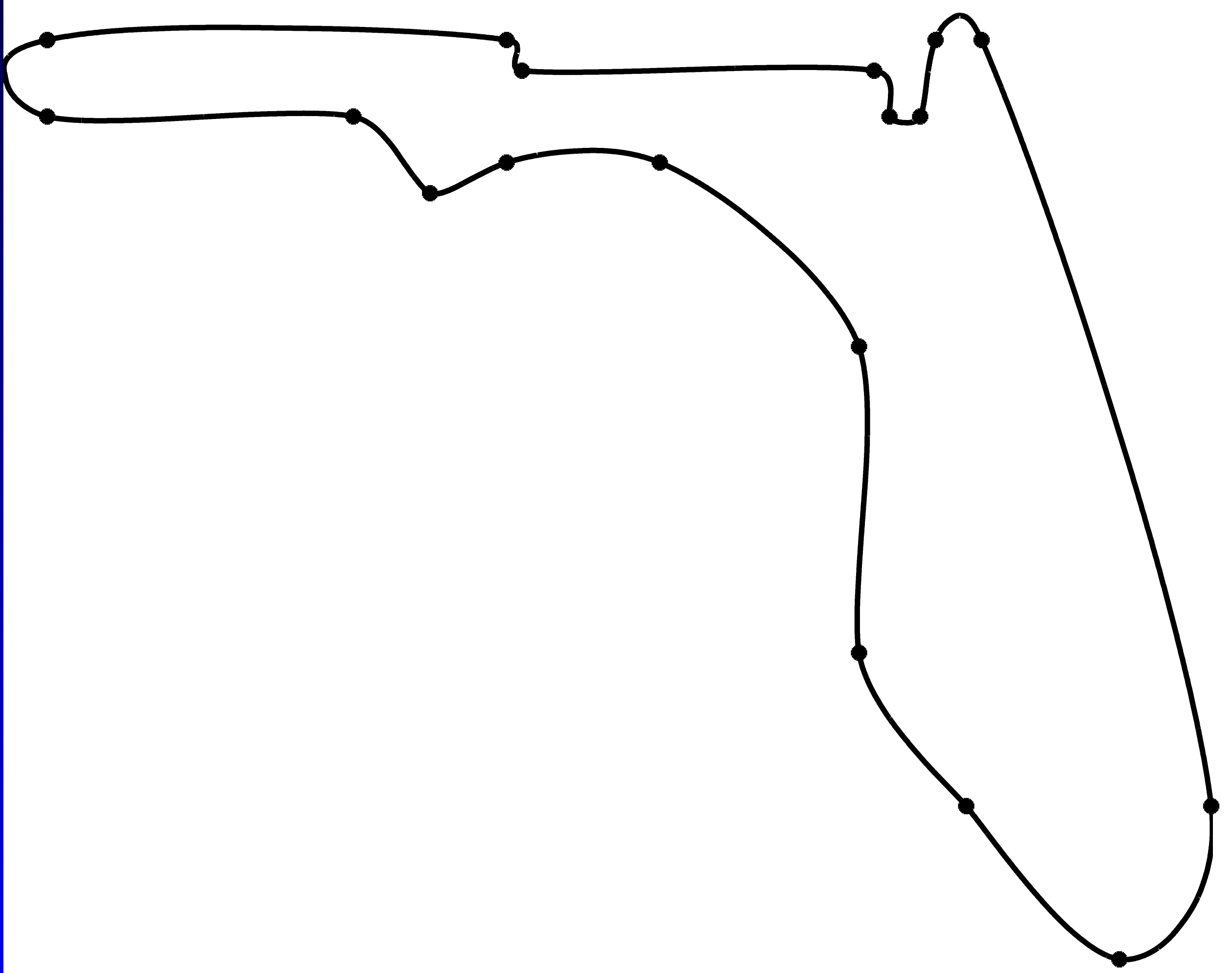
Tension scaled by length of segment.

Variational four-point scheme (K05)



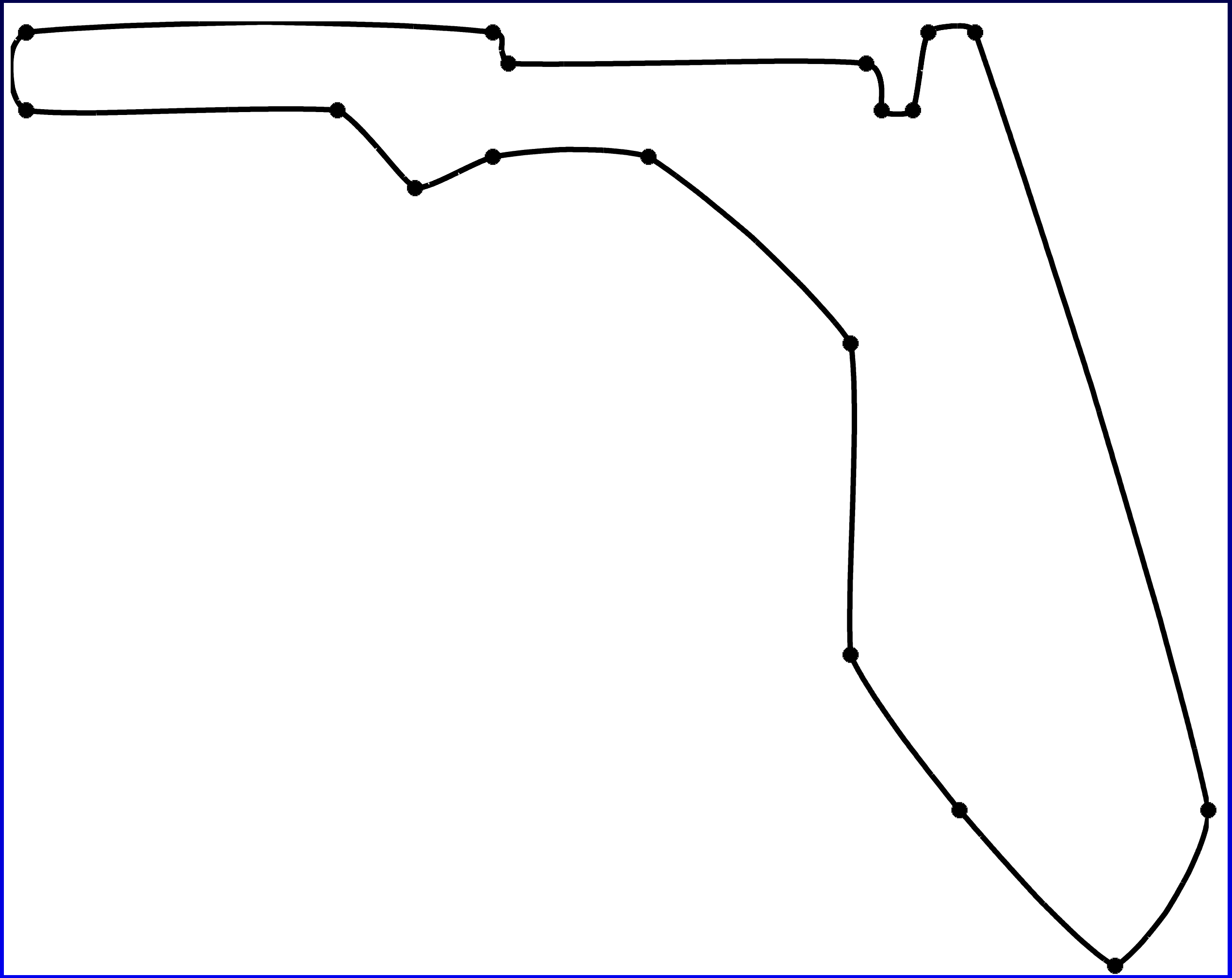
Variational four-point scheme (K05)

e=.2, alpha==1



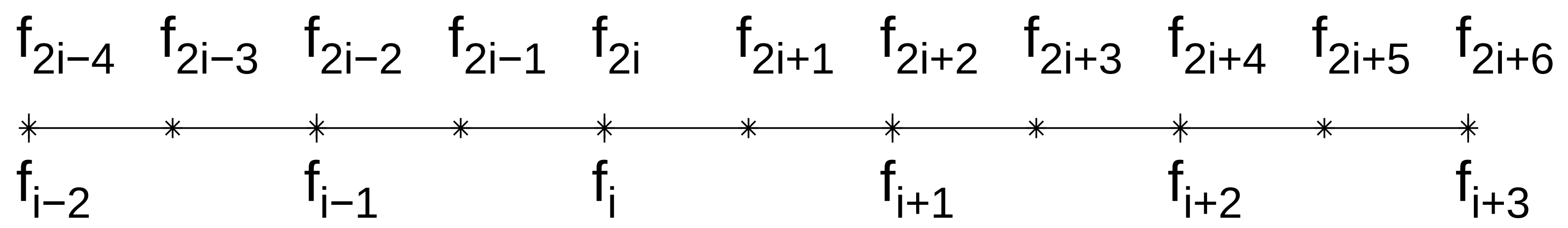
Variational four-point scheme (K05)

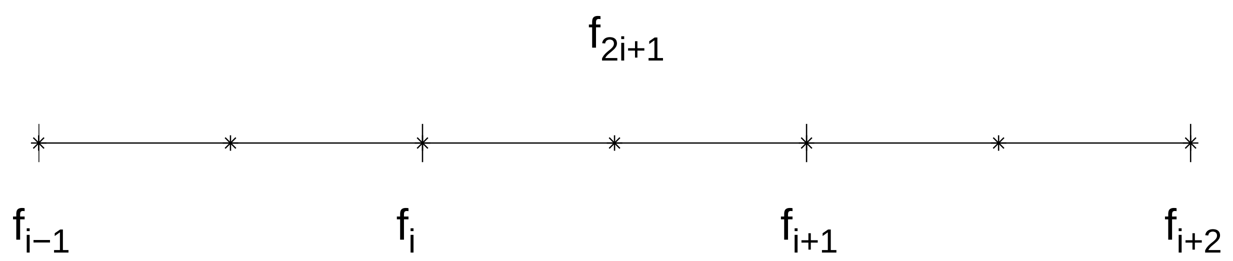
e=.2, alpha 1.1*Centripetal

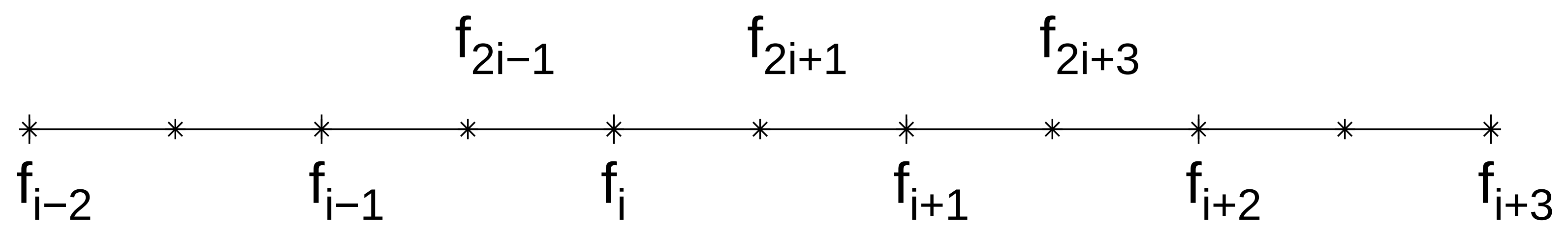


Variational six-point scheme (K05)

Global, 4-pt and 6-pt Subdivision:



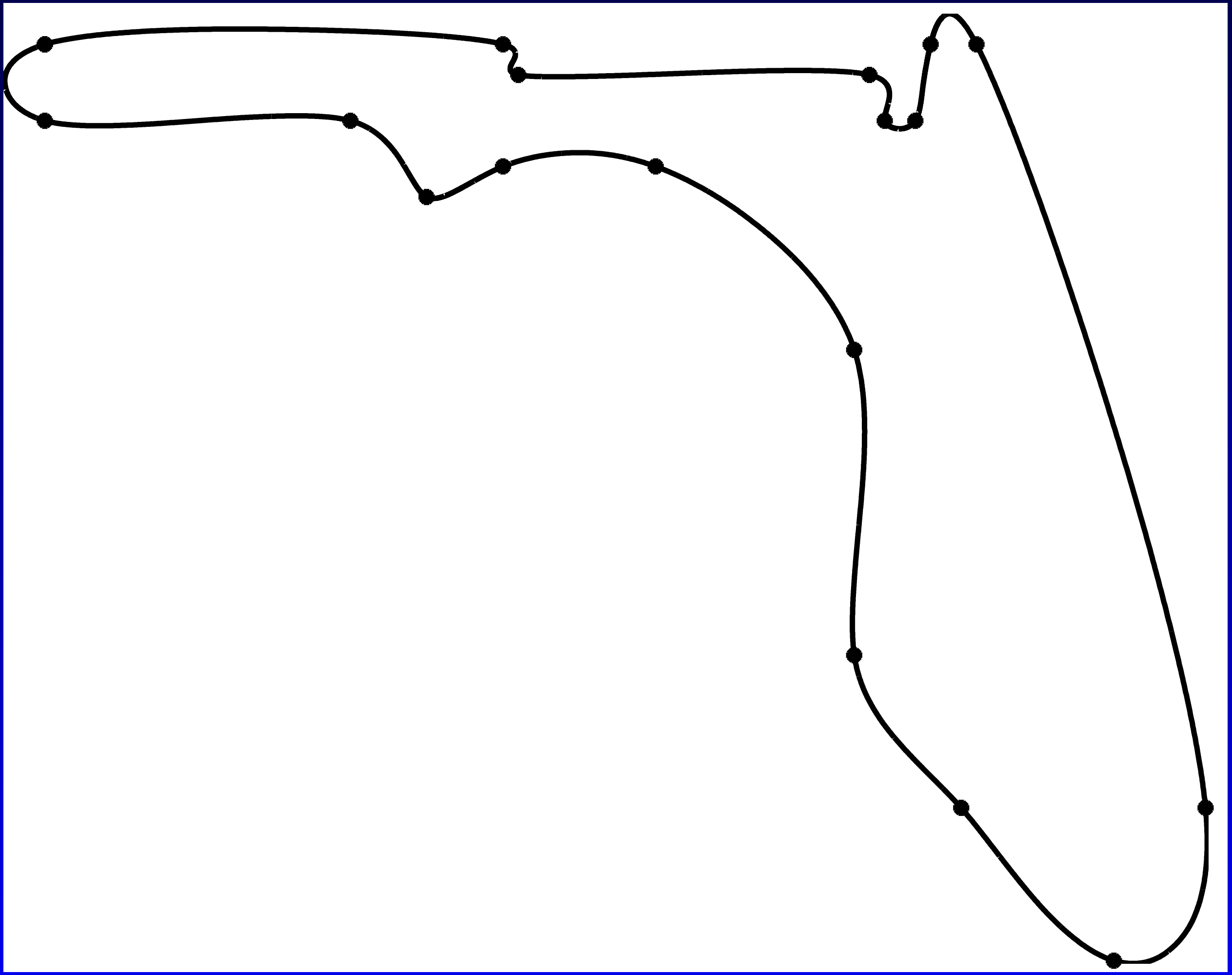




Idea: Insert Three points at a time to determine the middle point (Solve 3x3 systems).

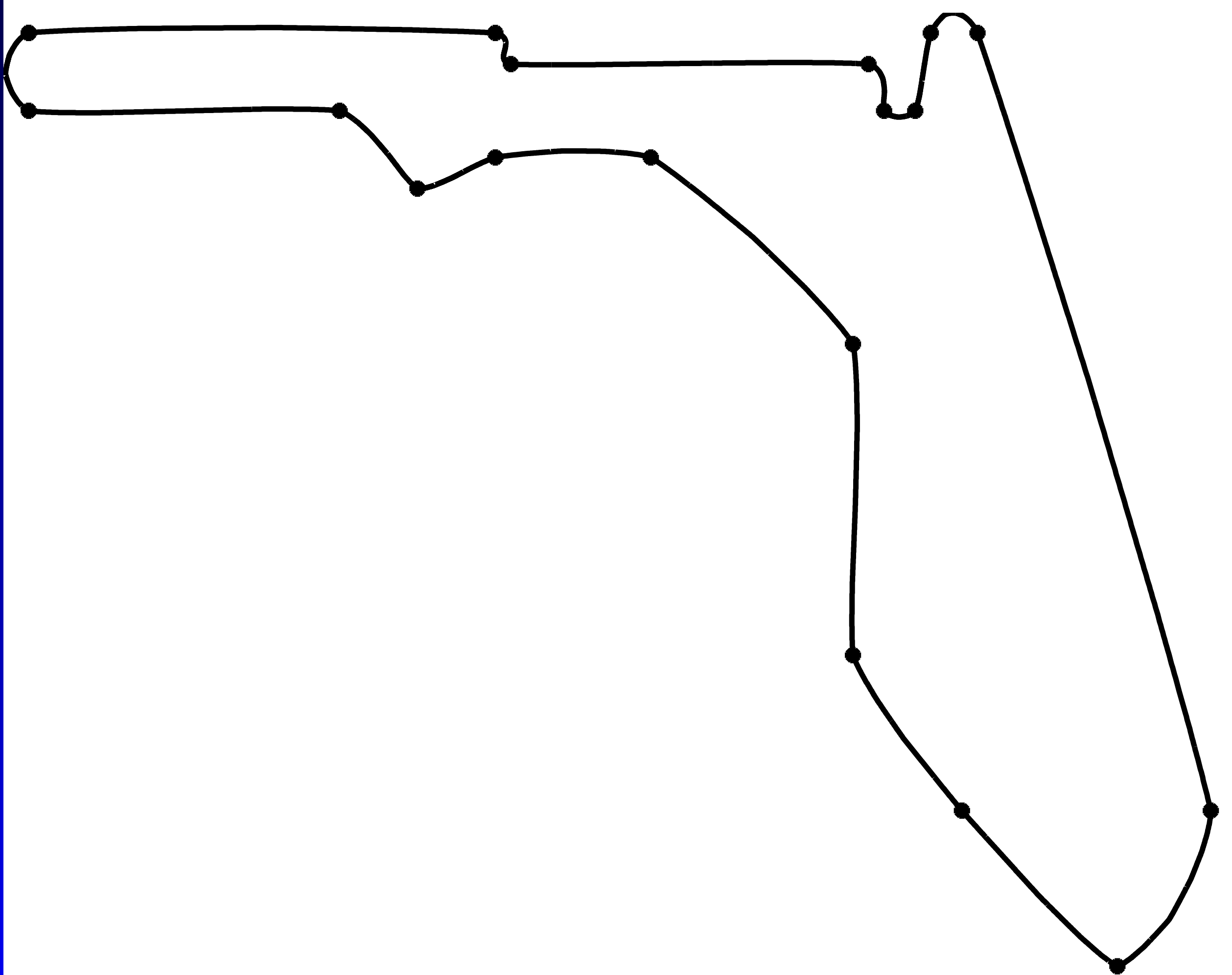
Variational six-point scheme (K05)

e=.2, alpha = uniform



Variational six-point scheme (K05)

e=.2, alpha = 1.1*Centripetal



Discretized Hölder Regularity

Lipschitz Continuous function:

$$|f(x) - f(y)| \le C|x - y|, \quad \omega_f(h) \le C\,h$$

Holder Regularity $H_{l+\alpha}$:

$$|f^{(l)}(x) - f^{(l)}(y)| \le C|x - y|^\alpha, \quad \omega_{f^{(l)}}(h) \le C\,h^\alpha$$

Discretization:

$$\left|l!\,\Delta^l f_{i+1}^k - l!\,\Delta^l f_i^k\right| \le C\,|h_i^k|^\alpha$$

Discretized Hölder Regularity

As in Kuijt et al., let

$$\rho_i^k := l! \left| \Delta^l f_{i+1}^k - \Delta^l f_i^k \right| .$$

Then

$$\frac{\rho_i^{k+1}}{\rho_i^k} \approx \frac{C(h_i^{k+1})^{\alpha_i}}{C(h_i^k)^{\alpha_i}} = \left(\frac{h_i^{k+1}}{h_i^k} \right)^{\alpha_i} ,$$

and so

$$\alpha_i \approx \log \left(\frac{\rho_i^{k+1}}{\rho_i^k} \right) \Big/ \log \left(\frac{h_i^{k+1}}{h_i^k} \right) .$$

Use arc length parametrization (a regular parametrization).

Discretized Hölder Regularity

Smoothness of Non-Uniform Local Variational Schemes.

Based on ‘Florida’ example shown earlier.

Method	Tension α	Parametrization	Regularity
MDL04	DNA	Chordal e=1	1.9
Four-Point	Uniform	Initially Chordal e=1	1.9
Four-Point	1.1*Centripetal	Centripetal e=.2	1.3
Six-Point	Uniform	Initially Centripetal e=.2	2.6
Six-Point	1.1*Centripetal	Centripetal e=.2	1.3

Part VI

Variational Subdivision Surfaces

Variational Surfaces – Kobbelt

The *thin-plate spline* minimizes the functional:

$$\int\int f_{uu}^2 + 2\, f_{uv}^2 + f_{vv}^2 \,\mathrm{d}A,$$

Discretization functional over surfaces:

$$E(p) := \sum_i \frac{1}{m_i} \sum_{j=1}^{m_i} \frac{1}{m_{ij}} \sum_{k=1}^{m_{ij}} |p_{ijk} - 2\,p_{ij} + p_i|^2.$$

Variational problem:

$$\text{minimize}_p \,\{E(p) : p_i = q_i \text{ for } i \in I_1\}.$$

Variational Surfaces – Kobbelt

Optimality (Laplacian smoothing): For all $i \notin I_1$,

$$p_i = p_i - \frac{1}{\nu_i} U^2(p_i)$$

with

$$U(p_i) := \frac{1}{m_i} \sum_{j=1}^{m_i} p_{ij} - p_i,$$

$$U^2(p_i) := \frac{1}{m_i} \sum_{j=1}^{m_i} U(p_{ij}) - U(p_i) = 0,$$

$$\nu_i := 1 + \frac{1}{m_i} \sum \frac{1}{m_{ij}}.$$

Variational Surfaces – Kobbelt

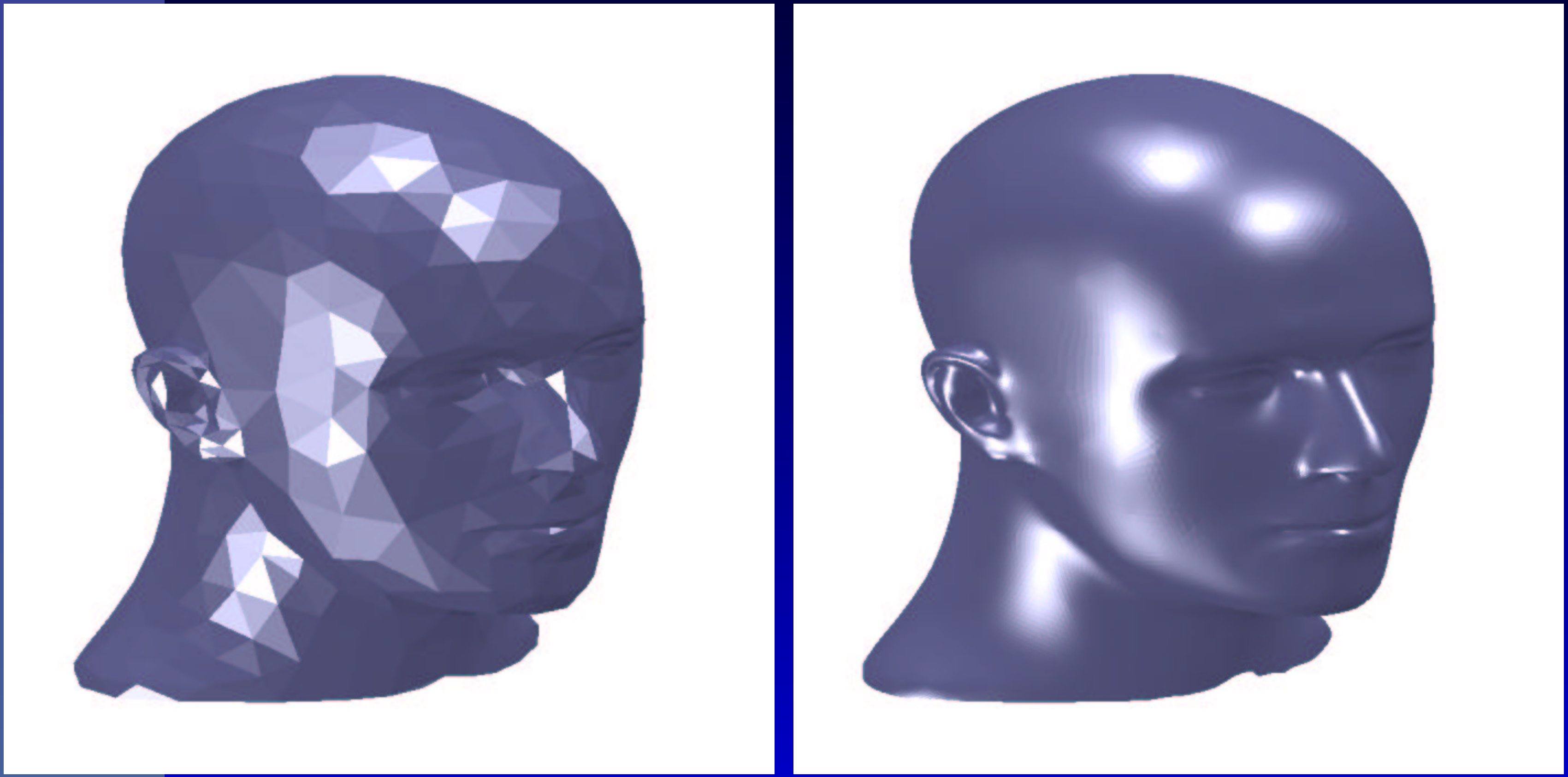


Figure 0.0: Interpolated Mannequin Head Data

Smoothing Subdivision Surfaces – Kersey (2004)

Let $\{I_1, I_2, I_3\}$ be a partition of the index set $i{=}1{:}n$, corresponding to those vertices that are interpolated, smoothed/near-interpolated, or unconstrained, respectively.

Mixed smoothing and interpolation:

$$\text{minimize}_p \left\{E(p) + \sum_{i \in I_2} w_i |p_i - q_i|^2 : p_i = q_i \text{ for } i \in I_1\right\},$$

Near-interpolation:

$$\text{minimize}_p \left\{E(p) : \ |p_i - q_i| \leq e_i \text{ for } i \in I_2,\ p_i = q_i \ \text{ for } i \in I_1\right\}.$$

Smoothing Subdivision Surfaces – Kersey (2004)

Mixed smoothing and interpolation:

$$\text{minimize}_p \; \{E(p) + \sum_{i \in I_2} w_i |p_i - q_i|^2 : p_i = q_i \text{ for } i \in I_1\}.$$

Optimality:

$$\boxed{\begin{array}{l} p_i = q_i, \;\; \text{for } i \in I_1 \\ U^2(p_i) + w_i\,(p_i - q_i) = 0 \text{ for } i \in I_2 \\ U^2(p_i) = 0 \text{ for } i \in I_3 \end{array}}$$

Smoothing Subdivision Surfaces – Kersey (2004)

Mixed smoothing and interpolation:

$$\text{minimize}_p \left\{ E(p) + \sum_{i \in I_2} w_i |p_i - q_i|^2 : p_i = q_i \text{ for } i \in I_1 \right\}.$$

Iteration:

$$\boxed{\begin{aligned} & p_i = q_i, \ \text{for } i \in I_1 \\ & p_i \leftarrow \tfrac{1}{\nu_i + w_i}\left(w_i\, q_i + \nu_i\, p_i - U^2(p_i)\right), \ \text{for } i \in I_2 \\ & p_i \leftarrow p_i - \tfrac{1}{\nu_i} U^2(p_i), \ \text{for } i \in I_3. \end{aligned}}$$

Smoothing Subdivision Surfaces – Kersey (2004)

Near-interpolation:

$$\text{minimize}_p \, \{E(p) : \, |p_i - q_i| \le e_i \text{ for } i \in I_2, \; p_i = q_i \text{ for } i \in$$

Iteration:

$$\boxed{\begin{array}{l}
p_i = q_i, \;\; \text{for } i \in I_1 \\[1em]
p_i \leftarrow \frac{1}{\nu_i + w_i} \left(w_i \, q_i + \nu_i \, p_i - U^2(p_i) \right), \;\; \text{for } i \in I_2 \\[1em]
p_i \leftarrow p_i - \frac{1}{\nu_i} U^2(p_i), \;\; \text{for } i \in I_3 \\[1em]
w_i \leftarrow \frac{|p_i - q_i|}{e_i} w_i, \;\; \text{for } i \in I_2.
\end{array}}$$

Examples

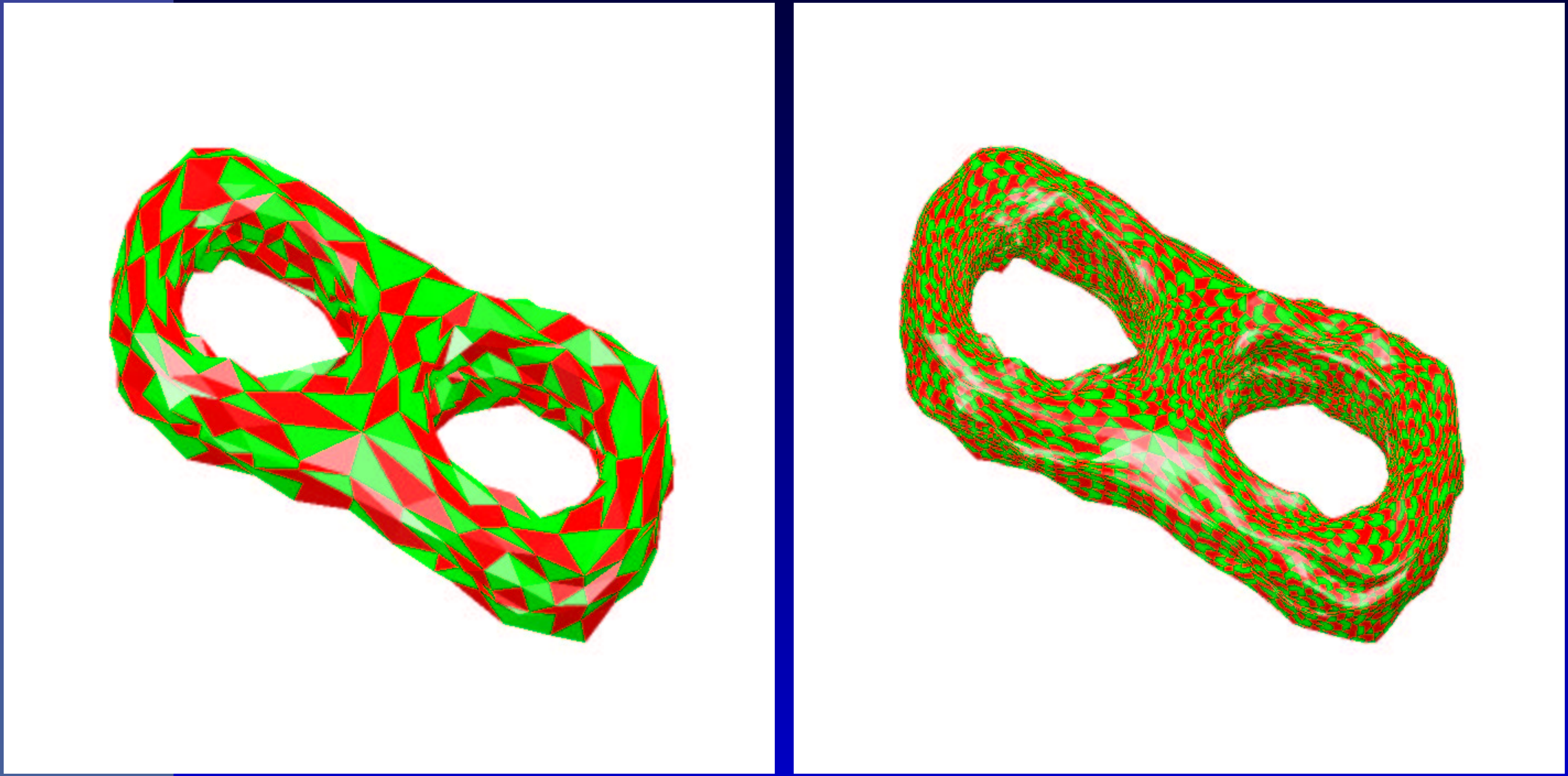


Figure 0.1: Interpolated Noisy Torus

Examples

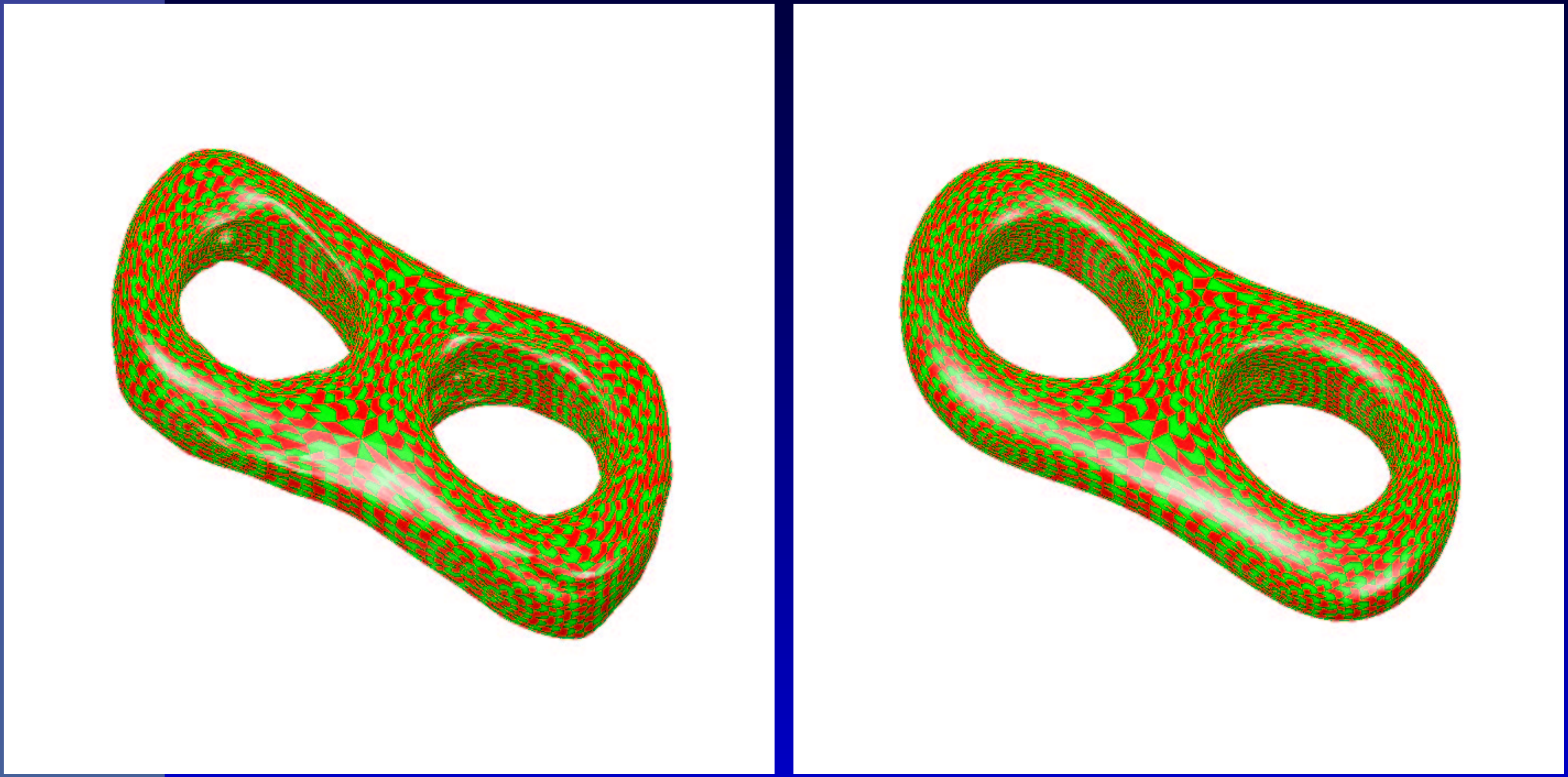


Figure 0.1: Near-Interpolated Noisy Torus

Examples

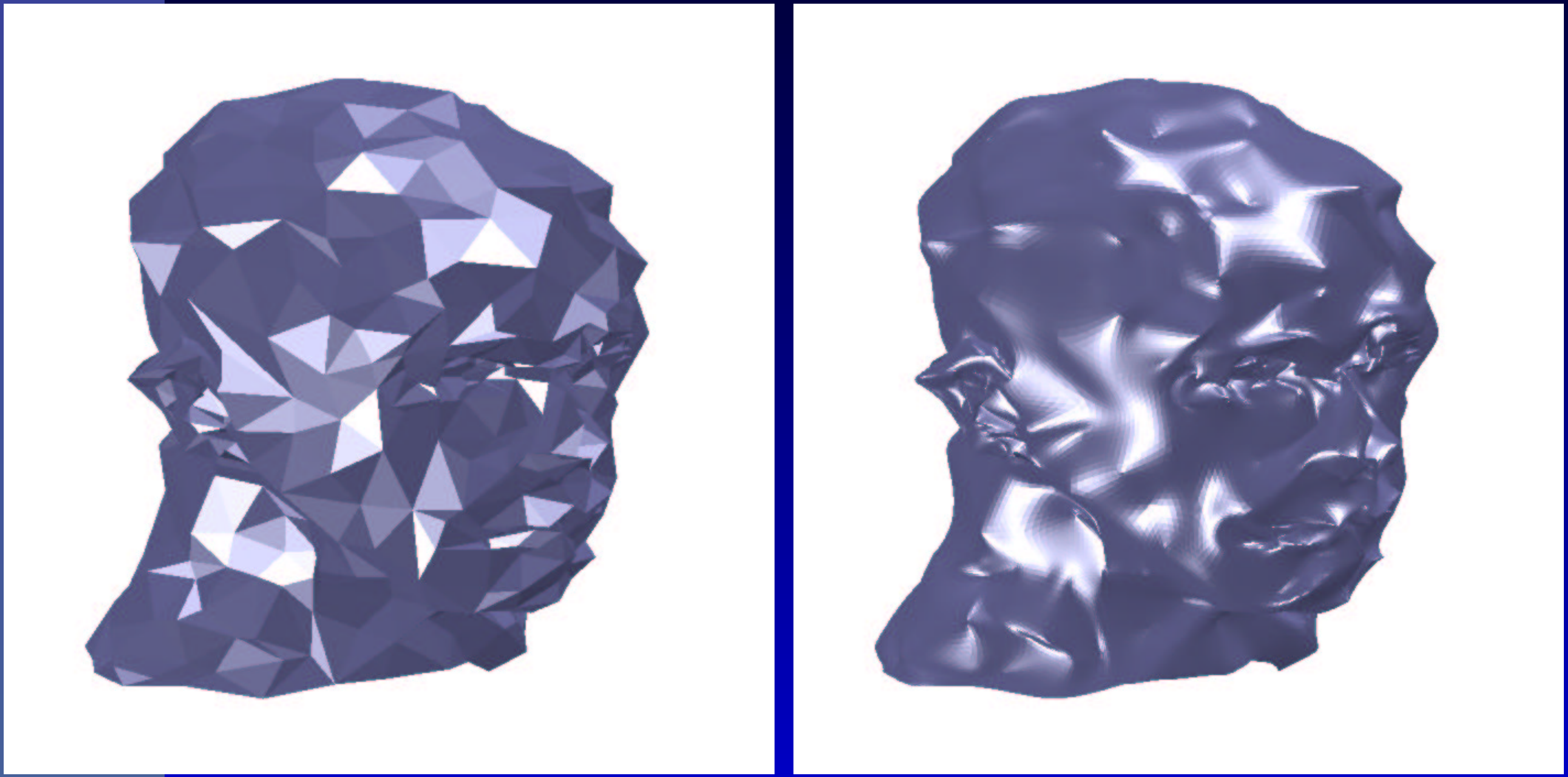


Figure 0.1: Interpolated Noisy Mannequin Head Data

Examples

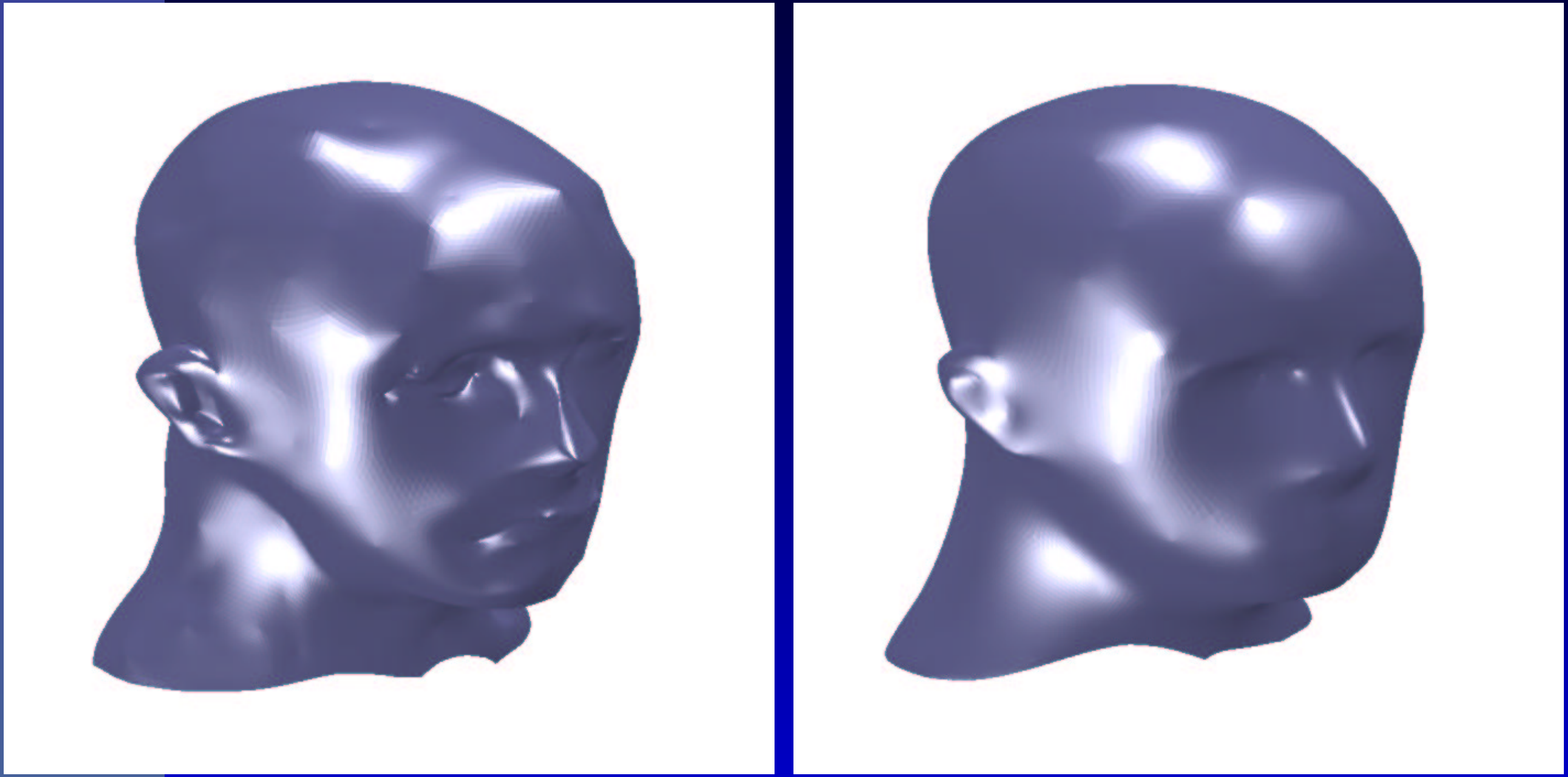


Figure 0.1: Near-Interpolated Noisy Mannequin Head Data

The End